8

CHAPTER

Linear Algebra – I

(Matrices, Vectors, Determinants, Linear System)

8.1 : Introduction

In modern engineering Mathematics Linear Algebra (Matrix theory) is used in various fields. Matrices are used througout mathematics to express relationship between elements in sets. The study of matrices originated from the idea of solution of various types of system of linear equations which occurs in many engineering processes; for example, for finding out the solution sets of a system of linear equations and linear transformations etc. Matrices provide an important tool in the study and development of linear Algebra. In this Chapter we shall introduce the concept of matrix and discuss some basic operations (like addition multiplication, inversion etc.) on matrices. The study of these basic operations play a very important role in the development of the theory of matrices. Also the matrices occur in state -space presentation of linear system models in applied systems engineering and control systems.

8.2 : Basic Concepts

***Def*ⁿ :** A matrix is a rectangular array of numbers. A matrix with 'm' rows and n columns is called m × n matrix. The plural of matrix is matrices. A rectangular array of elements of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \ldots\ldots & a_{mn} \end{bmatrix}_{m \times n}$$

is knows as a matrix with m-rows & n-columns.

We can also write it as $[a_{ij}]_{mxn}$ or $(a_{ij})_{mxn}$

→ Horizontal lines are called rows of the matrix

→ Vertical lines are called columns of the matrix

→ a_{ij} denotes, the element at ith row and jth column.

Different types of matrices :

Row matrix : A matrix having a single row is called a row matrix. It is of the form

$[a_{ij}]_{1\times m} = [a_{11}\ a_{12}\ \ldots\ldots\ a_{1m}]_{1\times m}$

Column matrix : A matrix having a single column of the form

$$[a_{ij}]_{n\times 1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}_{n\times 1}$$ is called a column matrix.

Rectangular matrix : A matrix of order m×n is said to be a rectangular matrix if $m \neq n$.

Square matrix : A matrix is said to be a square matrix if

No. of rows = No. of columns

$$A = \begin{bmatrix} a_{11} & a_{12}\ \ldots\ldots & a_{1n} \\ a_{21} & a_{22}\ \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2}\ \ldots\ldots & a_{nn} \end{bmatrix}_{n\times n} \quad \textbf{\textit{Ex :}} \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}_{2\times 2}, (1)_{1\times 1}, \begin{pmatrix} 2 & 3 & 5 \\ 2 & 5 & 3 \\ 4 & 2 & 8 \end{pmatrix}_{3\times 3}$$

Diagonal matrix :

A square matrix is said to be a diagonal matrix if all the elements not occured in the leading diagonal are zeros.

Example : $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Scalar matrix : A matrix is said to be a scalar matrix if all the diagonal elements are same.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Null matrix : A matrix is said to be a null matrix if all the entries are zero.

Unit matrix / Identity matrix : A square matrix is said to be a unit matrix if all the entries in the leading diagonal are unity and rest of others are zeros.

Or A scalar matrix is said to be a unit matrix if all the elements in the leading diagonal are unity.

Example : $I_1 = (1),\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Triangular matrix :

(a) *Lower Triangular matrix :* Lower triangular matrices can have non zero entries only on and below the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not. Otherwise A square matrix (a_{ij}) is called a lower triangular matrix if $a_{ij} = 0$, $i < j$ *i.e.*, elements above the leading diagonal are all zeros.

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 7 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \; i.e. \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{pmatrix}$$

Upper Triangular matrix :

Upper triangular matrices are square matrices that can have non zero entries only on and above the main diagonal, where as any entry below the diagonal must be zero. Otherwise A square matrix (a_{ij}) is called an upper triangular matrix if $a_{ij} = 0$, $i > j$ *i.e.* elements below the leading diagonal are zeros.

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

Horizontal matrix : A, m × n matrix is called a horizontal matrix if m < n.

Example : $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, horizontal matrix

Vertical matrix : A, m × n matrix is called a vertical matrix if m > n. $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 4 & 9 \\ 8 & 2 & 5 \end{pmatrix}$

Submatrix of a matrix : Any matrix is obtained by omitting some rows and some columns from the matrix 'A' is called a submatrix of the matrix 'A'.

Singular matrix : A square matrix 'A' is said to be a singular matrix if det A = 0

Non-singular matrix : A square matrix 'A' is said to be non singular if det A ≠ 0

Symmetric matrix : A square matrix 'A' is said to be a symmetric matrix if $A^T = A$

Skew symmetric matrix : A square matrix 'A' is said to be a skew symmetric if $A^T = -A$

Conjugate of a matrix : If A is a matrix having complex numbers as its elements, the matrix obtained form A by replacing each element of 'A' by its conjugate is called the conjugate of A and is dented by $\overline{A}$.

Example : $A = \begin{pmatrix} a+ib & c+id \\ p+iq & r+is \end{pmatrix}$ then $\overline{A} = \begin{pmatrix} a-ib & c-id \\ p-iq & r-is \end{pmatrix}$

Hermitian matrix : A square matrix A is said to be Hermitian if $(\overline{A})^T = A$

Skew hermitian matrix : A square matrix A is said to be skew hermitian if $(\overline{A})^T = -A$

Idempotant matrix : A square matrix 'A' is called an idempotant matrix if $A^2 = A$.

Involutory matrix : A square matrix 'A'is said to be Involutory if $A^2 = I$ where $I \rightarrow$ Identity matrix.

Nilpotant matrix : A square matrix 'A' is called Nilpotant if there exists a +ve integer 'm' s.t. $A^m = 0$ if m is the least +ve integer s.t. $A^m = 0$, then m is called 'index' of the nilpotant matrix.

8.3 : Addition of Matrix; Multiplication by Scalar

Equality of two matrix : Two matrix A & B are said to be equal if

(1) A & B have the same order i.e. they have same number of rows and same number of columns.

(2) Element in the corresponding position of A & B are same.

Example : $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

if a = p, b = q, c = r & d = s

Addition of matrix : Let $A = [a_{ij}]$ be matrix of order m × n

and $B = [b_{ij}]$ be a matrix of order m × n then their addition is defined as

$A + B = [a_{ij} + b_{ij}]_{m \times n}$, i = 1, 2, ...m, j = 1, 2 ....n

i.e., Addition of two matrices of same order is obtained by adding the elements in the corresponding position.

Example : $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 2 & 8 \\ 3 & 5 & 2 \end{pmatrix}$

then $A + B = \begin{pmatrix} 1+4 & 2+2 & 3+8 \\ 4+3 & 5+5 & 6+2 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 11 \\ 7 & 10 & 8 \end{pmatrix}$

Properties : If A, B, C are three matrix of order m × n then

(i) A + B = B + A [commutative]

(ii) (A+B) +C = A + (B + C) (Associative)

(iii) A + 0 = A, (0 $\rightarrow$ Null matrix)

(iv) A + (–A) = 0, (– A is the inverse of A)

Subtraction of matrix : The subtraction of two matrix A & B of same order is defined as

$$A - B = A + (-B)$$

Where – B is the –'ve' matrix of the matrix B.

Multiplication of a matrix by a scalar : Let $A = [a_{ij}]_{m\times n}$ be a matrix of order m × n. Then for any scalar q, 'Aq' is defined as 'Aq' $= [q\, a_{ij}]_{m\times n} = qA$

$$= \begin{bmatrix} qa_{11} & qa_{12} \;\ldots\ldots & qa_{1n} \\ qa_{21} & qa_{22} \;\ldots\ldots & qa_{2n} \\ qa_{m1} & qa_{m2} \;\ldots\ldots & qa_{mn} \end{bmatrix}$$

8.4 : Transpose of a Matrix

Let A be a matrix of order m×n. Then transpose of A is a matrix of order n × m is denoted by A^T or A' is obtained by interchanging the rows of A into columns and columns of A into rows.

i.e Let $A = [a_{ij}]_{m \times n}$ then $A^T = [a_{ji}]_{n \times m}$

$i = 1, 2 \ldots\ldots m,\ j = 1, 2, \ldots\ldots n$

Example : $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 7 & 4 \end{pmatrix}_{2 \times 3}$ then $A^T = \begin{pmatrix} 1 & 2 \\ 3 & 7 \\ 5 & 4 \end{pmatrix}_{3 \times 2}$

Note : 1. Transpose of a row matrix is a column matrix,

2. Transpose of a column matrix is a row matrix.

Def^n: Symmetric matrix :

A real square marix $A = [a_{ij}]$ is said to be symmetric if it is equal to its transpose i.e. $A^T = A$.

Example : $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -2 & 4 \\ -3 & 4 & 3 \end{bmatrix}$, (symmetric), Here $A = A^T$

Def^n : Skew symmetric matrix :

A real square matrix A is said to be a skew symmetric matrix if $A^T = -A$

Let $A = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$ are examples symmetric and skew symmetric matrices respectively.

Note : The elements in the principal diagonal of a skew symmetric matrix are all zero's.

Discuss : Let $A = [a_{ij}] \Rightarrow A^T = [a_{ji}]$

By the hypothes is $a_{ij} = -a_{ij}$

For diagonal element i = j

$\Rightarrow\ a_{ii} = -a_{ii} \quad \Rightarrow\ 2a_{ii} = 0 \quad \Rightarrow a_{ii} = 0$

$\therefore$ diagonal elements are zeros.

Note : 1. For every square matrix A;

$S = (A + A^T)$ is always symmetric and also $\dfrac{A + A^T}{2}$

Proof : Let $S = A + A^T \Rightarrow S^T = (A + A^T)^T$

$= A^T + (A^T)^T = A^T + A = A + A^T = S$

2. ***For every square matrix A,***

$R = (A - A^T)$ is always skew symmetric and also $\dfrac{A - A^T}{2}$

Proof : Let $R = A - A^T$

$\Rightarrow R^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -R$

$\therefore\ R^T = -R$

3. $A = \left(\dfrac{A + A^T}{2}\right) + \left(\dfrac{A - A^T}{2}\right)$

Remarks : Every square matrix can uniquely be expressed as the sum of symmetric as well as skew symmetric matrices.

Proof. : Let A be a square matrix

Consider $A = \dfrac{A + A^T}{2} + \dfrac{A - A^T}{2} = P + Q$ (Say)

Claim : P is symmetric and Q is skew symmetric

We have $P = \dfrac{A + A^T}{2} \Rightarrow P^T = \dfrac{1}{2}(A + A^T)^T = \dfrac{1}{2}(A^T + A) = P$ ($\because$ $P^T = P$)

$\therefore$ P is symmetric

Further $Q = \dfrac{A - A^T}{2} = \left(\dfrac{A - A^T}{2}\right)^T = \dfrac{1}{2}(A^T - A) = -\dfrac{1}{2}(A - A^T) = -Q$ ($\because Q^T = -Q$)

8.5 : Matrix Multiplication

Let $A = [a_{ij}]$ be a matrix of order $m \times n$ and $B = [b_{jk}]$ be matrix of order $n \times p$ so that the no. of columns in A is equal to the no. of rows in B, then the product AB is well defined and it will be matrix of order $m \times p$, whose elements are given by

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}\, b_{1k} + a_{i2} b_{2k} + ...+ a_{in}\, b_{nk} = (i, k)\text{th element of AB.}$$

Note : 1. In the product AB, A is called the pre-factor & B is called the post factor.

2. If the product AB is possible then it is not necessary that the product BA is possible.
3. If A be a matrix of order $m \times n$ then both AB & BA are defined only with B is a matrix of $n \times m$.
4. AB & BA are not equal in general.
5. In view of discussing the equality of AB & BA, 1st it is necessary that they must be the square matrices of same order then they may be equal or may not.

Properties of matrix multiplication :

For any three matrices A, B, C conformable for multiplication and scalar k

1. $k(AB) = (kA)B = A(kB)$
2. $A(BC) = (AB)C$ (Associative)
3. $(A+B)C = AC + BC$ (Distributive).
   $C(A+B) = CA+CB$

Properties : If A^T & B^T are transpose of A & B respectively

then (1) $(A+B)^T = A^T + B^T$

(2) $(A^T)^T = A$

(3) $(kA)^T = kA^T$, $k \rightarrow$ scalar

(4) $(AB)^T = B^T A^T$.

8.6 : Sub-matrix and minors

Any matrix obtained by omitting some rows or columns or both of a given m × n matrix 'A' is called a sub-matrix of A.

Thus $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is a submatrix of $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

The determinant of the square matrix of order r, obtained from a given m × n matix 'A' by omitting (m – r) rows and (n – r) columns, is called a minor of A. In other words, the determinants of the square submatrices of any matrix A are called minors of A.

Thus, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a minor of $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ of order 2.

Partitioned matrices : A matrix divided into sub-matrices by horizontal and/or vertical lines is called a partitioned matrix.

We consider the 3 × 4 matrix A as given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Some possible ways of partitioning it are as given below :

$$\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] \left[\begin{array}{cc:cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$$

A partitioned matrix can be represented economically by denoting each constituent sub matrix by a single matrix symbol.

The first example can be written as $\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$

Where $A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$ and $A_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

The second example can be written by $\begin{bmatrix} A_{11} & A_{12} \end{bmatrix}$

Where $A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $A_{12} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$

and so on.

Partitioning of matrices is useful to effect addition and multiplication by handling smaller matrices.

8.7 : Solutions of Simultaneous Linear Equations

Cramer's Rule :

A method is given below for solving three simultaneous linear equations in three unknowns. This method may also be applied to solve 'n' equations in 'n' unknowns.

Consider the system of equations.

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad ....(1)$$

Where the coefficients are real.

The coefficient of x, y, z as noted in equations (1) may be used to form the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Which is called the determinant of the system.

If $\Delta \neq 0$, the solution of (1) is given by $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$, where $\Delta_r; r = 1, 2, 3$ is the determinant obtained from Δ by replacing the r^{th} column by d_1, d_2, d_3.

Proof : $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

is called the determinant of the system.

If $\Delta \neq 0$, the solution of (1) is given by $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$, where $\Delta_r; r = 1, 2, 3$ is the determinant obtained from by replacing the r^{th} column in Δ by d_1, d_2, d_3.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore x.\Delta = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

or $\quad x\,\Delta = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix}$, replacing C_1 by $C_1 + yC_2 + zC_3$

$\therefore\ x.\,\Delta = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ by (1)

Hence $x.\,\Delta = \Delta_1$ or $x = \dfrac{\Delta_1}{\Delta}$

Similarly it can be shown that

$y = \dfrac{\Delta_2}{\Delta}$, where $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

and $z = \dfrac{\Delta_3}{\Delta}$, where $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

This method of finding the solution of the equation (1) by determinats is known as Cramer's rule.

Note :(1) Cramer's rule is not applicable when $\Delta = 0$

(2) If $\Delta \neq 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then the only solution of equation will be $x = y = z = 0$

(3) If $\Delta = 0$, but at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero then the system has no solution.

(4) If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, the system has infinite number of solutions.

Example – 1 : Solve the following by cramer's rule :

(i) $4x - y = 9,\ 5x + 2y = 8$

(i) $2x + y + 2z = 2,\ 3x + 2y + z = 2,\ -x + y + 3z = 6$

Solution :

(i) $4x - y = 9,\ 5x + 2y = 8$

$4x - y = 9,\ 5x + 2y = 8$

$\Delta = \begin{vmatrix} 4 & -1 \\ 5 & 2 \end{vmatrix} = 8 + 5 = 13$

$\Delta_1 = \begin{vmatrix} 9 & -1 \\ 8 & 2 \end{vmatrix} = 18 + 8 = 26$

$\Delta_2 = \begin{vmatrix} 4 & 9 \\ 5 & 8 \end{vmatrix} = 32 - 45 = -13$

$x = \dfrac{\Delta_1}{\Delta} = \dfrac{26}{13} = 2,\ y = \dfrac{\Delta_2}{\Delta} = \dfrac{-13}{13} = -1$

(ii) $2x + y + 2z = 2$
$3x + 2y + z = 2$
$-x + y + 3z = 6$

$$\Delta = \begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = 2\,(6-1) - 1\,(9+1) + 2\,(3+2)$$

$= 10 - 10 + 10 = 10$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 6 & 1 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 6\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \text{ expanding by } C_1$$

$= 2\,(6-1) - 2(3-2) + 6(1-4) = 2 \times 5 - 2\,(1) + 6\,(-3)$
$= 10 - 2 - 18 = -10$

$$\Delta_2 = \begin{vmatrix} 2 & 2 & 2 \\ 3 & 2 & 1 \\ -1 & 6 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 6 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= 2\,(6-6) - 3\,(6-12 - 1\,(2-4)$
$= 0 + 18 + 2 = 20$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ -1 & 1 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 1 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 1 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$

$= 2\,(12-2) - 3\,(6-2) - 1\,(2-4) = 2\,(10) - 3\,(4) - 1\,(-2)$
$= 20 - 12 + 2 = 10$

$$x = \frac{\Delta_1}{\Delta} = \frac{-10}{10} = -1, \quad y = \frac{\Delta_2}{\Delta} = \frac{20}{10} = 2, \quad z = z = \frac{\Delta_3}{\Delta} = \frac{10}{10} = 1$$

$\therefore$ The value of x , y and z are -1, 2 and 1 respectively.

8.8 : Rank of a Matrix

A number 'r' is said to be the rank of a non zero m × n matrix A, if

(i) there is atleast one ($r \times r$) sub matrix of A whose determinant is not equal to zero.

and (ii) the determinant of every (r +1) rowed square sub-matrix in A is zero.

In otherwords **rank of A** is the greatest possible positive integer 'r' such that 'A' has atleast one non zero minor of order r.

Otherwise, **rank of a matrix** which can be obtained by eleminating largest order of non vanishing minor of the matrix.

The rank 'r' of the matrix 'A' is denoted by r(A) or ρ (A). The rank of a non singular square matrix of order 'n' is n and that of a singular square matrix of order 'n' is less than n.

Obviously $\rho(A) \leq$ min (number of rows, number of columns)

$\rho(A) \leq$ min (m, n)

Rank of A = Rank of A^T, since A and A^T have identical minors.

Remark – 1 :

If all the elements of matrix A are zero, then $\rho(A) = 0$ i.e. the rank of a null matrix is assumed to be zero.

Remarks – 2 :

If an (m×n) matrix is reduced to the normal form by applying elementary row or column operations, then the order of identity matrix I_r is the rank of the matrix.

Remarks – 3 :

If a given matrix is transformed to it's Echelon form, then the number of non zero rows gives the rank of the matrix.

Example – 1 : Find the rank of

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $\rho(A) \leq$ min (3, 3)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1(15 - 16) - 2(10 - 12) + 3\,(8 - 9) = -1 + 4 - 3 = 0$$

Here A is a singular square matrix in which there is atleast one 2 × 2 sub -matrix, for example $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$

$|A_1| = \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 15 - 16 = -1 \neq 0$, whose determinant is not equal to zero.

Hence the rank of A is 2, which is less than the order 3 of the singular square matrix

Example – 2 : Find the rank of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{3} \\ \mathbf{3} & \mathbf{9} & \mathbf{12} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{1} \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

Rank of A $\le \min(3, 4) = 3$

Leaving one column one by one we get four minors of order 3 viz.

$$A_1=\begin{bmatrix}1 & 3 & 4\\ 3 & 9 & 12\\ 1 & 3 & 4\end{bmatrix}, A_2=\begin{bmatrix}1 & 3 & 3\\ 3 & 9 & 3\\ 1 & 3 & 1\end{bmatrix}, A_3=\begin{bmatrix}1 & 4 & 3\\ 3 & 12 & 3\\ 1 & 4 & 1\end{bmatrix}, A_4=\begin{bmatrix}3 & 4 & 3\\ 9 & 12 & 3\\ 3 & 4 & 1\end{bmatrix}$$

But in each case it's determinant is equal to zero.

But there is atleast one 2nd order minor $\begin{vmatrix}4 & 3\\ 12 & 3\end{vmatrix} = 12 - 36 = -24 \neq 0$

The order of this minor is 2, hence $\rho(A) = 2$

8.9 : Elementary Transformation

Elementary Matrices : The matrices obtained from a unit matrix 'I' after one or more elementary row / column operations are called elementary matrices.

Canonical Matrix : The matrix obtained by applying a series of elementary row/column operations such that there are some non zero rows in the top and the remaining rows consist of all zeros is called a canonical matrix.

By an elementary transformation of matrix, any one of the following operations are hold good.

(i) Interchange of any two rows or columns.

These are denoted by $R_i \leftrightarrow R_j$ or $(C_i \leftrightarrow C_j)$ or R_{ij} (or C_{ij}). When the i - th row (or column) is interchanged with j-th row (or column).

For example :

$$\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 5\\ 5 & 6 & 7\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}3 & 4 & 5\\ 1 & 2 & 3\\ 5 & 6 & 7\end{bmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{bmatrix}3 & 5 & 4\\ 1 & 3 & 2\\ 5 & 7 & 6\end{bmatrix}$$

(ii) Multiplication of the i-th row (or the ith column) by a non-zero number k.

These are denoted by $R_i \to kR_i$ (or $C_i \to kC_i$)

For example $\begin{bmatrix}1 & 3\\ 4 & 7\end{bmatrix} \xrightarrow{R_1 \to 2R_1} \begin{bmatrix}2 & 6\\ 4 & 7\end{bmatrix} \xrightarrow{C_1 \to 3C_1} \begin{bmatrix}6 & 6\\ 12 & 7\end{bmatrix}$

(iii) Adding a row or column by multiplying a non zero number k to a row or column. It is denoted by $R_i \to R_i + kR_j$ and $(C_i \to C_i + kC_j)$

For example : $\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix}1 & 3\\ 4 & 11\end{bmatrix} \xrightarrow{C_2 \to C_2 + 3C_1} \begin{bmatrix}1 & 6\\ 4 & 23\end{bmatrix}$

When applied to rows, the elementary operations are known as row operations and when applied to columns, they are known as column operations.

Note – 1 : Interchange of two rows does not alter the rank of a matrix.

Note –2 : The rank of a matrix is not altered by multiplying the elements of any row by a non zero number.

Note – 3 : The rank of a matrix is not alltered by adding k times the elements of a row to the corresponding elements of any other row, where k is any given number.

Cor- 1 : Combining the above theorems, we can say that elementary transformations donot alter the rank of a matrix.

This is the same as to say that pre-multiplication and post multiplication by any elementary matrix donot alter the rank of the matrix.

Cor- 2 : The rank of a matrix is not changed by pre-multiplication and post multiplication with a non-singular matrix.

Cor- 3 : Each elementary matrix is non-singular.

Cor- 4 : Two m × n matrices A and B will be equivalent, if and only if they have the same rank.

Cor- 5 : The rank of the product of any two matrices cannot exceed the rank of either of them.

8.10 : Row Reduced Echelon Matrix

Definition : An m × n matrix A is called a row reduced echelon matrix if the following conditions are satisfied.

(1) The first non-zero element in each non-zero row is unity; which is called the leading entry of the row.

(2) All the non-zero rows if any, preceed the zero rows.

(3) The number of zeros preceeding the first non-zero element in a row is less than the number of such zero's in the succeeding rows.

Row- equivalent systems : Row-equivalent linear system have the same set of solutions.

Important result : The rank of a matrix in Echelon form is equal to the number of 'non-zero' rows of the matrix.

Illustration : (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(1) The first non-zero element in each row is 1.

(2) First two non-zero rows preceed the zero rows.

(3) The number of zeros in R_1, R_2 and R_3 are 0, 1, 3.

(i.e. No zeros in R_1 one zeros in R_2 and three zeros in R_3)

i.e. A is a Row Reduced Echelon matrix

(b) $A = \begin{bmatrix} 1 & 3 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is row-echelon form

Note : If a matrix is in row reduced echelon form, then any column has a leading entry 1 must have zeros below the leading entry 1.

Example – 4 : Find the rank of the matrix A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{4} & \mathbf{6} & \mathbf{8} \end{bmatrix}$

Solution : Where $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}$ $[R_2 \to R_2 - 3R_1 \text{and } R_3 \to R_3 - 4R_1]$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ $[R_3 \to R_3 - R_2]$ (Here no of non zero rows is two)

Hence the rank of the matrix is 2.

Example – 5 : Find the rank of the matrix

$$A = \begin{bmatrix} \mathbf{6} & \mathbf{1} & \mathbf{3} & \mathbf{8} \\ \mathbf{4} & \mathbf{2} & \mathbf{6} & \mathbf{-1} \\ \mathbf{10} & \mathbf{3} & \mathbf{9} & \mathbf{7} \\ \mathbf{16} & \mathbf{4} & \mathbf{12} & \mathbf{15} \end{bmatrix}$$

Solution : Rank of $A \leq \min(4, 4)$

We have $|A| = \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix} \sim \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 6 & 1 & 3 & 8 \\ 4 & 1 & 3 & 8 \end{vmatrix} = 0$ $[R_3 \to R_3 - R_2, R_4 \to R_4 - R_3]$

[Since R_1 and R_3 are identical]

The rank of the matrix is not 4.

Consider a minor of order 3 which

$\begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 10 & 3 & 9 \end{vmatrix} \sim \begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 1 & 3 \end{vmatrix} = 0$ $[R_3 \to R_3 - R_2]$

In the same way it can be shown that all the minors of order 3 are zero. Consider the minor $\begin{bmatrix} 6 & 1 \\ 4 & 2 \end{bmatrix}$ of order 2. Its determinant is not zero. Hence the rank of the following matrix is 2.

Example – 6 : Find the rank of the given matrix $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{0} \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$, $\rho(A) \le \min(4, 4) = 4$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}, [C_3 \to C_3 - C_1, C_4 \to C_4 - C_1]$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, [R_3 \to R_3 - R_1,\ R_4 \to R_4 - R_1]$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [R_3 \to R_3 - 3R_2,\ R_4 \to R_4 - R_2]$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [C_3 \to C_3 + 3C_2,\ C_4 \to C_4 + C_2]$$

Hence rank of the matrix is 2.

8.11 : Linear Systems of Equations (Existence, uniqueness)

A linear equation in 'n' unknows $x_1, x_2.......x_n$ is an equation of the form $a_1x_1 + a_2x_2 + ...+ a_nx_n = b$ .....(1)

In the above equation if b = 0, then it is called a homogeneous linear equation. In contrsat (1) is called a non-homogeneous linear equation.

Consider a set of 'm' non-homogeneous linear equations in n-unknowns.

$$\begin{aligned} &a_{11}x_1+a_{12}x_2+ \dots\dots\dots + a_{1n}x_n = b_1 \\ &a_{21}x_1+a_{22}x_2+ \dots\dots\dots + a_{2n}x_n = b_2 \\ &\dots \quad \dots \quad \dots \qquad \dots \quad \dots.\dots \\ &\dots \quad \dots \quad \dots \qquad \dots \quad \dots.\dots \\ &a_{m1}x_1+a_{m2}x_2+ \dots\dots+ a_{mn}x_n = b_m \end{aligned} \qquad \dots..(2)$$

If atleast one set of values of $x_1x_2\dots\dots x_n$ can be found satisfying all the equations, then the equations are said to be **consistent**. If no such set exists, then the equations are said to be **inconsistent**.

In the former cases, the values of the unknowns for which all the equations are satisfied are said to constitute the solutions of the equations.

Consider the sets of equation

$$\left.\begin{aligned}2x+5y&=9\\ x-y&=1\end{aligned}\right\}\dots\dots\dots(i) \qquad \left.\begin{aligned}x+2y&=7\\ 4x+8y&=28\end{aligned}\right\}\dots\dots\dots(ii) \qquad \left.\begin{aligned}2x+3y&=5\\ 4x+6y&=-8\end{aligned}\right\}\dots\dots\dots(iii)$$

The first set has a unique set solution (x = 2, y = 1) the second has an infinite number of sets solutions (x = 7 – 2y, y = c, for arbitrary c) and the third has none. In the above example the sets (i) and (ii) are consistent while the set (iii) **is inconsistent**. A consistent system has either just one set of solutions or an infinite number of sets of solutions.

The associated system of homogeneous linear equations is given by

$$\begin{aligned} &a_{11}x_1+ a_{12}x_2+ \dots\dots\dots + a_{1n}x_n = 0 \\ &a_{21}x_1+ a_{22}x_2+ \dots\dots\dots + a_{2n}x_n = 0 \\ &\dots \quad \dots \quad \dots \qquad \dots \quad \dots \; \dots \\ &\dots \quad \dots \quad \dots \qquad \dots \quad \dots \; \dots \\ &a_{m1}x_1+a_{m2}x_2+ \dots\dots+ a_{mn}x_n = 0 \end{aligned} \qquad \dots.(3)$$

The (m × n) matrix A given by

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots. & \dots. & \dots & \dots. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = A$$

is called **coefficient matrix** denoted by A . The matrix $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots. & \dots. & \dots & \dots. & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$ is called the **augumented matrix** and is denoted by A_b or (A, b)

The system of non-homogeneous equations (2) may be put in the form AX = B where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ and A is given by (4)

In order to solve the equations we use the method of elimination which is, in fact, row operations on both A and B and reduce the coefficient matrix to the triangular form or diagonal form or normal form, the associated homogeneous system is given by AX = 0.

8.12 : Consistency of System of Linear Equations

Consider the following 'm' equations with 'n' unknowns :

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2 \\
&\dots \quad \dots \quad \dots \qquad \dots \quad \dots \quad \dots \quad \dots\dots \ (1) \\
&a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_m
\end{aligned}$$

Which is in matrix form AX = B, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Here A is called **coefficient matrix** and B is called right hand side matrix, with the help of A and B, consider K = [A/B] i.e.

$$K = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

which is known as **augumented matrix**.

If the elements in B are not all zero, Then (1) is called **Non-homogeneous system**, otherwise it is called **homogeneous system**.

A solution of (1) defined by a set of values of the variable x_1, x_2 ....x_n which satisfy simultaneously all the equation of (1).

If the system given by (1) has a solution, it is called **consistent system**, otherwise the system is called **inconsistent**.

In fact, a consistent system has either unique solution or infinitely many solutions obviously any homogenous system of equations is always a consisent system, since $x_i = 0$ for all i is a solution which is known as **Trivial solution**.

Let the rank of A = r, and the rank of K = r_1, of the system (1)

Rouche's theorem :

The system of equation (1) is consistent if and only if the coefficients matrix 'A' and the augmented matrix 'K' are of the same rank otherwise the system is inconsistent.

(1) Rank of A = rank of K = r (r ≤ the smaller of the numbers m and n). The equation (1) can by suitable row operations, be reduced to

$b_{11}x_1 + b_{12}x_2 + ... + b_{1n}x_n = k_1$

$0.x_1 + b_{22}x_2 + ... + b_{2n}x_n = k_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$0.x_1 + 0.x_2 + ... + b_{rn}x_n = k_r \rightarrow$

and the remaining $(m - r)$ equations being all of the form $0.x_1 + 0.x_2 + \ldots + \ldots 0.x_n = 0$

(2) The equations (ii) will have a solution, though n – r of the unknown may be choosen arbitarly. The solution will be unique when r = n. Hence the equations (i) are consistent.

(3) Rank of 'A' < rank of K. In particular, let the rank of K be r + 1. In this case the equations (i) will reduce, by suitable row operations to

$b_{11}x_1 + b_{12}x_2 + ... + b_{1n}x_n = k_1$

$0.x_1 + b_{22}x_2 + ... + b_{2n}x_n = k_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$0.x_1 + 0.x_2 + ... + b_{rn}x_n = k_r$

$0.x_1 + 0.x_2 + ... + 0.x_n = k_{r+1}$

and remaining m – (r + 1) equations are of the form $0.x_1 + 0.x_2 + ... + ...0.x_n = 0$. Clearly the (r+1)th equation can not be satisfied by any set of values for the unknowns. Hence the equations (i) are **inconsistent**.

(For Non-homogeneous, system)

(a) If $r \neq r_1 \Rightarrow$ Equations are inconsistent and there is no solution.

(b) If $r = r_1 = n \Rightarrow$ Equations are consistent, and there is a unique solution.

(c) If $r = r_1 < n \Rightarrow$ Equations are consistent but there exists infinite number of solution.

Illustrative Examples

Example – 1 : ***Solve***

$x + 2y - z = 3$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

Solution : First we check the consistency of the equations.

Writing in the matrix form, we have

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

The given equation which can be represented in Augmented matrix form

$$K = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 5 & -4 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & -8 \\ 0 & 5 & -4 & -4 \end{bmatrix} (c_2 \leftrightarrow c_3) \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -7 & 8 \\ 0 & 0 & 3 & 12 \end{bmatrix} (R_2 \to R_2 - R_3)$$

(ρ (A) = 3.) Obviously rank of $\rho(k)$ matrix is also 3.

ρ (A) = ρ (K) $\Rightarrow$ equations are consistent.

Next to find its solution

$x - y + 2z = 3, \quad 5y - 7z = 8, \quad 3z = 12$

By backward substitution we get $z = 4$.

$5y - 28 = -8, 5y = 20, y = 4$

$x - 4 + 8 = 3, x + 4 = 3, x = -1$

or $\quad x = -1, y = 4, z = 4$ be the required solutions.

Example – 2 : ***For what values of λ the system $x + 2y = 1$; $5x + \lambda y = 5$ has (i) unique solution; (ii) infinite number of solutions.***

Solution : There are two equation and two unknowns in the given system of equations.

Coefficient matrix $A = \begin{pmatrix} 1 & 2 \\ 5 & \lambda \end{pmatrix}$

The augumented matrix $K = \begin{pmatrix} 1 & 2 & 1 \\ 5 & \lambda & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & \lambda - 10 & 0 \end{pmatrix} R_2 \to R_2 - 5R_1$

There arises two cases : $\lambda - 10 \neq 0$ or $\lambda - 10 = 0$

Case (i) $\lambda - 10 \neq 0$

Then rank (K) = rank A = 2 = the number of unknowns.

Hence the equation are consistent and have unique solution if $\lambda \neq 10$.

Case (ii) $\lambda - 10 = 0$

Then rank A = rank (K) = 1, which is less than the number of unknowns.

$\therefore$ The equations are consistent and have infinite number of solutions if $\lambda = 10$.

Example – 3 : ***Solve the following system of equations completely***

$$2x - y + 3z = 3$$
$$x + 2y - z - 5w = 4$$
$$x + 3y - 2z - 7w = 5$$

Solution : Writing the equation in matrix form AX = B, we have

$$\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 2 & -1 & -5 \\ 1 & 3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Here the augmented matrix is $(K) = \begin{bmatrix} 2 & -1 & 3 & 0 & 3 \\ 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \end{bmatrix}$ $(R_2 \leftrightarrow R_1)$

To find the rank of [K], change it to Echelon form.

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 2 & -1 & 3 & 0 & 3 \\ 1 & 3 & -2 & -7 & 5 \end{bmatrix} (R_2 \to R_2 - 2R_3 \text{ and } R_3 \to R_3 - R_2)$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & -5 & 5 & 10 & -5 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix} \left[R_2 \to R_2 \times \frac{-1}{5}\right] \sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix} [R_3 \to R_3 - R_2]$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (R_3 \to R_3 - R_2)$$

Here number of non-zero rows = 2

$\therefore \rho(A) = \rho(K) = 2$

Hence the system is consistent

Since $\rho(A) = 2 < 4$ (number of unknowns), the solution will contain $4 - 2 = 2$ arbitary constants

The equations corresponding to the matrix are

$x + 2y - z - 5w = 4$ ....(1)

and $y - z - 2w = 1$..........(2)

Taking $z = k_1$, $w = k_2$,

From (2), $y = 1 + z + 2w = 1 + k_1 + 2k_2$

and $x = 4 - 2y + z + 5w$

$= 4 - 2(1 + k_1 + 2k_2) + k_1 + 5k_2, = 2 - k_1 + k_2$

Hence $x = 2 - k_1 + k_2$, $y = 1 + k_1 + 2k_2$, $z = k_1$, $w = k_2$

where k_1 and k_2 are arbitary constants.

Example – 4 : ***For what value of λ, does the system***

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has (i) no solution , (ii) unique solution (iii) more than one solution.

Solution :

(i) Let $A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Matrix equation is AX = B.

We find the ranks of [A] and [AB]

Augument matrix is

$$(K) = \begin{bmatrix} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 1 & \lambda & 1 \end{bmatrix} [R_1 \leftrightarrow R_1(-1)]$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -1 \\ 3 & -1 & 2 & 1 \\ 0 & 1 & \lambda & 1 \end{bmatrix} (R_1 \leftrightarrow R_1 - 3R_1) \sim \begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 5 & 5 & 4 \\ 0 & 1 & \lambda & 1 \end{bmatrix} \left[R_2 \leftrightarrow R_2 \times \left(\frac{1}{5}\right)\right]$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & \frac{4}{5} \\ 0 & 1 & \lambda & 1 \end{bmatrix} [R_3 \leftrightarrow R_3 - R_2] \sim \begin{bmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 4/5 \\ 0 & 0 & \lambda - 1 & 1/5 \end{bmatrix} \quad \text{.............(1)}$$

Case (i) If $l = \lambda$, then from (1),

ρ (A) = 2 and ρ (K) = 3

$\therefore$ ρ (A) $\neq$ ρ (K)

Hence the system is inconsistent.

(ii) If $\lambda \neq 1$, then from (1)

ρ (A) = ρ (K) = 3 = number of unknowns

Hence the system has unique solution

$\therefore$ From (1), the given system of equations reduces to

$$(\lambda - 1)z = \frac{1}{5} \Rightarrow z = \frac{1}{5(\lambda - 1)}, \quad y + z = \frac{4}{5} \Rightarrow y = \frac{4}{5} - z = \frac{4\lambda - 5}{5(\lambda - 1)}$$

and $x - 2y - z = -1, \Rightarrow x = -1 + 2y + z = \dfrac{3\lambda - 4}{5(\lambda - 1)}$

Here $x = \dfrac{3\lambda - 4}{5(\lambda - 1)}, y = \dfrac{4\lambda - 5}{5(\lambda - 1)}, z = \dfrac{1}{5(\lambda - 1)}$.

For different values of λ we get different solutions.