CHAPTER 9

Linear Algebra – II

(Matrix Eigen Value Problems)

9.1 : The Characteristic Equation of a Matrix

Definitions : Let A be any nth order square matrix and λ some scalar,

(a) **Characteristic Matrix**

The matrix $(A-\lambda I)$ is called the characteristic matrix of A (where I is a unit matrix of order n).

$$A-\lambda I=\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}$$

(b) **Characteristic Polynomial**

The determinant $|A-\lambda I|$ when expanded yields a polynomial $\phi(\lambda)$ of degree n in λ and is called the characteristic polynomial of the matrix A.

(c) **Characteristic Equation**

The equation $\phi(\lambda)=|A-\lambda I|=0$ is called the characteristic equation of matrix A.

(d) **Characteristic Roots**

The roots of a characteristic equation $|A-\lambda I|=0$

$\lambda_1,\lambda_2,\dots\dots\dots\lambda_n$ (say) are called characteristic roots of the matrix A. Characteristic roots are also called **latent** roots or **eigen-values.**

(e) **Spectrum.** The set of characteristic roots of a matrix A is called the spectrum of matrix A .

Example – 1 : Find the characteristic polynomial of the following matrix $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 2 \end{bmatrix}$.

Solution : Characteristic matrix of any matrix A is given by $A-\lambda I$.

$$A-\lambda I=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 3\\ -2 & 1 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2-\lambda & -1 & 1\\ 1 & 2-\lambda & 3\\ -2 & 1 & 2-\lambda\end{bmatrix}$$

The characteristic polynomial is $|A-\lambda I|$

$$|A-\lambda I|=\begin{vmatrix}2-\lambda & -1 & 1\\ 1 & 2-\lambda & 3\\ -2 & 1 & 2-\lambda\end{vmatrix}$$

$$=-\lambda^3+6\lambda^2-12\lambda+15\cdot$$

Example – 2 : Find the characteristic roots and spectrum of matrix $\mathbf{A}=\begin{bmatrix}\mathbf{8} & \mathbf{-6} & \mathbf{2}\\ \mathbf{-6} & \mathbf{7} & \mathbf{-4}\\ \mathbf{2} & \mathbf{-4} & \mathbf{3}\end{bmatrix}$ **.**

Solution : Characteristic matrix is $A-\lambda I$

$$A-\lambda I=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}8-\lambda & -6 & 2\\ -6 & 7-\lambda & -4\\ 2 & -4 & 3-\lambda\end{bmatrix}$$

Characteristic equation in $|A-\lambda I|=0$

i.e, $\begin{vmatrix}8-\lambda & -6 & 2\\ -6 & 7-\lambda & -4\\ 2 & -4 & 3-\lambda\end{vmatrix}=0$

$\Rightarrow \quad \lambda^3-18\lambda^2+45\lambda=0 \qquad \Rightarrow \qquad \lambda(\lambda^2-18\lambda+45)=0$

$\Rightarrow \quad \lambda(\lambda-15)(\lambda-3)=0$

$\Rightarrow \quad \lambda=0,\ 3,\ 15$. Hence the characteristic roots of A are 0, 3, 15 and the spectrum is $\{0, 3, 15\}$.

9.2 : Eigen Values & Eigen Vectors

Let $A=[a_{ij}]$ be a m × n matrix.

Consider the vector equation Ax = λx .... (1)

Where λ may be real or complex : Here the zero vector x=0 is a trivial solution of (1).

The value of λ for which (1) has a solution $x \neq 0$ is called the characteristic value or eigen value of the matrix A and the corresponding solution $x \neq 0$ of (1) are called eigen vectors or characteristic vector of A corresponding to the eigen value λ.

The set of eigen values are called spectrum of A. The largest of the absolute values of the eigen values of A is called the spectral radius of A.

The problem of determing the eigen values & eigen vectors of a matrix is called an eigen value problem.

How to solve :

Let $A=[a_{ij}]$ be a square matrix.

1. Consider the characteristic matrix as $(A-\lambda I)$ where I is an identity.
2. Write the charateristic equation as det $(A-\lambda I)=0$
3. Solving the characteristic equation we can fixed the valus of λ_1 called the eigen values. Let, λ_1, λ_2, λ_3 be the eigen value of A.
4. Corresponding to $\lambda_1, \lambda_2, \lambda_3$ we can find there eigen vectors satisfying

$Ax=\lambda_1 x$, $Ax=\lambda_2 x$ & $Ax=\lambda_3 x$ respectively.

Important properties of Eigen Values

Theorem – 1 : (a) The product of the eigen values of a matrix $A_{n\times n}$ is equal to its determinant.
(b) The sum of the eigen values of a matrix is the trace of the matrix.

Proof : (a) If $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$ are n eigen values of A

$$|A-\lambda I)=(-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots.(\lambda-\lambda_n)$$

Put, $\lambda=0$,

$$|A|=(-1)^n.(-\lambda_1)(-\lambda_2)\ldots.(-\lambda_n)=(-1)^n.(-1)^n.\lambda_1.\lambda_2\ldots\lambda_n$$
$$=(-1)^{2n}\lambda_1, \lambda_2\ldots\ldots..\lambda_n.$$
$$=\lambda_1.\lambda_2\ldots\lambda_n$$

From this result, we can say that a matrix is singular if it has at least one zero eigen value and non singular if all its eigen values are non-zero.

(b) Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$\Rightarrow \quad -\lambda^3+\lambda^2(a_{11}+a_{22}+a_{33})\ldots=0 \ldots\ldots\ldots\ldots\ldots.. \text{(i)}$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then

$$|A-\lambda I|=(-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=0$$

$$\Rightarrow \quad -\lambda^3+\lambda^2(\lambda_1+\lambda_2+\lambda_3)=0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$\therefore$ Comparing (i) and (ii), we get

$$\lambda_1+\lambda_2+\lambda_3=a_{11}+a_{22}+a_{33}$$

Theorem – 2 : If λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

Proof : If x be the eigen vector corresponding to eigen value λ, then

$Ax=\lambda x$

Premultiply both sides by A^{-1}.

$$A^{-1}.(Ax)=A^{-1}(\lambda x) \Rightarrow (A^{-1}A).x=\lambda A^{-1}x$$

$\Rightarrow \quad \frac{1}{\lambda}.x = A^{-1}x \;\Rightarrow\; \frac{1}{\lambda}$ is an eigen value of A^{-1}.

In general,

If $\lambda_1, \lambda_2, ... \lambda_n$ be the eigen values of A then

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ... \frac{1}{\lambda_n}$ be the eigen values of A^{-1}.

Theorem – 3 : If λ be an eigen value of a matrix A then λ^m be an eigen value of a matrix A^m, (m > 0).

Proof: If x be the eigen vector corresponding to the eigen values λ, then

$Ax = \lambda x$ .... (i)

Premultiply by A on both sides,

$$A.(Ax) = A(\lambda x)$$

$$\Rightarrow \quad (A.A)x = \lambda.Ax \;\Rightarrow\; A^2x = \lambda.Ax \text{ by (i)}$$

$$\Rightarrow \quad A^2x = \lambda^2 x \text{ ....(ii)}$$

Again pre - multiply by A both sides,

$$A.(A^2x = A.(\lambda^2 x) \Rightarrow (A.A^2).x = \lambda^2.Ax$$

$$\Rightarrow \quad A^3x = \lambda^2.\lambda x \text{ by (i)} \;\Rightarrow\; A^3x = \lambda^3 x$$

Similarly, $A^m x = \lambda^m x$ which shows that λ^m is the eigen value of $A^m.(m > 0)$

In general,

If $\lambda_1, \lambda_2, ... \lambda_n$ by the eigen values of A, then $\lambda_1^m, \lambda_2^m, ... \lambda_n^m$ be the eigen values of $A^m.(m > 0)$.

Theorem – 4 : If λ be an eigen value of a non -singular matrix A, show that $|A|/\lambda$ is an eigen value of the matrix adj A.

Proof : If x be the eigen vectors corresponding to the eigen value λ then

$Ax = \lambda x$

Pre-multiply both sides by (adj.A).

$$(adj\;\; A).(Ax) = (adj\;\; A)\,(\lambda x)$$

$$\Rightarrow \quad (adj\;\; A).(A)x = \lambda(adj\;\; A).x$$

$$\Rightarrow \quad |A|.Ix = \lambda(adj\;\; A)\,x \qquad (\because \text{(adj A) A} = |A|.I)$$

$$\Rightarrow \quad \frac{|A|}{\lambda}.x = (adj\;\; A)\,.\,x$$

which shows that $\frac{|A|}{\lambda}$ is the eigen value of adj A.

Theorem –5: *(a)* λ is characteristic root of a matrix A, iff there exists a non-zero vector X such that AX = λX.

Proof : Let A be an $n \times n$ matrix . If λ be a characteristic root of A, then $|A - \lambda I| = 0$

that is , the characteristic matrix $(A - \lambda I)$ is singular.

Hence there exists a non-zero solution of the system

$(A - \lambda I)\,X = 0$ or, $AX = \lambda\,X$.

Conversely, let there exist a non-zero vector X such that

$AX = \lambda X$.

Thus there exists a non-zero solution of the equation

$(A - \lambda I)X = 0$.

Therefore the coefficient matrix $(A - \lambda I)$ must be singular.

This implies that $|A - \lambda I| = 0$

that is , λ is a characteistic root of the matrix A.

(b) Corresponding to an eigen value of a matrix there will correspond different eigen vectors of the matrix but corresponding to one eigen vector of a matrix there cannot correspond more than one eigen value of it.

Proof : Since the system of equations. $(A - \lambda I)X = 0$

is homogeneous, it is obvious that, if X be an eigen vector of A, then kX, where $k(\neq 0)$ is any scalar, is also an eigen vector of A corresponding to the same eigen value.

Let $X(\neq 0)$ be the eigen vector of a matrix A corresponding to the eigen value λ. Then X satisfies the equation.

$AX = \lambda X$.

Let k be any non-zero scalar so that $kX \neq 0$ [by (1)].

Now $A(kX) = k(AX) = k(\lambda X) = \lambda(kX)$.

Thus the non-zero vector kX satisfies the equation (1). Hence kX is also an eigen vector of the matrix A corresponding to the eigen value λ.

Thus there are more than one eigen vector of A corresponding to the same eigen value of A.

On the other hand, if λ_1 and λ_2 be two eigen values corresponding to the same eigen vector X of A, then $AX = \lambda_1 X$ and $AX = \lambda_2 X$

that is , $\lambda_1 X = \lambda_2 X$ or, $(\lambda_1 - \lambda_2)X = 0$.

But $X \neq 0$, therefore $\lambda_1 = \lambda_2$.

Thus the eigen vector X of a matrix A cannot correspond to more than one eigen value of A.

(c) Two eigen vectors of a square matrix A over a field F corresponding to two distinct eigen values of A are linearly independent. ***[B.P.U.T. – 2013***

Proof : Let X_1 and X_2 be two eigen vectors of A corresponding to two distinct eigen values λ_1 and λ_2 respectively. Then $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$ ....(1)

Let us consider the relation $a_1X_1 + a_2X_2 = 0$ .... (2)

where a_1 and a_2 are scalars.

Then we have $a_1AX_1 + a_2AX_2 = 0$ ....(2)

or, $a_1\lambda_1X_1 + a_2\lambda_2X_2 = 0$, from (1).

Also $a_1\lambda_1X_1 + a_2\lambda_1X_2 = 0$, from (2).

Subtracting, $a_2(\lambda_1 - \lambda_2)X_2 = 0$.

This gives $a_2 = 0$, since $\lambda_1 - \lambda_2 \neq 0$ and $X_2 \neq 0$.

Similarly it can be shown that $a_1 = 0$, which proves that X_1 and X_2 are linearly independent.

In general, a set of eigen vectos of an $n \times n$ matrix A, one each corresponding to different eigen values of A, is linearly independent.

(d) The eigen values of a unitary matrix are of unit modulus.

Proof : Let A be a unitary matrix, so that $A^{\Theta}A = I$

where we denote the transpose conjugate of A by A^{Θ}.

If X be the eigen vector of A corresponding to the eigen value λ, then we have

$$AX = \lambda X \text{ .... } (2)$$

Taking transpose conjugate of both sides of (2), we get

$$[AX]^{\Theta} = [\lambda x]^{\Theta}$$

or, $X^{\Theta}A^{\Theta} = \bar{\lambda}X^{\Theta}$ .... (3)

From (2) and (3), we have

$$X^{\Theta}A^{\Theta}AX = \bar{\lambda}X^{\Theta}\lambda X$$

or, $X^{\Theta}(A^{\Theta}A)X = \bar{\lambda}\lambda X^{\Theta}X$

or, $X^{\Theta}IX = \bar{\lambda}\lambda X^{\Theta}X$

or, $(1 - \bar{\lambda}\lambda)X^{\Theta}X = 0$

Now, since $X^{\Theta}X \neq 0$, from (4), we have $\bar{\lambda}\lambda = 1$

that is, $|\lambda|^2 = 1$, giving $|\lambda| = 1$.

(e) The eigen values of a real symmetric matrix are real.

Proof : Let A be an $n \times n$ real symmetric matrix. Assume that some of the eigen values λ of A are complex numbers. Then, for the $n \times n$ identity matrix I, we have $|A - \lambda I| = 0$.

Therefore there exists a non-null solution X_1 of the system of homogeneous equations

$$(A - \lambda I)X = 0$$

or, $(A - \lambda I)X_1 = 0$ or $AX_1 = \lambda X_1$.

Taking transpose of the conjugate, we have

$$[\overline{AX_1}]^T = [\overline{\lambda X_1}]^T$$

or, $\overline{X_1}^T\overline{A}^T = \bar{\lambda}\,\overline{X_1}^T$ or $\overline{X_1}^T A = \bar{\lambda}\,\overline{X_1}^T$

as A is real symmetric, therefore $\overline{A}^T = A^T = A$

Multiplying from the right by X_1, we get

$$\overline{X}_1^T AX_1 = \bar{\lambda}\,\overline{X}_1^T X_1 \quad \text{or} \quad \overline{X}_1^T \lambda X_1 = \bar{\lambda}\,\overline{X}_1^T X_1$$

or, $\lambda\overline{X}_1^T X_1 = \bar{\lambda}\,\overline{X}_1^T X_1$ or $(\lambda - \bar{\lambda})\overline{X}_1^T X_1 = 0$

Now X_1 is a non-null matrix

Therefore $\overline{X}_1^T X_1 \neq 0$

Hence, $\lambda - \bar{\lambda} = 0$, that is, $\lambda = \bar{\lambda}$

Therefore λ is purely real. This proves the theorem.

Cor. The eigen values of a real skew symmetric matrix are either purely imaginary or zero.

(Here $\overline{A}^T = A^T = -A$).

(*f*) The eigen values of a skew - Hermitian matrix are either purely imaginary or zero.

Proof : Let A be a skew-Hermitian matrix so that iA is a Hermitian matrix. If X be the eigen vector of A corresponding to the eigen value λ, then we have

$$AX = \lambda X$$

or, $i(AX) = i(\lambda X)$

or, $(iA)X = (i\lambda).X$

Thus $i\lambda$ is an eigen value of the matrix iA. But iA is a Hermitian matrix. Hence the eigen values of iA are all real . Therefore λ is either purely imaginary or zero.

(g) The eigen vectors coresponding to two distinct eigen values of a real symmetric matrix are orthogonal. ***[B.P.U.T. – 2011***

Proof : Let A be a real symmetric matrix. Let X_1 and X_2 be two eigen vectors of A corresponding to two distinct eigen values λ_1 and λ_2 . Hence we have

$$AX_1 = \lambda_1 X_1 \text{ and } AX_2 = \lambda_2 X_2 . \qquad \text{.... (1)}$$

Now , since $AX_1 = \lambda_1 X_1$, we have $[AX_1]^T = \lambda_1 X_1{}^T, \lambda$ being real that is , $X_1{}^T A = \lambda_1 X_1{}^T$ since $A = A^T$, A being symmetric.

Post - multiplying both sides by X_2, we get

$$X_1{}^T AX_2 = \lambda_1 X_1{}^T X_2$$

or, $X_1{}^T \lambda_2 X_2 = \lambda_1 X_1{}^T X_2$, by (1)

or, $(\lambda_1 - \lambda_2) X_1{}^T X_2 = 0$.

But $\lambda_1 \neq \lambda_2$; therefore $X_1{}^T X_2 = 0$,

that is, X_1 is orthogonal to X_2, since $X_1 \neq 0$.

(h) The eigen values of an $n \times n$ matrix A and $P^{-1}AP$ are the same where P is an $n \times n$ non-singular matrix.

Proof : Let $B = P^{-1}AP$, where P is an invertible matrix.

We have $B - \lambda I = P^{-1}AP - \lambda I.$

Now $P^{-1}(\lambda I)P = \lambda P^{-1}IP = \lambda I.$

Therefore $B - \lambda I = P^{-1}AP - P^{-1}(\lambda I)P = P^{-1}(A - \lambda I)P$

Hence $|B - \lambda I| = |P^{-1}||(A - \lambda I||P|,$ where $|A| = \det A$

$= |P^{-1}||P||A - \lambda I| = |P^{-1}P||A - \lambda I|$

$= |I||A - \lambda I| = |A - \lambda I|$

Thus the matrices A and B have the same characteristic polynomial and hence they will have same eigen values.

(i) The eigen values of an orthogonal matrix are of unit moduls.

Proof : Let A be an orthogonal matrix defined over the field F. Then

$$A^T A = I \qquad \text{...(1)}$$

where A^T is the transpose of A.

Let λ be an eigen value of A, x being the corresponding eigen vector. Hence

$$AX = \lambda X \qquad \text{.....(2)}$$

Taking transpose of both sides of (2), we get

$$[AX]^T = [\lambda X]^T$$

or, $\quad X^T A^T = \lambda X^T$. ..... (3)

From (2) and (3), we have

$$X^T A^T AX = \lambda X^T \lambda X$$

or, $\quad X^T (A^T A) X = \lambda\lambda X^T X \quad$ or $\quad X^T IX = \lambda^2 X^T X$, by (1)

Therefore $\quad (1-\lambda^2) X^T X = 0$

Now X being the eigen vector, X and X^T are non-zero vectors.

Hence $\quad 1-\lambda^2 = 0$

or, $\quad \lambda^2 = 1$, giving $|\lambda| = 1$.

(j) If A be square matrix, then the sum of the characteristic roots of A is equal to the trace of A.

Proof : Let A be the squre matrix $[a_{ij}]$ of order n. Then we have

$$T_r(A) = \sum_{i=1}^{n} a_{ij}$$

If $\alpha_1, \alpha_2, ..., \alpha_n$ be the characteristic roots of A, then

$$|A - \lambda I| = (-1)^n (\lambda - \alpha_1)(\lambda - \alpha_2).....(\lambda - \alpha_n) \qquad ..... \qquad (1)$$

The coefficient of λ^{n-1} in $|A - \lambda I|$ is $(-1)^{n-1} \sum_{i=1}^{n} a_{ij}$

and the coefficient of λ^{n-1} on the right-hand side of (1) is $(-1)^{n+1} \sum_{i=1}^{n} a_{ij}$

Therefore $\quad (-1)^{n-1} \sum_{i=1}^{n} a_{ij} = (-1)^{n+1} \sum_{i=1}^{n} \alpha_i$

which gives $\quad T_r(A) = \sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} \alpha_i$

(k) If λ is an eigen value of A, then λ^2 is eigen value of A^2.

Proof : Let λ be an eigen value of A. $\therefore$ $AX = \lambda X$.

Where X is an non-zero column matrix.

Now $A^2X = A(AX) = A(\lambda X) = \lambda AX = \lambda(\lambda X) = \lambda^2 X$

Hence λ^2 is an eigen values of A.

Illustrative Examples

Example –1 : ***Find the eigen values and eigen vectors of the following matrix*** $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-3} \end{bmatrix}$

Solution : Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2}$ be a matrix of order –2

Consider the characteristic equation

$\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\ (-3-\lambda) = 0$$

$\Rightarrow\ \lambda = 1$ and $\lambda = -3$, are the eigen values of A.

Hence the spectrum of A is {4, –6}

To find the eigen vector $\lambda_1 = 1$

Let $X = (x_1\ x_2)^T \neq 0$ be a eigen vector cover to eigen value $\lambda_1 = 1$

$\Rightarrow\ (A - \lambda I)\ X = 0$

$$\Rightarrow \begin{pmatrix} 1-1 & 0 \\ 0 & -3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 0.x_1 + 0.x_2 = 0$ ... (1) , $0.x_1 - 4.x_2 = 0$ ... (2)

$\Rightarrow$ equation (2) $\Rightarrow\ -\ 4x_2 = 0\ \Rightarrow\ x_2 = 0 \Rightarrow x_1 = 1$

$$\therefore\quad (x_1, x_2)^T = (x_1, 0)^T = x_1(1,0)^T \quad \therefore \text{ an eigen vector is } X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore\ [1,0]^T$ is a eigen vector of A corresponding to the eigen value $\lambda_1 = 1$ and so also any constant multiple of $[1,0]^T$.

To find the eigen value $\lambda_2 = -3$

Let $x = [x_1, x_2]^T$ to be a eigen value of A corresponding to the eigen value $\lambda_2 = -3$

$\Rightarrow\ (A - \lambda_2 I)\ x = 0$

$$\Rightarrow \begin{pmatrix} 1+3 & 0 \\ 0 & -3+3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow\ 4x_1 = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 1$

$\because\ [x_1\ x_2]^T = [0\ x_2]^T = x_2[0\ 1]^T$

$$\therefore \text{ an eigen vector is } X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Here $[0\ 1]^T$ is a eigen value of A corresponding to the eigen value $\lambda_2 = -3$ and so also any constant multiple of [0 1]

Therefore the desired eigen values are 1 and – 3 and the corresponding eigen value are $[1\ 0]^T$ & $[\ 0\ 1]^T$ and so any constant multiple of the eigen vectors.

Example –2 : ***Find the eigen values and eigen vectors of*** $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{i} \end{bmatrix}$

Find the algebraic and the geometric multiplicities of the eigen values.

Solution : Here $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & i-\lambda \end{vmatrix} = 0$ gives $(1-\lambda)(i-\lambda) = 0.$

Therefore, the eigen values of A are 1 and i. These are simple eigen values, their algebraic multiplicaties being 1.

Let corresponding to the eigen value 1, the eigen vector be $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Now X will be given by a non-zero solution of the equation $(A - 1.I)X = 0$, which gives

$$\begin{bmatrix} 0 & 0 \\ 0 & i-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ (i-1)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here the rank of the characteristic matrix is 1 and the equations will have 2 – 1=1 independent solution. Thus the geometric multiplicity is 1.

Thus $x_2 = 0$ and $x_1 = k$, where k is any non-zero number.

Therefore, $X = \begin{bmatrix} k \\ 0 \end{bmatrix}$ where $k \neq 0$, is the eigen vector.

Similarly, for the eigen value i, we have $(A - iI)X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Thus $\begin{bmatrix} 1-i & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}'$ that is , $\begin{bmatrix} (1-i)x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Here the equations will have one independent solution.

Thus $x_1 = 0$ and $x_2 = c$, where is any non-zero number.

Therefore $X = \begin{bmatrix} 0 \\ c \end{bmatrix}$ is the eigen vector, where $c \neq 0$.

Here we see that, for both the eigen values,

algebraic multiplicity = geometric multiplicity .

Example – 3 : Find the eigen values and the corresponding eigen vectors of the matrix

$$\begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$$

Find the algebraic and the geometric multiplicities of the eigen values.

Solution : Let A be the given matrix. We have

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 2-\lambda \\ 2 & -1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 6-\lambda & -2 & 0 \\ -4 & 4-\lambda & 0 \\ 2 & -1 & 2-\lambda \end{vmatrix} \quad (R_2 \to R_2 - R_3)$$

$$(\because C_2 \to C_2 - C_3)$$

$$= (2-\lambda)\begin{vmatrix} 6-\lambda & -2 \\ -4 & 4-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 10\lambda + 16) = -(\lambda-2)^2(\lambda-8).$$

Thus the characteristic equation of the matrix A is $|A - \lambda I| = 0$, that is , $(\lambda-2)^2(\lambda-8) = 0$
Hence the eigen values of A are 2, 2 and 8.
The eigen vector of the matrix A corresponding to the eigen value λ is given by the non-zero solution of the equation

$(A - \lambda I)X = 0$ .... (1)

For $\lambda = 2$, We have $\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the eigen vector

or, $\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ or, $\begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $(\because R_1 \leftrightarrow R_2)$

or, $\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $(\because R_2 \to R_2 + 2R_1 \text{ and } R_3 \to R_3 + R_1)$

Thus the rank of the characteristic matrix is 1 and hence the equations have 3 – 1 = 2 linearly independent solutions.

From above, we have $\begin{bmatrix} -2x_1 + x_2 - x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Or, $-2x_1 + x_2 - x_3 = 0$

Two independent solutions are $X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

Every non-zero multiple of these column vectors is an eigen vector of A corresponding to the eigen value 2.

For $\lambda = 8$, substituting in (1), the corresponding eigen vector is given by

$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ or, $\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

or, $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, $(\because R_2 \to R_2 - R_1, R_3 \to R_3 - R_2 + 2R_1.)$

Thus the chracteristic matrix is of rank 2 and hence the number of independent solutions of the homogeneous equtions will be 3 – 2=1.

Then we have $\begin{bmatrix} -2x_1 - 2x_2 + 2x_3 \\ -3x_2 - 3x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The equations are $-2x_1 - 2x_2 + 2x_3 = 0$ and $-3x_2 - 3x_3 = 0$.

These give $x_2 = -x_3$

or, $\frac{x_2}{-1} = \frac{x_3}{1} = k$ (say) or, $x_2 = -k, x_3 = k.$

Then we have $x_1 = 2k$.

Therefore $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$.

Hence $x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 8$.

Any non-zero multiple of this column vector is an eigen vector of A corresponding to the eigen value 8.

We see that 2 is an eigen value of algebraic multiplicity 2 and for that we have two linearly independent solutons.

Thus, for $\lambda = 2$, algebraic mutiplicity = geometric multiplicity = 1.

Therefore in both cases the eigen value is regular.

9.3 : Characteristic Vector or Eigen Vector

Any solution of the equation $AX = \lambda X$, other than X = 0 corresponding to some particular value of λ is called the Eigen Vector or Characteristic Vector.

Theorem – 1 : Prove that a characteristic vector X of a matrix A cannot correspond to more than one. Characteristic value of A.

Proof : Let us suppose that the characteristic vector X is corresponding to two characteristic values λ_1 and λ_2, then

$AX = \lambda_1 X$ and $AX = \lambda_2 X$ ... (1)

From (1) and (2), ... (2)

$\Rightarrow \lambda_1 X = \lambda_2 X \quad \Rightarrow \lambda_1 X - \lambda_2 X = 0$

$\Rightarrow (\lambda_1 - \lambda_2)X = 0 \Rightarrow (\lambda_1 - \lambda_2) = 0$

or $\lambda_1 = \lambda_2$.

Theorem – 2 : Prove that any square matrix A and its transpose A have same eigen values.

Proof : Characteristic equation of A is

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{22} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

Characteristic equation of A'.

$$|A'-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{21} & .... & a_{n1} \\ a_{12} & a_{22}-\lambda & .... & a_{n2} \\ : & : & .... & ..... \\ a_{1n} & a_{2n} & .... & a_{nn}-\lambda \end{vmatrix} = 0$$

But $|A'| = |A| \Rightarrow |A-\lambda I| = A'-\lambda I|$

$\Rightarrow$ Characteristic equation of A and A' are same.

Theorem – 3 : Prove that the matrix A and $B^{-1}AB$ have the same latent roots.

Proof : The matrices A and $B^{-1}AB$ will have same latent roots if their characteristic equations are same.

Let $B^{-1}AB = C$

Then characteristic equation of $B^{-1}AB$ can be written as

$C - \lambda I = B^{-1}A.B - \lambda I = B^{-1}AB - B^{-1}\lambda I.B = B^{-1}(A-\lambda I)B$

$|C - \lambda I| = |B^{-1}(A-\lambda I)B|$

$= |B^{-1}|A-\lambda I|\,|B| = |A-\lambda I|\,)|B^{-1}|\,|B|) = |A-\lambda I|\,(|B^{-1}B|) = |A-\lambda I|\ \ |I|\ = |A-\lambda I|$

i.e, C and A have same characteristic equations and thus same latent roots.

Some Results

1. If there is n distinct eigen values of A, we get n linearly independent eigen vectors.
2. If two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values.
3. Eigen vector x_i corrsponding to a eigen value λ_i is not unique but it can be one of the vectors cx_i (c is a scalar) .
4. An eigen vector cannot correspond to two different eigen values.
5. If x_1 and x_2 are the eigen vectors corresponding to distinct eigen vlaues of a real symmetric matrix of order three, then the cross product of x_1 and x_2 is the third eigen vector.
6. The matrix $[A - \lambda I]$ is singular.
7. If $\lambda_1, \lambda_2, ...\lambda_n$ be the eigen values of A and K is a scalar then
   (a) the eigen values of KA are $K\lambda_1, K\lambda_2, ...K\lambda_n$.
   (b) the eigen values of $A \pm KI$ are $\lambda_1 \pm K, \lambda_2 \pm K, ..., \lambda_n \pm K$.

Illustrative Examples

Example – 1 : Find the eigen values of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{-4} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{bmatrix}$

Solution : Let λ be an eigen value of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

$\therefore$ Characteristic equation of A is $|A - \lambda I| = 0$

Here $A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{bmatrix}$

Now $|A - \lambda I| = 0 \quad \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$

Expanding along C_1, we get $(1-\lambda)(-4-\lambda)(7-\lambda) = 0$

$\Rightarrow \lambda = 1, -4, 7$ are the eigen values of matrix A.

Example – 2 : Find the eigen values and eigen vectors of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \end{bmatrix}$**.**

Solution : The characteristic equation of matrix A is $|A - \lambda I| = 0$

Here $A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$

Now $|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0$ Expanding along R_1

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda) - 2] + 0 - 1(2 - 4 + 2\lambda) = 0$

$\Rightarrow (1-\lambda)(4 - 5\lambda + \lambda^2) + 2 - 2\lambda = 0 \Rightarrow 4 - 5\lambda + \lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 + 2 - 2\lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Clearl $\lambda = 1$ satisfies it.

Dividing $\lambda^3 - 6\lambda^2 + 11\lambda - 6$ by synthetic division, we get

$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\Rightarrow \qquad \lambda = 1, 2, 3$

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

Hence eigen values of A are 1, 2, 3.

For $\lambda = 1$, Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \neq 0$ be the eigen vector of A.

Then $(A - \lambda I) X_1 = 0 \Rightarrow (A - I) X_1 = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Operating $R_3 \to R_3 - 2R_2$, we get

$\Rightarrow\ -x_1 = 0$ and $x_1 + y_1 + z_1 = 0 \Rightarrow\ x_1 = 0$ and $y_1 = -z_1 = 1$ (say)

$$\Rightarrow \quad X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ is an eigen vector of A corresponding to } \lambda = 1.$$

For $\lambda = 2$, let $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \neq 0$ be an eigen vector of A.

$\therefore \quad (A - \lambda I)\, X_2 = 0 \Rightarrow (A - 2I)\, X_2 = 0$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \to R_1 + R_2$, we get

$\Rightarrow\ x_2 + z_2 = 0$ and $2x_2 + 2y_2 + z_2 = 0$

$\Rightarrow\ x_2 = -z_2 = 2$ (say) $\quad \therefore\ 2(2) + 2y_2 - 2 = 0 \Rightarrow y_2 = -1$

$$\therefore \quad X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \text{ is an eigen vector of A corresponding to } \lambda = 2.$$

For $\lambda = 3$, let $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \neq 0$ be an eigen vector of A.

$\therefore \quad (A - \lambda I)\, X_3 = 0 \Rightarrow (A - 3I)\, X_3 = 0$

$$\therefore \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_1 \to R_1 + 2R_2$ and $R_3 \to R_3 - 2R_2$, we get

Operating $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow\ -2y_3 + z_3 = 0$ and $x_3 - y_3 + z_3 = 0 \Rightarrow z_3 = 2y_3 = 2$ (say) $\Rightarrow y_3 = 1,\ z_3 = 2$

$\therefore\quad x_3 - 1 + 2 = 0\ x_3 = -1$

$$\therefore \quad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \text{ is eigen vector of A corresponding to } \lambda = 3.$$