MODULE – III

- DIFFERNTIAL CALCULUS
- INTEGRAL CALCULUS
- COORDINATE GEOMETRY

CHAPTER 12 DIFFERNTIAL CALCULUS

Calculus is a study of infinitesimally small (very very small) quantities which are so small that they cannot be measured or calculated by ordinary means and methods.

Differential calculus. The branch of calculus which deals with the rate of change of one quantity relative to the other at some point (instant) is called.

Differential Calculus.

Indeterminant form $\frac{0}{0}$. We know 0 × anything = 0,

$\therefore \quad \frac{0}{0}$ = Any thing.

This show $\frac{0}{0}$ is not unique, but may be any number and is therefore indeterminant.

[**Note.** The division by zero is meaningless and undefined].

The expression x tends to a, means, the set of inifinite (real) values of x which are very close to a and are slightly less than a but $x \neq a$.

Limit of a function. Let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ when } x \neq 1$$

$$= \frac{(x+1)(x-1)}{(x-1)} = x + 1$$

Let x approach 1 through values less than 1,

Putting x = 99, 99, 999

$f(x)$ = 1.9, 1.99, 1.999 ....(i)

Let x approach 1 through values less than 1,

Putting x = 1.1, 1.01, 1.001, ...

$f(x)$ 2.1, 2.01, 2.001,.... ...(ii)

X takes values nearer and nearer to 1, remaining always less than 1 [as in (i)] or greater than 1 [as in (ii)] $f(x)$ takes values nearer and nearer to 2. The difference between $f(x)$ and 2 i.e. $|f(x)\ 2|$ can be made small as we please by giving to x values close to 1. We say $f(x)$ and 2 *i.e.*, $|f(x)-2|$ can be made as small as we please by giving to x values close to 1. We say $f(x)$ tends to the limit 2 as x tends to 1.

(i) *Left handed and right handed limits.*

The limiting value of $f(x)$ as $x \to a-$ is called the left-hand limit of $f(x)$ and is written as

$\underset{x\to a+}{Lt} f(x)$.

The limiting value of $f(x)$ as $x \to a-$ is called the left-hand limit of $f(x)$ and is written as

$\underset{x\to a-}{Lt} f(x)$.

(ii) *Existence of the limit of a function.* $\underset{x\to a}{Lt} f(x)$ is said to exists and are equal.

Thus if $\underset{x\to a-}{Lt} f(x) = \underset{x\to a+}{Lt} f(x) = f(x) = 1$, only then we say that $\underset{x\to a}{Lt} f(x)$ exists and is $= I$.

Distinction between the value an the limit of a function. The value of a function $f(x)$ at $x = a$ is obtained by putting $x = a$. The limit of the function $f(x)$ as $x \to a$ is obtained by considering the values of x very close to a. Thus $\underset{x\to a-}{Lt} f(x)$ may exist even if the function is not defined at $x = a$.

THEOREMS ON LIMITS

Theorem – 1. The limit of a constant quantity is the constant itself.

i.e. $\underset{x\to a}{Lt} = c = a$

where c is a constant.

2. The limit of the sum of two (or more) function is equal to the sum of their limits.

   i.e. $\underset{x\to a}{Lt} [\psi(x) + \phi(x)] = \underset{x\to a}{Lt} \psi(x) + \underset{x\to a}{Lt} (x)$.

3. The limit of the difference of two functions is equal to the difference of their limits.

   i.e. $\underset{x\to a}{Lt} [\psi(x) - \phi(x)] = \underset{x\to a}{Lt} \psi(x) - \underset{x\to a}{Lt} (x)$.

4. The limit of the product of two functions is equal to the product of their limits.

   $i \underset{x\to a}{Lt}\left[\psi(x)\phi(x)\right] = \left[\underset{x\to a}{Lt} \psi(x)\right]\left[\underset{x\to a}{Lt} \phi(x)\right]$

5. The limit of the quotient of two function is equal to the quotient of their limits, provided the limit of the denominator is not zero.

$$\underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)} = \frac{\underset{x\to a}{Lt} \phi(x)}{\underset{x\to a}{Lt} \psi(x)}$$

   provided $\underset{x\to a}{Lt} \psi(x) \neq 0$.

6. The limit of the product of a constant and a function is equal to the product of the constant and limit of the function.

   i.e. $\underset{x\to a}{Lt}$ [c.ψ (x)] =c [$\underset{x\to a}{Lt}$ ψ (x)]

   where c is a constant and ψ (x) is any function of x.

   Methods for finding the limits of a function 1st method (*By factorization*).

   When $f(x)$ is of the form $\frac{g(x)}{h(x)}$.

(i) Factorise $g(x)$ and $h(x)$ and cancel the common factors.

(ii) Put the value of x.

Example Evaluate

$$\underset{x\to a}{Lt}\frac{\sqrt{x}-\sqrt{a}}{x-a}$$

Sol. $\underset{x\to a}{Lt}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\underset{x\to a}{Lt}\frac{\sqrt{x}-\sqrt{a}}{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}-\sqrt{a}\right)}$ [Factorising]

$$\underset{x\to a}{Lt}\frac{1}{\sqrt{x}-\sqrt{a}} \quad [\because x\to \text{a} \therefore x\ a\neq 0]$$

$$=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

2nd Method (by substitution)

To evaluate $\underset{x\to a}{Lt}\frac{g(x)}{h(x)}$

(i) Put $x = a + h$, where ($h \neq 0$) is very small As $x \to a$ then $h \to 0$.

(ii) Simplify the numerator and denominator so as to cancel through-out as $h \neq 0$.

(iii) Put $h = 0$ and get the required limit.

Example : *Evaluate* $\underset{x\to a}{Lt}\frac{x^n-1}{x-1}$

Solution : Put x = 1 h, where h is very samll.

Since $x \to 1 \therefore 1 + h \to 1$ *i.e.* $h \to 0$

$$\therefore \underset{x\to a}{Lt}\frac{x^n-1}{x-1}=\underset{x\to a}{Lt}\frac{(1+h)^n-1}{(1+h)-1}$$

$$\underset{x\to a}{Lt}\frac{\left[\left(1+nh\frac{n(n-1)}{2!}h^2+\ldots\right)-1\right]}{(1+h)-1}$$ Binomial theorem]

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$$= \underset{x \to a}{Lt} \frac{nh \frac{n(n-1)}{2!} h^2 + \ldots}{2}$$

$$= \underset{x \to a}{Lt} \frac{n \frac{n(n-1)}{2!} h + \ldots}{2!}$$

[Cancelling h as $h \neq 0$]

= n + 0 + 0 + ...

= n.

Derivative of differential co-efficient of a function. If $y = f(x)$ is a function of x and δx, a small change in the value of x, then $= \underset{x \to a}{Lt} \frac{\delta y}{\delta x} = \underset{x \to a}{Lt} \frac{f(x + \delta x) - f(x)}{\delta x}$ if it exists, is called the differential coefficient or derivative of $y = f(x)$ and is denoted by $\frac{dy}{dx}$ or $f(x)$.

$\frac{dy}{dx}$ is read as 'dee y be dee x'

The symbol $\frac{d}{dx}$ stands for the operation of differentiation with repect to x.

Thus $\frac{d}{dx} = \frac{d}{dx}(y)$

The process of finding $\frac{dy}{dx}$ is called differentiation.

The process of finding the derivative of a function by using the definition of derivative as a limit is called differrentiation from first principles or differentiation *ab*-initio or differentiation by delta method.

Working rule.

(1) Denote the given function by y i.e. let $y = f(x)$.

(2) Let δx be a small change in x and δy the corresponding change in y, to that

$$y + \delta y = f(x + \delta x).$$

(3) Find δy by subtracting y from v + δy. Thus

$$\delta y + f(x + \delta x) - f(x)$$

(4) Divide both sides by δx to obtain the difference quotient.

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

(5) Find the limit of $\frac{\delta y}{\delta x}$ as $\delta z \to 0$

$$\frac{dy}{dx} = \underset{\delta_x \to a}{Lt} \frac{\delta y}{\delta x}$$

Derivative of x^n, where n ∈ R

$$\frac{dy}{dx} = \left(x^n\right) = nx^{n-1}$$

[Write power before x and subtract one from the power]

$$e.g.\ \frac{d}{dx} = \left(x^8\right) = 8x^{8-1} = 8x^7$$

$\frac{d}{dx}(x) = 1$ *i.e.* rate of change of any variable w.*r.t.* itself is 1.

Be careful $\frac{d}{dx}(y)$ is $\frac{d}{dx}a$ d not 1.

Fundamental theorems on differentiation

Theorem 1. Derivative of a constant is zero.

$$i.e.,\ \frac{d}{dx}(c) = 0$$

$$e.g.,\ \frac{d}{dx}(\pi) = 0\ ;\ \frac{d}{dx}(9) = 0\,.$$

Theorem 2. The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.

$$i.e.,\ \frac{d}{dx}[cf(x)] = \text{c}\ \frac{d}{dx}\ [f(x)]$$

$$e.g.\ \frac{d}{dx}\left(5x^7\right) = 5\frac{d}{dx}\left(x^2\right) = 5\left(7x^{7-1}\right)$$

$$= 5(7x^6)\ 35x^6.$$

Theorem 3. The derivative of the algebraic sum of any finite number to functions is the algebraic sum of the derivatives.

$$i.e.,\ \frac{d}{dx}[f(x) + g(x) - (x) + ...]$$

$$d = \frac{d}{dx}\ f(x) + \frac{d}{dx} g(x) - \frac{d}{dx}\ h(x) + ...$$

$$e.g. = \frac{d}{dx}\ (x^2 - 4x + 8) = \frac{d}{dx}\ (x^2) - \frac{d}{dx}\ (4x) + \frac{d}{dx}\ (8)$$

Theorem – 4. An additive constant disappears in differentation

$i.e.\ \frac{d}{dx}[f(x)+c] = \frac{d}{dx}[f(x)]$

where c is a constant

e.g. $\frac{d}{dx}[x^{19}+16] = \frac{d}{dx}x^{10} = 10^{9}.$

Theorem – 5. Derivative of product of two functions : the differential coefficient of the product of two functions is the sum of the products of each function with the derivative of the other.

$e.g. = \frac{d}{dx}(u.v) = u\frac{d}{dx} + v\frac{d}{dx}$

where u and v are differentiable functions of x.

Working rule. The differential coefficient of the product of two functions = first function × differentail coefficient of the second + second function × the differential coefficient of the first.

Cor. $\frac{d}{dx}(uvw) = ur\frac{dw}{dx} + \mathrm{uw}\frac{dv}{dx} + vw\frac{du}{dx}$

where u, v, w are functions of x and their derivative exists.

Thorem – 6. The differential coefficient of the quotient of two functions is.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} u\frac{dv}{dx}}{v^2}$$

where u and v are derivable functions of x.

In words

$$= \frac{\text{Denom.}\times\text{DerivativeofNum} - \text{Num.}\times\text{ Derivative of Denom}}{(\text{Denom})^2}$$

Important Definitions

(1) **Velocity,** $\boldsymbol{v} = \frac{ds}{dt}$ where s is the displacement in time t seconds.

(2) **Acceleration** $= \frac{dv}{dt}$ *i.e.* Rate of change of velocity.

(3) **Marginal cost** $\frac{dC}{dx}$ *i.e.* where C is the cost of producing x units of quantity.

(4) **Marginal Revenue** $= \frac{dR}{dx}$, where R is the revenue obtained by selling x units of commodity.

(5) Marginal Profit $= \frac{dP}{dx}$, where P is the profit obtained by selling x units of a commodity.

Funcation of a function. If y is a function of u and u in turn is a function of x then y is called a *function of a function or a composite function.*

Theorem. If y is a function of u and u in turn is a function of x, then y is a function of x and

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

Note 2. If $y = u^n$ and $u = g(x) = a$ function of x.

$$\Rightarrow \frac{dy}{dx} = nu^{n-1} \frac{d}{dx}(u)$$

Remember $\frac{dy}{dx}$ [A function of $x]^n = n$ (function)$^{n-1} \times$ diff.coeff. of the function w.r.t.x.

Theorem. $\frac{dy}{dx} \times \frac{dx}{dy} = 1.$

Diffparametric equations. If x and y be expressed in terms of any variable parameter t then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Geometrical meaning of the derivative

The derivative at a point P of a curve is the role slope of the tangent to the curve at the point P.

The slope of the path or curve at any point P is defined at the slope of the tangent at P

$$\left[\frac{d}{dx}\right)_{at\ P}$$

If the tangent to the curve at a point is parallel to the axis of x, then $\psi = 0^o$.

$$\therefore \frac{d}{dx} = \tan y = \tan 0^o = 0$$

If the tangent to the curve at a point is perpendicular to the axis of x (or parallel to the axis of y.) then $\psi = 90^o$.

$$\therefore \frac{d}{dx} = \tan \psi = \tan 90^o = \infty.$$

Equations of the tangent and the normal to the curve below the equation of the curve is

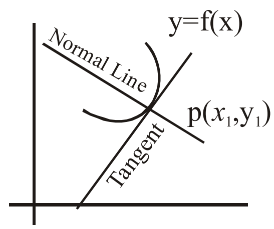
$$y = f(x)$$

$\frac{d}{dx}$ = slop of tangent at (x, y)

$\therefore$ Slop of tangent at $P(x_1, y_1)$

$$\left(\frac{dy}{dx}\right)_{at(x_1, y_1)} = m$$

The equation of tangent at (x_1, y_1) is



Slope of normal line at a given point is the negative of the reciprocal of the slope of the tangent line at that point.

Slope of normal at $P(x_1, y_1) = -\frac{1}{m}$ $(m \neq 0)$.

The equation of the normal at (x_1, y_1) is

$(y - y_1) = -\frac{1}{m}(x - x_1)$.

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CHAPTER 13

INTEGRAL CALCULUS

Integration is the inverse operation of differentiatio.

In differential calculus, we are given a function and we are required to differentiate it. In integral calculus we are required to find a function whose differential coefficient (or derivative) is given.

If the differential coefficient of a function $f(x)$ w.r.t. x is F(x) then we say that the integral (or primitive) of F(x) w.r.t. x is $f(x)$.

Symbolically. If $\frac{d}{dx}\left[f(x)\right] = F(x)$ then $\int F(x)dx = f(x)$ and is read as "integral of F(x) w.r.t.x is $f(x)$".

The symbol ∫ (elongated S) is called **the sign of integration**, $f(x)$ is called the **integrand** and the process of finding $f(x)$ is called integration.

Constant of integration. If $\frac{d}{dx}[f(x)] = F(x)$, then $\int F(x)\ dx = f(x) + c$ where c is an arbitrary constant.

The constant c is called the constant of integration. The constant of integration is generally omitted.

[**Note.** Different methods of integrating a function may give different answers apparently, but any two such answers differ only by a constant.]

Standard Forms

(i) $\int x^n dx = \frac{x^N + 1}{n+1}\left[n \neq -1\right]$

i.e., increase the index of x by 1 and divide by the new index.

(ii) $\int \frac{1}{x} dx = \log x$

[**Note:** In (i) if n = – 1 $\int x^n dx = \int x^{-1} dx = \int \frac{1}{x} dx = \log x$ [$\because$ of (ii)]

(iii) $\int e^{ax} dx = \frac{e^{ax}}{a}; fe^{x} dx = e^{x}$

(iv) $\int a^{mx} dx = \frac{a^{mx}}{m}; \int a^{x} dx = \frac{a^{x}}{\log a}$

(v) $\int \sin x \, dx = -\cos x$

(vi) $\int \cos x \, dx = \sin x$

(vii) $\int \sec^2 x \, dx = \tan x$

(viii) $\int \operatorname{cosec}^2 x \, dx = -\cot x$

(ix) $\int \sec x \tan x \, dx = \sec x$

(x) $\int \operatorname{cosec} x . \cot x = -\operatorname{cosec} x$

(xi) $\int 1 dx = \int x^0 dx = \frac{x^{0+1}}{0+1} = x.$ [as a special caseof form (i)]

Important extension of elementary forms

(i) All the results of the above list hold good when x is replaced by $x + a$ [a being a constant] in any formula.

i.e., $\int (x+a)^n \, dx = \frac{(x+a)^{n+1}}{n+1} [n \neq -1]$

$\int \sec (x + 6) \tan (x + 6) \, dx = \sec (x + 6)$

(ii) If x be replaced by ax + b [a and b being constants] on both sides of any standard result of the above table of integrals, the standard form remains true, provided the result of R.H.S. is divided by 'a' the coefficient of x

i.e., $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)a} [n \neq -1]$

$$\int \cos ec^2 (8x+3) dx = -\frac{\cot(8x+3)}{8}$$

THEOREMS ON INTEGRATION

Theorem – 1 : The processes of differential and integration neutralises each other

i.e. $\frac{d}{dx}\left[\int f(x) dx\right] = f(x)$

Theorem – 2 : The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function.

i.e. $\int cf(x) \, dx = c \int f(x) \, dx$

e.g. $\int 5x^3 dx = 5\int x^3 dx = \frac{5x^{3+1}}{3+1} = \frac{5}{4} x^4$

Theorekm – 3 : If *u, v, w.....* (finite number) are functions of *x*, then

$$\int (u + v - w + ...)dx = \int udx + \int vdx - \int wdx$$

e.g. $\int\left(a + \frac{b}{x} + \frac{c}{x^2}\right)dx = a\int 1dx + b\int \frac{1}{x}dx + c\int \frac{1}{x^2}dx$

$$= ax + \text{b} \log x + \text{c}\ \frac{x^{-2+1}}{-2+1} = ax + \text{b} \log x - \frac{c}{x}$$

[If the degree of the numerator of the integrand is equal or greater than that of denominator, divide the numerator by the denominator until the degree of the remainder is less than that of denominator, e.g.

$$\int \frac{x^3}{x+1}dx = \int\left(x^2 - x + 1 - \frac{1}{x+1}\right)dx = \frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1)$$

$x+1)\ x^3\ (x^2 - x + 1$

$\underline{x^3 + x^2}$

$-x^2$

$\underline{-x^2 - x}$

x

$\underline{x + 1}$

-1

Definite Integral. If ϕ (*x*) be an integral of *f*(*x*) then the quantity ϕ (b) – ϕ (a) is called the definite integral of *f*(*x*) between the limits a and b is written as $\int_a^b f(x)dx$. It is real as integral from a to b of *f*(*x*) dx. 'a' is called the lower limit and b the upper limit ϕ(b) – ϕ(a) is written as $[\phi(x)]_a^b$

Thus $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$.

Rule to evaluate $\int_a^b f(x)dx$

1. Integrate $\int f(x)$ dx.
2. In the result first put x = upper limit (b), and then x = lower limit (a).
3. Subtract the second result from the first.

e.g., $\int_3^4 \frac{dx}{x} = [\log x]_3^4 = \log 4 - \log 3 = \log\frac{4}{3}$ $\left[\because \log m - \log n = \log\frac{m}{n}\right]$

Integration by substitution. Integration of many functions become simple by the substitution of a new variable. In other words, many integrals can be evaluated in a simple way by changing the variable of the given integrand say from the given variable x to the new variable z, the two variable x, z being connected by some relation.

This process of integration is called integration by substitution.

For example . Let I = $\int \sin^3 x \cos x \, dx$

Put $\sin x = t$, then $\cos x = \frac{dt}{dx}$ [Differentiation w.r.t x]

or $\cos x \, dx = dt$

$\therefore$ $I = \int t^3, dt = \frac{t^4}{4} + c = \frac{1}{4}\sin^4 x + c = \frac{1}{4}\sin^4 x.$

Two important forms of integrals

Theorem – 1. $\int \frac{f(x)dx}{f(x)} = \log\left[f(x)\right]$

The integral of a fraction whose numerator is differential co-efficient of the denominator is log [denom.]

e.g. $\int \frac{2x+3}{x^2+3x+7} dx$

$$\frac{d}{dx}\left(x^2+3x+7\right) = 2x+3$$

$$\therefore \int \frac{2x+3}{x^2+3x+7} dx = \log(x^2+3x+7)$$

$$\left[\because \frac{f(x)}{f(x)} dx = \log\left[f(x)\right]\right]$$

Theorem – 2. $\int\left[f(x)\right]^n f(x)dx = \frac{\left[f(x)\right]^{n+1}}{n+1}$

(Provided $n \neq -1$)

Thus if the integrand is the product of power of a function $f(x)$ and its derivative $f'(x)$ then the integral is obtained by increasing the index of $f(x)$ by a and dividing the result by the new index.

e.g., $\int(ax^2 + bx + c)^5 (2ax + b) \, dx$ [Form f {$f(x)$n $f(x)$ dx]

$$= \frac{\left(ax^2+bx+c\right)^{5+1}}{5+1}$$ [Here $f(x) = (ax^2 + bx + c)$ $f(x) = 2ax + b$]

$$= \frac{\left(ax^2+bx+c\right)^6}{6}$$

The following notes should not be forgotten

(1) $\int \tan x \, dx = -\log \cos x = \log \sec x$.

(2) $\int \cot x \, dx = \log \sin x$.

(3) $\int \text{cosec } x dx = \log \tan \frac{x}{2}$

(4) $\int \text{cosec } x \, dx = \log (\sec x - \cot x)$.

(5) $\int \sec x dx = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

(6) $\int \sec x dx = \log (\sec x + \tan x)$.

Five standard forms

(i) $\int \frac{dx}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a}$

(ii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

(iii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right)$

(iv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \sinh^{-1}\left(\frac{x}{a}\right)$

(v) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$

To integrate a fraction whos numerator is 1 and denominator is homogeneous function of the second degree in cos x and sin x.

1. Divide the numerator and denominator by $\cos^2 x$.
2. Put $\tan x = z$.

e.g. $\int \frac{d\theta}{a^2 \sin^2 \theta + b \cos^2 \theta}$ (Dividing nur and denom. by $\cos^2\theta$)

$$\int \frac{\sec^2 \theta \; d\theta}{a^2 \tan^2 \theta + b^2} = \frac{1}{a^2} \int \frac{dz}{z^2 + \frac{b^2}{a^2}} \quad \text{put } \tan z = \theta$$

[**Note.** To make the coeff. of z^2 unity]

$$= \frac{1}{a^2} \int \frac{dz}{z^2 + \left(\frac{d}{a}\right)^2} \quad \left[\text{From } \int \frac{dx}{x^2 + x^2} \text{ Here 'a' } = \frac{b}{a}\right]$$

$$= \frac{1}{a^2} \cdot \frac{1}{\frac{b}{a}} \tan^{-1}\left(\frac{z}{\frac{b}{a}}\right) = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}z\right)$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan\theta\right) \quad [\because\ z = \tan\theta]$$

Integration by parts

If u and v be two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int \frac{du}{dx}\left(\int v dx\right)dx$$

In words. Inegral of the product of two functions = 1st function × integral of 2nd-integral of [Diff. Coeff. of 1st × Integral of 2nd].

Rule to choose the factor of differentiation or the first function. If one factor of the product is a power of x take it as the first function provided the integral of the second function is handy. If however the integral of the second function is not readily available [in case of inverse circular function or inverse hyperbolic functions or a logarithmic function] in that case, take that factor as the first function.

If the integrand is a single inverse circular function (or hyperbolic function) or a single logarithm, take that functions the function).

(1) as the second function.

$\int e^n [f(x) + f'(x)]\, dx = e^x f(x)$

e.g., $\int e^n [(\sin x + \cos x)\, dx$ [Here $f(x) = \sin x, f'(x) = \cos x = e^x \sin x$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int \sqrt{a^2 + x^2}\,dx = \frac{x\sqrt{x^2 + x^2}}{2} + \frac{a^2}{2}\sin_h^{-1}\frac{x}{a}$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

i.e., for $\int \sqrt{a^2} - x^2 dx, \int \sqrt{a^2} + x^2 dx, \int \sqrt{x^2 - a^2}\,dx$

Integral $= \dfrac{x \times \text{integrand}}{2} \cdot \dfrac{+a^2 or - a^2 as\ \text{int}\,egrand}{2}$ × integral of [reciprocal of integrand]

Two standard integrals

$$\left.\begin{aligned} \int e^{ax}\sin bx dx &= \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos x) \\ \int e^{ax}\cos bx dx &= \frac{e^{ax}}{a^2 + b^2}(a\cos bx - b\sin x) \end{aligned}\right] \text{1st Form}$$

$$\left.\begin{aligned}\int e^{ax}\sin bx\,dx &= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)\\ \int e^{ax}\cos bx\,dx &= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)\end{aligned}\right]\text{ 2nd Form}$$

Important Note. $e^{\log f(x)} = f(x)$.

INTEGRATION OF RATIONAL FUNCTIONS

Two standard forms

(i) $\int \frac{dx}{x^2-a^2}\left[x^2>a^2\right] = \frac{1}{2a}\log\frac{x-a}{x+a}$

(ii) $\int \frac{dx}{a^2-x^2}\left[x^2<a^2\right] = \frac{1}{2a}\log\frac{a+x}{a-x}$

$\int \frac{dx}{ax^2+bx+c}$ (a be + ve then).

Case I. When $b^2 > 4ac$ then

$$= \frac{1}{\sqrt{b^2-4ac}}\log\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b-\sqrt{b^2-4ac}}\right|$$

Case II. When $b^2 < 4ac$ then $= \frac{1}{4ac-b^2}\tan^{-1}\left[\frac{2ax+b}{4ac-b^2}\right]$

To integrate $\int \frac{dx}{Quadratic}$

1. Make the coefficient of x^2 unity by taking the numerical coefficient of x^2 outside.

2. Complete the square in terms containing x by adding and subtracting the square of half the coefficient of x.

3. Use the proper standard form.

[**Note.** If in a numberical problem, the discriminant of quadratic in denominator [b^2–4ac] is +ve and a perfect square, factorise the denominator and resolve it into partial fractions then integrate.

e.g. $\int \frac{dx}{2x^2-2x+1} = \frac{1}{2}\int \frac{dx}{x^2-x+\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+\frac{1}{2}}$

$$= \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}} = \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x-\frac{1}{2}}{\frac{1}{2}} = \tan^{-1}(2x-1).$$

Method to integrate $\int \frac{Linear}{Quadratic} dx.$

1. Put linear = $\lambda \frac{d}{dx}$ (quadratic) + μ
2. Equate the coefficient of x and constant terms on both sides to find λ and μ.
   The above two steps are taken to break the given fraction into two fraction such that in one the numberator is the derivative of denominator and in the other, numerator is a constant.

e.g. $\int \frac{2x}{x^2+2x+2} dx$

Let $2x = \lambda \frac{d}{dx}\left(x^2+2x+2\right) + \mu$

i.e, $2x = \lambda\,(2x + 2) + \mu$

Equating coeffi. of x, $2 = 2\lambda$, $\therefore \lambda = 1$

Equation constant term, $0 = 2\lambda + \mu$

$\mu = -2\lambda = 2$

$$\therefore \int \frac{2x}{x^2+2x+2} dx = \int \frac{\lambda(2x+2)+\mu}{x^2+2x+2} dx = \lambda \int \frac{2x+2}{x^2+2x+2} dx + \mu \int \frac{dx}{x^2+2x+2} dx$$

$$= \lambda \log\left(x^2+2x+2\right) + \mu \int \frac{dx}{(x+1)^2+1}$$

$$= \log (x^2 + 2x + 2) - 2 \tan^{-1} (x + 1) \quad [\because \lambda = 1, \mu = -2]$$

Integration of irrational functions

Rule to integrate $\int \frac{dx}{\sqrt{Quadratic}}$ or $\int \sqrt{Quadratic}\,dx$

1. Make the coeff. of x^2 unity by taking its numerical coeffi. outside the square root sign.
2. Complete the square in terms containing x by adding and subtracting the square of half the coeffi. of x.
3. Use proper standard form.

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CHAPTER 14

COORDINATE GEOMETRY

Ordered Pairs

Most of the equations we shall be considering represent a relation between the two variables x and y, so that any numerical value for x leads to one or more numerical values of y. Thus, if the relation is given by the equation $y - x = 5$ or $y = x + 5$ then the replacement of x *byn* 2 means that y must be replaced by 7 we can represent this result or solution by writing it out in full as $x = 2$, $y = 7$, or we may adopt an abbreviated form (2, 7) called an **ordered pair,** so that whenever we write (p, q) this means x

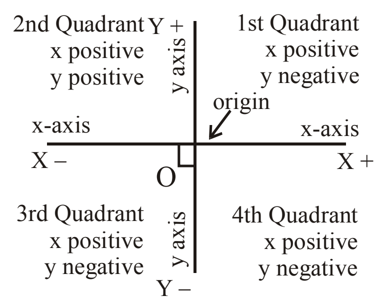


fig.(a)

= p and $y = q$, for example, another solution of the same equation $y = x + 5$ is given by $x = -1$, $y = 4$, which may be written as the ordered pair (–1, 4). By suggesting more and more values of x we build up a set of solutions, or ordered pairs, that will also satisfy the equations y = x + 5.

We now seek to give a diagrammatical representation of the complete relation between x and y, which avoids the need to find so many of the ordered pair solutions. For example, if $y = x + 5$can be represented by a straight line (and it can), then we shall only need to find two poins in order to obtain the whole line which represents the relation. As we did last time with regard to the number and type of solutions of the simultaneous equations, such diagrams will clarify that which algebra tends to obscure.

Rectangular Coordinates

We can associate an ordered pair with a point in a plane, such as a piece of paper, by choosing two perpendicular reference lines in the paper and taking measurements from these lines, with then usual convention that measurements on each side of the reference lines correspond to positive or negaive values. Consider Fig (a), in which the two perpendiculare reference lines are X^-OX^+ and Y^-OX^+ is the x axis and Y^-OY^+ is the y axis.

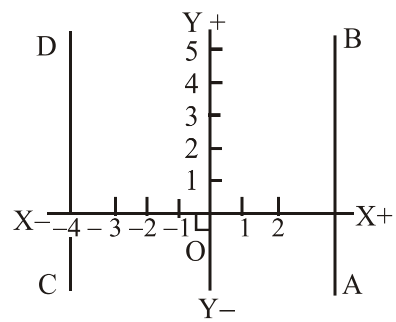


fig.(b)

Graduating the axes to some choosen scale leads to the marked diagram of Fig (b). (Throughout coordinate geometry it is assumed that the reader uses A4 size metric graph paper). Observe that the same scale has been chosen for both axes. This may not always be convenient and, consequently, it is very important to understand that when we measure a lihne on the paper we must be aware of the scales in use. The axes divide the plane into four regions **called quadrants** which are numbered from 1 to 4 anti-clockwise. The point of intersection of the axes is the point from which all the graduations start so we call this point O, the origin.

Now consider the line AB in Fig. (b) which is parallel to Y^-OY^+. Every point on the line AB is 3 unit from Y^-OY^+ because AB has also been drawn through the point marked 3 on the x axis. The line CD is likewise parallel to Y^-OY^+ but, since it has been drawn through the point marked –4 on the x axis, we say that every point on CD is 4 units from Y^-OY^+ and on the left; or alternatively – 4 units from Y^-OY^+.

In order to fix a particular point on the line AB we refer to its distance from X^-OX^+ and these distances will be given by drawing lines of rank (i.e. ordinates) parallel to X^-OX^+, which leads us to Fig (c), where we have two such lines, PQ and RS, which inersect AB and CD in the points K, L, M and N as shows. Thus every point on PQ is 5 units from X^-OX^+ because it has been drawn through the point marked 5 on the y axis.

Now the point L lies on PQ and AB, which means that its position may be associated with the ordered pair (3, 5) that is, 3 units measured in a direction parallel to the x axis and 5 units measured in a direction parallel to the y axis. Since these measurements are obtained from the ordinate lines (lines of rank) parallel to the two axes we call the two measurements the coordinates of the point. Thus the point L has an x coordinate of 3 and a y coordinate of 5. Similarly the coordinates of the point K are given by the ordered pair (–4, 5) i.e., K is the point which is –4 units measured in a direction parallel to the x axis and 5 units measured in a direction parallel to the y axis. The x coordinate of the point K may also be described as 4 units meaured in the negative direction parallel to the x axis. Similary, the point N is (–4, –2) and M is the point (3, –2)

Inequalities

Figs. (b) and (c) concentrated on points on a straight line or the point of intersection of two lines, i.e., K, L, M and N. However, the straight line AB, for example, did much more than harness a set of points onto the line; it also divided then plane of the paper. The set of all points on the squared paper is called the **coordinate plane,** and this plane is divided into three sets by the straight line AB: the set of points on the **half plane** on the left; the set of points on the line itselfl; and the points on the **half plane** on the right. Thus all the points to the left of AB have an x coorddinate less than 3, i.e., $x < 3$, and all the points to the right of AB have an x coordinate greater then 3, i.e., $x > 3$. We cannot make a similare statement about the y coordinate of these points because this is not restricted by the line AB.

If we now consider the effect on the coordinate plane of the straight line RS we see that the y coordinate of any point on this line is always – 2; indeed, we may say that $y = -2$ is the equation of the line R. However, the line also divides the coordinate plane into two other sets of points; namely, those for which $y > -2$, i.e. all points above the line; and those for which $y < -2$, i.e., all points below the line.

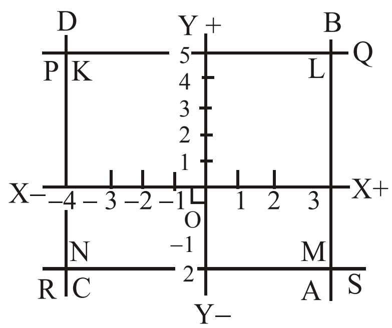


fig.(c)

If we combine the effects of drawing the lines AB and RS then we see that they partition the coordinate plane into several

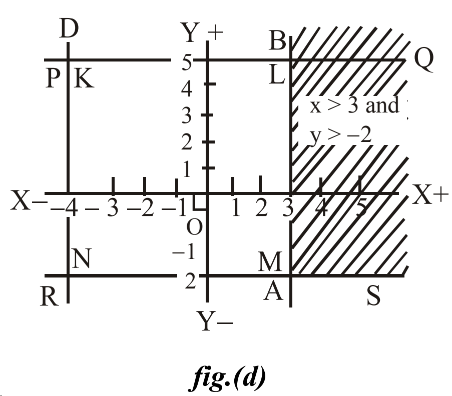


fig.(d)

regions. The shaded region in Fig (d) corresponds to $x > 3$ and $y > 2$ or, to put it another way, the shaded region corresponds to the set of all points with an x coordinate greater then 3 and a y coordinate greater than –2. Examples of such points are (6, –1), (10, 7), (3.01, – 1.93), etc. Perhaps the reader can now see how the inequalities will be dealt with. Similarly the interior region of the square KLMN corresponds to $x > -4$, $x < 3$, $y > -2$, $y > 5$, all four inequalities being applied at the same time. We can shorten this statement by writing $-4 < x < 3$ and $-2 < y < 5$. Typical points which belong to this region are (2, 4), (2.9, –1.9), (–3, –1) and so on.

The Distance Between Two Points

We have so far referred to particular points by giving their coordinates, such as (3, 5) or (–2, 1). The general point in the coordinate palne has been referred to as the point (x, y); that is, we make (x, y) represent the general point in the plane and we then make it a particular point by giving numerical values to x and y. There are some occasions when we wish to discuss one two or more points without giving their coordinates numerical values. We could refer to these points as (a, b); (c, d); (p, q); etc. but, rather than use so many different letters, we choose a subscript notation illustrated by referring to the points as (x_1, y_1), (x_2, y_2), (x_3, y_3) and so on. This way we always know not only which coordinate is being referred useful to extend this economy to the point letters themselves so that instead of referring to points. A, B.... we refer to the point P_1 with coordinates (x_1, y_1) the point P_2 with coordinates (x_2, y_2) and so on. Now it follows that since the position of two points is indicated by their coordinates then the distance between the points should be expressed in terms of these coordinates. Indeed, we may go so far as to define the distance d between two points without referring to any diagrams as follows :

Definition. The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined by $d = [(x_2 - x_1)^2 + (y_2 - y_1)^2]$, and this distance will be called the length P_1P_2 or the length of the straight line segment P_1P_2.

Note that this definition applies only when we are using a rectangular coordinate system, i.e. the axes are perpendicular.

Example. Find the lengths of the sides of the triangle formed by joining the points $P_1(4, 3)$, $P_2(7, -2)$, $P_3(-1, -5)$.

Solution : $P_1P_2 = \sqrt{[(7-4)^2 + (-2-3)^2]} = \sqrt{}\,[(9+25) = \sqrt{34}$

= 5.83 (correct to 2 decimal places)

$P_2P_3 = \sqrt{[(-1-7)^2 + (-5-(-2))^2]} = (64 + 9) = 73$

= 8.54 (correct to 2 decimal places)

$P_3P_1 = \sqrt{[(4-(-1))^2 + (3-(-5))^2]} = (25 + 64) = 89$

= 9.43 (correct to 2 decimal places)

The lengths of the sides of the triangle are 5.83, 8.54, 9.43 units (correct to 2 decimal places).

We again make the observation as we did last that the distance between two points P_1 and P_2 or the length P_1P_2 is dependent only upon the coordinates of the points P_1 and P_2 and not the scales we choose for the x and y axes. This is why we chose to give a definition for distance, before discussing the same result from a diagram as follows.

Consider finding the length of the line segment AB joining the points A (2, 1) and B (5, 7) given in Fig. (a). We see that the line through A and B parallel to the axes from a right-angled triangle ACB. Further more, C is the point (5, 1).

From Pythagora's Theorem we have

$$AB^2 = AC^2 + CB^2$$

$$\therefore \quad AB^2 = (5-2)^2 + (7-1)^2 = 9 + 36$$

$$\therefore \quad AB = \sqrt{45} = 6.71 \text{ units (correct to 2decimal places).}$$

Now consider fig. (b) where the scale for the *x* axis is 6 mm to 2 unit. Again we have the result AC = 3 units, CB = 6 units, that is, CB and AC are calculated in units, not mm. In consequence the length of AB is still 6.71 units. Furthermore, it must also be understood that halving the scale along the y axis does not result in halving distances such as AB; indeed, only when both the *x* and *y* scales are halved together would a distance like AB be halved. Finally, to find the angle of inclination of the line AB to the *x* axis, the angle CAB is measured by using a protractor only when the *x* and *y* scales are the same, otherwise we must work from the result

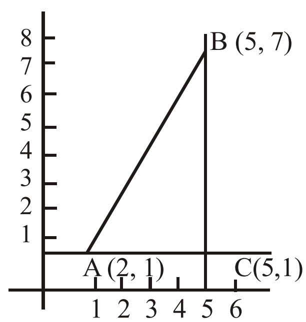
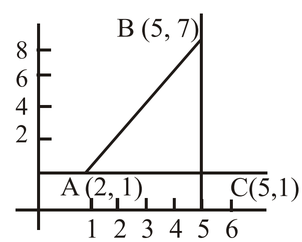


Fig (a)



Fig(b)

$$\tan \angle CAB = \frac{7-1}{5-2} = \frac{6}{3} = 2$$

We then consult tangent tables to reveal that $\angle$ CAB = 63º26′ and we note that that this is the result for both Fig. (a) and (b).

Observe further that if we write tan $\angle$CAB = CB/AC then we mean that CB is the distance 6 units and AC is the distance 3 units, i.e., not the actual measurement in terms of millimeters in Fig. (a) or (b). Tan CAB is called the gradient of the line AB and we usually denote this by the letter m, i.e. m = tan $\angle$CAB for the line AB.

The Midpoint of a Straight Line Segment

Again we could define this without the aid of a diagram by saying that the midpoint M (x, y) of the line segement joining $P_1(x_1, y_1)$ and $P_1(x_1, y_1)$ is given by $x = \frac{1}{2}(x_1 + x_2)$, $y = \frac{1}{2}(y_1 + y_2)$. However, since we are now aware of the meaning of length on a graph irrespective of the scales, we shall consider obtaining the required midpoint from the diagram in Fig.(a).

We first observe the simply cases of the straight line segment being parallel to one or other of the axes. In Fig (c) the coordinates of the point C are (4, 2), and therefore the point E has coordinates $\left(\frac{1}{2}(1+4), 2\right) or \left(2\frac{1}{2}\right)$. Similary the x coordinate of F is obviously 4 and its y coordinate (7 +2) = 4 $\frac{1}{2}$ i.e. F is the (4, $4\frac{1}{2}$).

Now M has the same x coordinate as E and the same y coordinate as F son M is the point ($2\frac{1}{2}$, $4\frac{1}{2}$).

Alternatively, suppose M is the point (u, v) then, because E is the midpoint of AC, we have AE = EC in the form u –1 = 4 – u, an equation which has the solution u = (4 + 1) / 2 = 2– Similarly, becuase F is the midpoint of CB we have CF = FB is the form v – 2= 7 – v, which has the the solution v = (7 + 2)/2 = $4\frac{1}{2}$.

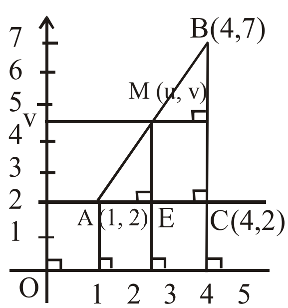
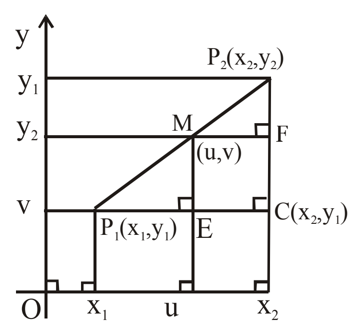


Fig. (c)



Fig(d)

Finally, M is the point (2.$\frac{1}{2}$, $4\frac{1}{2}$) as before.

As another numberical examples, the midpoint of the line segment joining the points (–2, 3) and (4, 7) is $\left(\frac{-2+4}{2}, \frac{3+7}{2}\right)$ i.e., (1, 5).

If we now move to the general example of Fig. (D), and once again let the point M (u,v) be the midpoint of the line segment joining $P_1(1, y_1)$ and $P_2(x_2, y_2)$, then with $C(x_2, y_1)$ and E the midpoint of P_1C we get

$$P_1E = u - x_1 = EC = x_2 - u.$$

$$\therefore\ u - x_1 = x_2 - u.$$

$$2u = x_1 + x_2$$

$$u = \frac{1}{2}(x_1 + x_2)$$

Similarly from $CF = FP_2$ we have $v - y_1 = v_2 - v$

$$2v = y_1 + y_2,\ v = \frac{1}{2}(y_1 + y_2)$$

Hence the midpoint of P_1P_2 is the point $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$

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