7

CHAPTER

Power Series

7.1 : Introduction

We have seen that homogeneous linear differential equations with constant coefficients can be solved by algebraic methods and their solutions are elementary functions known from the calculus (*e.g.,* $\sin x$, $\cos x$, e^x, $ln\ x$, x^n etc.)

But many differential equations arising from physical problems may have variable co-efficients and may not permit a general solution in terms of known elementary functions of calculus. We need method to solve these equations whose solutions may be non-elementary functions.

Legendre's equation, Bessel's equation (see chapter 8 and chapter 9) are examples of differential equations which play an important role in **engineering mathematics** and for which **power series expansions** can be used to obtain information about the form of the solution. The basic concepts and definitions involved in solving second order linear differential equations by **power series** are discussed in this chapter.

7.2 : Power Series and its radius of Convergence

A series of the form $a_0 + a_1 x + a_2 x^2 + \ldots$ where $a_0, a_1, a_2, \ldots$ are real numbers, is called a **power series.** It is called a power-series about 0. The general form of a power series about $x_0 \in \mathbb{R}$ is

$$a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots, \text{ where } a_0, a_1, a_2, \ldots \in \mathbb{R}$$

which is the set of all real numbers. Since the general form reduces to the form $a_0 + a_1 x + a_2 x^2 + \ldots$ by the substitution $x' = x - x_0$, we shall study the nature of power series by considering power series about 0.

The power series $a_0 + a_1 x + a_2 x^2 + \ldots.$ is denoted as $\sum_{n=0}^{\infty} a_n x^n$.

The series $\sum_{n=0}^{\infty} a_n x^n$ is said to be **convergent** at $x = c \in \mathbb{R}$ if the series $\sum_{n=0}^{\infty} a_n c^n$ is convergent *i.e.* if there exists a real number l (called the sum of the series.)

such that $\lim\limits_{n \to \infty} (a_0 + a_1 c + a_2 c^2 + ... + a_n c^n) = l$ and we write

$$a_0 + a_1 c + a_2 c^2 + ... \qquad = l$$

The power series $\sum\limits_{n=0}^{\infty} a_n x^n$ may not be convergent for all real x.

Examples :

(i) The series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ is convergent for all real values of x and it represents e^x.

(ii) The series $1 - x + x^2 - x^3 + ...$ is convergent if $|x| < |$ and is not convergent if $|x| > |$ and this series represents $\frac{1}{1+x}$ for $-| < x < |$

(iii) The power series $1 + x + 2!\, x^2 + 3!\, x^3 + ...$ is convergent only at $x = 0$.

Note : A series is said to be divergent if it is not convergent.

Remark : It appears that some power series converges for all $x \in \mathbb{R}$. They are called **everywhere convergent** power series. Some power series converge only for $x = 0$. They are called **nowhere convergent** power series. Some power series converge for some real x and diverge for the others. Also we observe that **every power series** of form $\sum\limits_{n=0}^{\infty} a_n x^n$ is convergent at $x = 0$.

Radius of convergence of a power series :

It can be proved (here we omit the proof) that if a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ be neither no-where convergent nor everywhere convergent, then there exists a positive real number R such the series $\sum\limits_{n=0}^{\infty} |a_n x^n|$ converges for all real x satisfying $|x| < R$ and $\sum\limits_{n=0}^{\infty} a_n x^n$ diverges for all real x satisfying $|x| > R$.

The series $\sum\limits_{n=0}^{\infty} a_n x^n$ is said to be **absolutely convergent if** $\sum\limits_{n=0}^{\infty} |a_n x^n|$ is convergent. Also it can be shown that any absolutely convergent series (not necessarily power series) is convergent. Then from the property of power series (stated above without proof) we can define the **radius of convergence** R of a power series as follows :

If $\sum\limits_{n=0}^{\infty} a_n x^n$ is neither no where convergent nor everywhere convergent then its radius of convergence R is a **real number** such that the series is convergent for all real x satisfying $|x| < \mathbb{R}$ and divergent for all real x satisfying $|x| > R$.

The open interval (–R, R) is called the interval of convergence.

If $\sum_{n=0}^{\infty} a_n x^n$ be nowhere convergent (*i.e.* convergent only for $x = 0$) we define R = 0 and it the power series be everywhere convergent we define R = ∞.

Note : It the radius of convergence is a real number R then the power series may or may not converge at $x = \pm R$. There are power series for which both R and –R are points of convergence, or both R and –R are points of divergence or one of R and –R is a point of convergence and the other is a point of divergence.

Determination of the radius of convergence :

Here we shall use **D'Alembert's ratio test** or **Cauchy's root test** for determining the radius of convergence of a power series.

D'Alembert's ratio test (Statement without proof)

Let $\sum_{n=1}^{\infty} u_n$ be an infinite series. Let $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l$.

where $l \in \mathbb{R}$ or $l = \infty$.

If $l \in R$ then the series $\sum_{n=1}^{\infty} u_n$ is absolutely convertengent (hence convergent) if $l < 1$ and divergent if $l > 1$. If $l = \infty$ then the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Cauchy's root test (Statement without proof)

Let $\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = l$

where $l \in \mathbb{R}$ or $l = \infty$.

If $l \in \mathbb{R}$ then $\sum_{n=1}^{\infty} u_n$ is absolutely convergent (hence convergent) if $l < 1$ and divergent if $l >$ 1. If $l = \infty$ then $\sum_{n=1}^{\infty} u_n$ is divergent.

Note : We observe that the above tests fail to give any conclusion if $l = 1$ or it $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$ and $\lim_{n\to\infty} |u_n|^{\frac{1}{n}}$ do not exist. In such case we can use **Cauchy-Hadamard test** (which is beyond the scope of our discussion) for determining the radius of convergence of a power series.

Example : ***Find the radius of convergence of the power series*** $\sum_{n=0}^{\infty} \frac{1}{4^n}(x-3)^{2n}$ **.**

[B.P.U.T. 2005

Solution : The given series is a power series about $x = 3$. Let $x - 3 = x'$. Then the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{4^n}(x')^{2n} .$$

$$\text{Let } u_{n+1} = \frac{1}{4^n}(x')^{2n}, n \geq 0$$

$$\text{Then } u_{n+1} = \frac{x^{2(n+1)}}{4^{n+1}}$$

$$\text{So } \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{1}{4}(x')^2\right| = \frac{1}{4}(x')^2$$

$$\text{Now } \lim_{n\to\infty} \frac{1}{4}(x')^2 = \frac{1}{4}x'^2. \quad \text{Hence } \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{4}(x')^2$$

Then by D'Alembert's ratio test, the series $\sum_{n=0}^{\infty} \frac{1}{4^n}(x')^{2n}$ is convergent of $\frac{1}{4}x'^2 < 1$ *i.e.* $x'^2 < 4$ and divergent of $\frac{1}{4}x'^2 > 1$ *i.e.* $(x')^2 > 4$.

Now $(x')^2 < 4 \quad \Rightarrow |x'| < 2$

and $(x')^2 > 4 \quad \Rightarrow |x'| > 2$

So the given power series is convergent if $|x - 3| < 2$ and divergent if $|x - 3| > 2$.

So the required radius of convergence is 2.

Remark : For more examples on power series see section 7.6.

7.3 : Ordinary and Singular point of a differential equation

We consider the second order linear differential equation

$$\frac{d^2 y}{dx^2} + \mathrm{P}(x)\frac{dy}{dx} + \mathrm{Q}(x)\,y = 0 \qquad .... (1)$$

A point x_0 is said to be an **ordinary point** of the differential equation (1) if the functions P(x), Q(x) both can be expressed as a power series about x_0 in a neighbourhood of x_0, otherwise.

x_0 is called a **singular point** or **singularity** of the differential equation.

A singularity x_0 of the differential equation (1) is called a **regular singularity** if $(x - x_0)$ P (x) and $(x - x_0)^2$ Q(x) both can be expanded as power series about x_0 in a neighbourhood of x_0. A singularity is called an **irregular singularity** of the differential equation if it is not a regular singularity.

Note : A function $f(x)$ is said to **analytic** at a point x_0 if $f(x)$ can be expanded as a power seried about x_0 in a neighbourhood of x_0.

Then we can say x_0 is an ordinary point of the differential equation (1) if both P (x), Q (x) are analytic at x_0.

Illustration – 1 : ***We consider the differential equation $(1 + x^2)\, y'' + 2x\, y' - 2y = 0$. This equation is equivalent to***

$$y'' + \frac{2x}{1+x^2} - \frac{2}{1+x^2}y = 0 \qquad \left[y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right]$$

Here $P(x) = \dfrac{2x}{1+x^2}$, $Q(x) = \dfrac{-2}{1+x^2}$

We see that x, $1 + x^2$ are polynomials and $1 + x^2 \neq 0$ for all real x and so P (x), Q (x) are both analytic at any point x_0. Then any point x_0 is an ordinary point of the above differential equation.

Illustration – 2 : ***The differential equation $(1 - x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 5y = 0$ can be expressed as***

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} - \frac{5}{1-x^2}y = 0,$$

where $P(x) = \dfrac{-2x}{1-x^2}$, $Q(x) = \dfrac{-5}{1-x^2}$

Here we see that P(x), Q (x) are not analytic at $x = \pm 1$ since $1 - x^2 = 0$ at these points.

[Derivatives of P(x), Q (x) will not exist at $x = \pm 1$ and so P(x), Q (x) can not be expanded in power series about $x = 1$ or about $x = -1$ since if a function $f(x)$ can be expanded in a power series about $x = x_0$ then derivatives of all orders of $f(x)$ must exist at $x = x_0$].

So 1, –1 are singular points of the above differential equation.

Here we find that $(x - 1)$ P (x), $(x - 1)^2$ Q(x) are analytic at $x = 1$ and $(x + 1)$ P (x), $(x + 1)^2$ Q (x) are analytic, at $x = -1$.

So 1, –1 are **regular singularities** of the above differential equation.

Illustration – 3 : We consider the differential equation $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + xy = 0$.

This equation can be expressed as

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0,$$

where $P(x) = \dfrac{2}{x}$, $Q(x) = 1$.

We see that P (x) is not analytic at $x = 0$.

So 0 is a singularity of the above equation. Also 0 is a regular singularity since $x \cdot P(x)$, x^2 Q (x) are analytic at $x = 0$.

Further we observe that any **other point** is an ordinary point of the above differential equation.

7.4 : Power series solution in the neighbourhood of an ordinary point of second order homogeneous linear differential equation.

We state below the theorem (without proof) on the **existence** of power series solution of

$$\frac{d^2y}{dx^2} + \mathrm{P}(x)\frac{dy}{dx} + \mathrm{Q}(x)\,y = 0 \qquad ...(1)$$

in the neighbourhood of an ordinary point.

Theorem. [Statement without proof].

Let x_0 be an ordinary point of the differential equation $\frac{d^2y}{dx^2} + \mathrm{P}(x)\frac{dy}{dx} + \mathrm{Q}(x)\,y = 0$ and let a_0 and a_1 be arbitrary but fixed constants. Then there exists a unique function $y(x)$ that is analytic at x_0, where $y(x)$ is a solution of (1) in a certain neighbourhood of this point, and satisfies the initial conditions $y(x_0) = a_0$, $y'(x_0) = a_1$. Furthermore, if the power series expansions of $\mathrm{P}(x)$, $\mathrm{Q}(x)$ are valid on an interval $(x_0 - \mathrm{R}, x_0 + \mathrm{R})$, $\mathrm{R} > 0$ then the power series expansion of this solution is also valid on the same interval.

Note : Here the interval a $(x_0\text{-R}, x_0 + \mathrm{R})$ may be the eutire real axis in which case we take $\mathrm{R} = \infty$. Now we shall explain the method of obtaining the power series solution valid in the neighbourhood of an ordinary point of a differential equation through an example.

Example. ***Solve :*** $\boldsymbol{\frac{d^2y}{dx^2} + xy = 0}$.

Solution : Here $x = 0$ is an ordinary point of the given differential equation.

For a solution in power series form, let us assume $y = \sum_{n=0}^{\infty} a_n x^n$.

i.e. $y = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots.$

Then $\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots\ldots\ldots.$

and $\frac{d^2y}{dx^2} = 2a_2 + 3.2\,a_3x + 4.3\,a_4x^2 + \ldots\ldots\ldots.$.

Substituting for y and $\frac{d^2y}{dx^2}$ in the given differential equation we get

$$2.1\,a_2 + 3.2a_3x + 4.3\,a_4x^2 + ... + n(n-1)\,a_nx^{n-2} + ... + x(a_0 + a_1x + a_2x^2 + ...) = 0.$$

$$\text{or, } 2.a_2 + (3.2\,a_3 + a_0)\,x + (4.3\,a_4 + a_1)\,x^2 + (5.4\,a_5 + a_2)\,x^3 + ...$$
$$+ [\,(n+2)(n+1)\,a_{n+2} + a_{n-1}]\,x^n + ... = 0. \qquad .... (1)$$

Now (1) can be regarded as an identity if $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of the given differential equation.

Then equating the coefficients of various powers of x to zero, we get.

$a_2 = 0$

$3.2\, a_3 + a_0 = 0$ giving $a_3 = -\dfrac{a_0}{3!}$,

$4.3\, a_4 + a_1 = 0$ giving $a_4 = --\dfrac{2a_1}{4!}$,

$5.4\, a_5 + a_2 = 0$

or, $5.4\, a_5 = -a_2 = 0$, giving

$a_5 = 0$ etc.

Thus $(n + 2)(n + 1)\, a_{n+2} + a_{n-1} = 0$

$$\text{or,}\quad a_{n+2} = -\frac{1}{(n+2)(n+1)} a_{n-1} \qquad \text{.... (2)}$$

Substituting $n = 4, 5, 6, \ldots\ldots\ldots$ in (2) we get

$a_6 = \dfrac{4a_0}{6!}, a_7 = \dfrac{5.2}{7!} a_1,$

$a_8 = 0,\ a_9 = -\dfrac{7.4}{9!} a_0$

and so on.

Hence the formal solution is

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{4x^6}{6!} - \frac{7.4}{9!}x^9 + \ldots\ldots\ldots\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{5.2x^7}{7!} + \ldots\ldots\ldots\right)$$

where it can be shown that the series

$$1 - \frac{x^3}{3!} + \frac{4x^6}{6!} - \frac{7.4x^9}{9!} + \ldots\ldots\ldots$$

and $x - \dfrac{2x^4}{4!} + \dfrac{5.2x^7}{7!} + \ldots\ldots\ldots$

are convergent for all real x.

So the required solution is

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{4x^6}{6!} - \frac{7.4}{9!}x^9 + \ldots\ldots\ldots\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{5.2}{7!}x^7 + \ldots\ldots\ldots\right)$$

where a_0, a_1 are arbitrary constants.

Remark. If a power series solution about an ordinary point x_0 be required we assume a solution of the from $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

Note : The above method can also be applied to a differential equation of order one about an ordinary point.

[A point x_0 is an odinary point of $\frac{dy}{dx} + \text{P}(x)\, y = 0$ if P (x) is analytic at x_0].

More examples on solution of differential equation by power series method are given in illustrative examples

7.5 : Solution about a regular singularity

It $x = x_0$ is a singular point of the differential equation

$$\frac{d^2 y}{dx^2} + \text{P}(x)\frac{dy}{dx} + \text{Q}(x)\, y = 0 \qquad ..... (1)$$

then there is no series of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ such that $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ may be a solution of (1).

However, with a slight modification of the power series method, solutions in a neighbourhood of regular singularity can be obtained. This method is known as **Frobenius method.**
We consider the case where $x = 0$ is a regular singularity of (1). In Frobenius method we assume a series solution of the form.

$$y = x \sum_{n=0}^{\infty} a_n x^n \text{ where } a_0 \neq 0. \qquad .... (2)$$

Then substituting $y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}$ obtained from (2) in the given differential equation, the coefficients of various powers of x are equated to zero. In this process, equating to zero the smallest power of x gives a quadratic equation in known as the **indicial equation.**
The roots of the indicial equation determine the complete solution depending on whether they are :

(i) distinct and not differing by an integer.
(ii) equal
(iii) distinct and differing by an integer.

Here we shall explain Frobenius method by an example only in case (i).

Illustration : ***Find a series solution of*** $2x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + (1 - x^2)y = 0$ ***about the regular singularity x = 0 by Frobenius method.***

Solution. We assume for the solution $y = x^{\rho}(c_0 + c_1 x + c_2 x^2 + ... + c_n x^n + ...)$
where $c_0 \neq 0$. ..... (1)

Then $y = \sum_{n=0}^{\infty} c_n\, x^{\rho + n}$ $c_0 \neq 0$.

So $\dfrac{dy}{dx} = \sum\limits_{n=0} c_n\,(\rho + n)\,x^{\rho+n-1}$

and $\dfrac{d^2y}{dx^2} = \sum\limits_{n=0} c_n(\rho+n)(\rho+n-1)\,x^{\rho+n-2}$

Substituting these expressions for $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ in the given differential equation we get

$$\{2c_0\,\rho\,(\rho-1) - c_0\,\rho + c_0\}\,x^{\rho} + \{2c_1\,(\rho+1)\,\rho - c_1\,(\rho+1) + c_1\}\,x^{\rho+1}$$
$$+ \{2c_2\,(\rho+2)\,(\rho+1) - c_2\,(\rho+2) + c_2 - c_0\}\,x^{\rho+2}$$
$$+ \{2c_3\,(\rho+3)\,(\rho+2) - c_3\,(\rho+3) + c_3 - c_1\}\,x^{\rho+3} + \ldots\ldots = 0$$

This is an identity. Equating the coefficient of the term of lowest power of x, that is of x^{ρ}, to zero we get the indicial equation as

$2c_0\,\rho\,(\rho-1) - c_0\,\rho + c_0 = 0$

or, $2\rho\,(\rho-1) - \rho + 1 = 0$, ($\because c_0 \neq 0$)

or, $2\rho^2 - 3\rho + 1 = 0$

or, $(2\rho - 1)(\rho - 1) = 0$

which gives $\rho = 1, \dfrac{1}{2}$

The roots of the indicial equation are different and the difference is not an integer.

Now equating the coefficients of the other powers of x to zero, we have

$2\,c_1\,(\rho+1)\,\rho - c_1\,(\rho+1) + c_1 = 0$

$2c_2\,(\rho+2)\,(\rho+1) - c_2\,(\rho+2) + c_2 - c_0 = 0$

$2c_3\,(\rho+3)\,(\rho+2) - c_3\,(\rho+3) + c_3 - c_1 = 0,$

....

$2c_m\,(\rho+m)\,(\rho+m-1) - c_m\,(\rho+m) + c_m - c_{m-2} = 0$ .... (2)

for $m \geq 2$.

The first equation gives $c_1 = 0$ for $(\rho+1)(2\rho-1) + 1 \neq 0$.

since here $\rho = 1, \dfrac{1}{2}$

From (2) we get $c_m = \dfrac{c_{m-2}}{(2\rho+2m-1)(\rho+m-1)}, m \geq 2$ ..... (3)

Putting $m = 2, 3, 4, \ldots\ldots$ in (3),

We get $c_2 = \dfrac{1}{(2\rho+3)(\rho+1)}c_0,$

$c_3 = \dfrac{1}{(2\rho+5)(\rho+2)}c_1 = 0,$ ($\because c_1 = 0$)

Similarly $c_5 = c_7 = c_9 = ... = 0$,

$$c_4 = \frac{1}{(2\rho+7)(\rho+3)} c_2$$

$$= \frac{1}{(2\rho+7)(2\rho+3)(\rho+3)(\rho+1)} c_0$$

and so on.

Putting these values in (1) we get,

$$y = x^{\rho}\left\{c_0 + \frac{c_0}{(2\rho+3)(\rho+1)}x^2 + \frac{c_0}{(2\rho+7)(2\rho+3)(\rho+3)(\rho+1)}x^4 + ...\right\} \quad .... (4)$$

We get two linearly independent solutions by putting $\rho = 1$ and $\rho = \frac{1}{2}$ in (4) and replacing c_0 by A when $\rho = 1$ and by B where $\rho = \frac{1}{2}$, where A, B are arbitrary constants. Thus the series solution of the given equation is

$$y = \text{A}\,x\left(1 + \frac{x^2}{2\cdot5} + \frac{x^4}{2\cdot4\cdot5\cdot9} + ...\right) + \text{B}\,x^{\frac{1}{2}}\left(1 + \frac{x^2}{2\cdot3} + \frac{x^4}{2\cdot3\cdot4\cdot7} + ...\right) \quad .... (5)$$

Note : Here $\text{P}(x) = -\frac{x}{2x^2} = -\frac{1}{2x}, x \neq 0$

$$\text{Q}(x) = \frac{1-x^2}{2x^2}, x \neq 0$$

Here $x\,\text{P}(x)$ and $x^2\,\text{Q}(x)$ are analytic at $x = 0$. If the power series expansions of $x\,\text{P}(x)$ and $x^2\,\text{Q}(x)$ are valid in $(-\text{R}, \text{R})$ then we can say that solution (5) is valid for $0 < |x| < \text{R}$ where $\text{R} > 0$.

Illustrative Examples

Example – 1 : ***What is the radius of convergence of the power series*** $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$ ***?***

[B.P.U.T. – 2007

Solution : The given series is a power series about $x = 2$.

If the series be expressed as $\sum_{n=0}^{\infty} a_n (x-2)^n$ then we find that $a_n = \frac{1}{n!}$, for $n \geq 0$.

Then if u_n be n - th term of the series we get

$u_{n+1} = a_n (x-2)^n$

$u_n = a_{n-1} (x-2)^{n-1}$, for $n \geq 1$.

So. $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{a_n}{a_{n-1}}(x-2)\right| = \left|\dfrac{a_n}{a_{n-1}}\right| |x-2|$

$= \left|\dfrac{1}{n!}(n-1)!\right| |x-2| = \dfrac{1}{n}|x-2|$

Then $\underset{n\to\infty}{\text{Lim}}\left|\dfrac{u_{n+1}}{u_n}\right|$ exists and is equal to

$\left(\underset{n\to\infty}{\text{Lim}}\dfrac{1}{n}\right)|x-2| \quad = 0 \cdot |x-2|$

$= 0 < 1.$ for all x.

So the given power series is convergent for all real values of x and hence the radius of convergence of the power series is ∞.

Example – 2 : ***Find the radius of convergence of the power series*** $\sum_{n=0}^{\infty}(-1)^n \cdot \dfrac{x^{2n}}{2^n}$.

[B.P.U.T. 2004

Solution : The power series is $1 - \dfrac{x^2}{2} + \dfrac{x^4}{2^2} + \ldots\ldots + \dfrac{(-1)^n}{2^n}x^{2n} + \ldots\ldots$

Let $u_n = \dfrac{(-1)^n \cdot x^{2n}}{2^n}$, $n = 0, 1, 2, \ldots\ldots$

Then $u_{n+1} = \dfrac{(-1)^{n+1} \cdot x^{2(n+1)}}{2^{n+1}}$

so $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{-1 \cdot x^2}{2}\right| = \dfrac{1}{2}x^2$

$\underset{n\to\infty}{\text{Lim}}\left|\dfrac{u_{n+1}}{u_n}\right| = \dfrac{1}{2}x^2$

Then the given power series is convergent if $\dfrac{1}{2}x^2 < 1$ and non-convergent for $\dfrac{1}{2}x^2 > 1$.

Now $\dfrac{1}{2}x^2 < 1 \quad \Leftrightarrow x^2 < 2$

$\Leftrightarrow |x| < \sqrt{2}$

So the required radius of convergence is $\sqrt{2}$.

Example – 3 : ***Find the radius of convergence of*** $\sum_{m=0}^{\infty}\frac{x^{2m}}{m!}$.

Solution : $\sum_{m=0}^{\infty}\frac{x^{2m}}{m!}$

Here $a_{n+1} = \frac{x^{2n}}{n!}$ $\qquad a_n = \frac{x^{2n-2}}{(n-1)!}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n}}{n!}\times\frac{(n-1)!}{x^{2n-2}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{n}\right| = 0 < 1$$

we know the radius of convergence $R^* = 0$

$$\Rightarrow R = \frac{1}{R^*} = \infty$$

Hence the radius of convergence $R = \infty$.

Example – 4: ***Find the radius of convergence of*** $\sum_{m=0}^{\infty}\frac{1}{3^m}(x-3)^{2m}$

Solution : $\sum_{m=0}^{\infty}\frac{1}{3^m}(x-3)^{2m}$

Here $a_{n+1} = \frac{1}{3^n}(x-3)^{2n}$

$a_n = \frac{1}{3^{n-1}}(x-3)^{2n-2}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{3_n}(x-3)^{2n}\times\frac{3^{n-1}}{(x-3)^{2n-2}}\right| = \lim_{n\to\infty}\left|\frac{1}{3}(x-3)^2\right| = \frac{1}{3}\left|(x-3)^2\right|$$

we know for convergency $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

$$\Rightarrow \frac{1}{3}\left|(x-3)^2\right| < 1 \Rightarrow \left|(x-3)^2\right| < 3$$

$$\Rightarrow \left|(x-3)^2\right| < \sqrt{3} \Rightarrow 3-\sqrt{3} < x < 3+\sqrt{3}$$

Hence the radius of convergence is $R = \sqrt{3}$.(Ans)

Example –5 : ***Find the radius of convergence of*** $\sum_{m=0}^{\infty}\frac{x^{2m+1}}{(2m+1)!}$

Solution : $\sum_{m=0}^{\infty}\frac{(-1)^m}{k^m}.x^{2m}$

Here $a_{n+1} = \frac{(-1)^n}{k^n}\cdot x^{2n}$ $\qquad a_n = \frac{(-1)^{n-1}}{k^{n-1}}\cdot x^{2n-2}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \lim_{n\to\infty}\left|\frac{(-1)^2}{k^2}.x^{2n}\,\frac{k^{n-1}}{(-1)^{n-1}.x^{2n-1}}\right| \quad = \lim_{n\to\infty}\left|\frac{-1}{k}x^2\right| = \frac{1}{k}\left|x^2\right|$$

we know for convergency $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

$$\Rightarrow \frac{1}{k}\left|x^2\right|<1 \;\Rightarrow \left|x^2\right|<\text{k} \;\Rightarrow \left|x^2\right|<\text{k} \;\;\Rightarrow \left|x^2\right|<\sqrt{|k|}$$

Hence the radius of convergence is $R=\sqrt{|k|}$.

Example – 6 : ***Find the radius of convergence of*** $\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^m .x^{2m}$

Solution : $\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^m .x^{2m}$

Here $a_{n+1} = \left(\frac{2}{3}\right)^n .x^{2n}$

$a_n = \left(\frac{2}{3}\right)^{n-1} .x^{2n-1}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{2}{3}\right)^n .x^{2n}}{\left(\frac{2}{3}\right)^{n-1} .x^{2n-2}}\right| = \lim_{n\to\infty}\left|\frac{2}{3}x^2\right| = \frac{2}{3}\left|x^2\right|$$

we know for convergency, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \;\;\Rightarrow \frac{2}{3}\left|x^2\right| < 1 \;\Rightarrow \frac{2}{3}\left|x^2\right| < 1$

$$\Rightarrow \frac{2}{3}\left|x^2\right| < \frac{3}{2} \Rightarrow |x| < \sqrt{\frac{3}{2}} \;\Rightarrow -\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$$

Hence the radius of convergence is $R=\sqrt{\frac{3}{2}}$

Example – 7 : ***Find a power series solution of the differential equation*** $\frac{dy}{dx} = -y.$

[B.P.U.T. 2005, B.P.U.T. 2007

Solution : The given equation is $\frac{dy}{dx} + y \quad = 0 \quad$ ..... (1)

Here $x = 0$ is an ordinary point of the differential equation. So we seek a solution of form.

$$y = a_0 + a_1 x + a_2 x^2 + \ldots.. + a_n x^n + \ldots.. \qquad \ldots. (2)$$

Then $\dfrac{dy}{dx} = a_1 + 2a_2\,x + ... + na_n\,x^{n-1} + ...$ .... (3)

Substituting (2) and (3) in (1) we get

$(a_0 + a_1) + (a_1 + 2a_2)\,x + (a_2 + 3a_3)\,x^2 + ...\ [a_n + (n+1)\,a_{n+1}]\,x^n + ... = 0$ .... (4)

In order that (2) may be a formal solution of s1), (4) must be an identity in x and so the coefficients of successive power of x can be equated to zero. Thus we get

$$a_0 + a_1 = 0$$
$$a_1 + 2a_2 = 0$$
$$a_2 + 3a_3 = 0$$
$$a_n + (n+1)\,a_{n-1} = 0$$

From the above relations we get

$$a_1 = -a_0,\ a_2 = -\frac{a_1}{2} = \frac{1}{2}a_0 = (-1)^2\,\frac{a_0}{2!}$$

$$a_3 = -\frac{1}{3}a_2 = -\frac{1}{2\cdot 3}a_0, = (-1)^3\cdot\frac{1}{3!}a_0$$

$$a_4 = -\frac{1}{4}a_3 = \frac{1}{2\cdot 3\cdot 4}a_0 = (-1)^4\cdot\frac{1}{4!}a_0$$

$$a_{n+1} = (-1)^{n+1}\,\frac{1}{(n+1)!}a_0$$

Then $y = a_0 - a_0\,x + a_0\,\dfrac{x^2}{2!} - a_0\,\dfrac{x^3}{3!} + ... + (-1)^{n+1}\cdot a_0\,\dfrac{x^{n+1}}{(n+1)!} + ...$

or $y = a_0\left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + ...\right)$ .... (5)

We have shown formally that (5) satisfies the given differential equation where the constant a_0 can be taken arbitrarily. So the required power series solution is given by (5).

Now we know that the series $1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + ...$ is convergent for all values of x and it represents the exponential function e^{-x}.

Hence the required solution is $y = a_0\,e^{-x}$, where a_0 is arbitrary constant.

Note : The above solution can be easily obtained by the method of separation of variables.

Example – 8 : ***Find a power series solution of y″ + y = 0, given that y (0) = 0. [B.P.U.T. 2012***

Solution : Here $x = 0$ is an ordinary point of the given differential equation. For a power series solution we assume $y = a_0 + a_1\,x + a_2\,x^2 + ...$ .... (1)

Then $\dfrac{dy}{dx} = a_1 + 2a_2\, x + 3a_3\, x^2 + \ldots\ldots..$ .... (2)

and $\dfrac{d^2y}{dx^2} = 2a_2 + 3 \cdot 2\, a_3\, x + 4{\cdot}3\, a_4\, x^2 \ldots\ldots$ .... (3)

Substituting (2) and (3) in the given differential equation we get

$$(2a_2 + 3{\cdot}2\, a_3\, x + 4{\cdot}3\, a_4\, x^2 + \ldots.) + (a_0 + a_1\, x + a_2\, x^2 + \ldots.) = 0$$

or, $(2a_2 + a_0) + (3{\cdot}2\, a_3 + a_1)\, x + (4{\cdot}3\, a_4 + a_2)\, x^2 + \ldots\ldots = 0$ .... (4)

In order that (1) may be a formal solution of (1), (4) must be an identity in x and so the coefficients of the successive powers of x can be equated to zero. Thus we get

$$\begin{aligned} 2a_2 + a_0 &= 0 \\ 3{\cdot}2\, a_3 + a_1 &= 0 \\ 4{\cdot}3\, a_4 + a_2 &= 0 \\ (n+1) \cdot n\, a_{n+1} + a_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

From above we get $a_2 = -\dfrac{a_0}{2}, a_3 = -\dfrac{a_1}{3{\cdot}2},$

$$a_4 = -\frac{1}{4{\cdot}3} a_2 = (-1)^2 \cdot \frac{1}{4{\cdot}32} a_0,$$

$$a_5 = -\frac{1}{5{\cdot}4} a_3 = (-1)^2 \cdot \frac{1}{5{\cdot}4{\cdot}3{\cdot}2} a_1$$

and so on.

[Here we observe that a_2, a_4, a_6, ... are expressed in terms of a_0 while a_3, a_5, ... are expresed in terms of a_1].

Hence $y = a_0 + a_1 x - \dfrac{a_0}{\lfloor 2} x^2 - \dfrac{a_1}{\lfloor 3} x^3 + \dfrac{a_0}{\lfloor 4} x^4 + \dfrac{a_1}{\lfloor 5} x^5 - \ldots\ldots\ldots\ldots\ldots.$

or, $y = a_0 \left(1 - \dfrac{x^2}{\lfloor 2} + \dfrac{x^4}{\lfloor 4} - \ldots\ldots..\right) + a_1 \left(x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \ldots\ldots..\right)$ .... (5)

where the values of the constants a_0, a_1 can be taken arbitrarily. We have shown formally that (5) satisfies the given differential equation. Now it can be shown that the power series

$$1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots\ldots\ldots\ldots.$$

and $x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} - \ldots\ldots\ldots\ldots$

are convergent for all values of x.

For the first series $a_0 = 1$, $a_{2n} = (-1)^n \cdot \dfrac{1}{\lfloor 2n}$, $a_{2n-1} = 0$ for $n \geq 1$.

Now $\lim\limits_{n\to\infty} \left| \dfrac{a_{2n+2}\ x^{2n+2}}{a_{2n}\ x^{2n}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{2n+2}}{(-1)^{2n}} \cdot \frac{\lfloor 2n}{\lfloor 2n+2} x^2 \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)\,(2n+1)} \right|$$

$= 0 < 1$, for all x.

So the series $1 - \dfrac{x^2}{\lfloor 2} + \dfrac{x^4}{\lfloor 4} - \ldots\ldots\ldots$ is convergent for all x. Similarly it can be shown that the series $x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} \ldots\ldots\ldots$ is convergent for all x.

So (5) gives the power series solution of the given differential equation.
Now it is given that $y(0) = 0$ Then from (5) we get $0 = a_0$.
So the required power series solution is

$$y = a_1 \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots\ldots\ldots \right)$$

where (a_1 is arbitrary constant) valid for all x.

Note : Here we note that

$$x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots\ldots\ldots\ldots \qquad \text{and}$$

$$1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots\ldots\ldots\ldots$$

represent respectively $\sin x$ and $\cos x$.

So the required solution in the closed form is given by $y = a_1 \sin x$ which can be obtained easily by the method of solving ordinary linear differential equations with constant co-efficients.

Example – 5 : ***Find a power series solution in powers of x of the following differential equation $(1 - x^2)\, y'' = 2xy$.*** ***[B.P.U.T. 2012,2011***

Solution : The co-efficient of y'' is $1 - x^2$ which is a polynomial in x and $1 - x^2 \neq 0$ for $x = 0$. So $x = 0$ is an ordinary point of the given differential equation. Then for a power series solution of the given differential equation we assume.

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \qquad \ldots (1)$$

Then $y' = \dfrac{dy}{dx} = a_1 + 2a_2\, x + ... + n\, a_n\, x^{n-1} + ...$ .... (2)

and $y'' = \dfrac{d^2y}{dx^2} = 2a_2 + 3{\cdot}2a_3\, x + ... + n\,(n-1)\, a_n\, x^{n-2} + ...$ .... (3)

Substituting (1), (2), (3) in the given differential equation.

$$(1-x^2)\frac{d^2y}{dx^2} - 2xy = 0, \text{ we get}$$

$[2a_2 + 3{\cdot}2\, a_3\, x + ... + n\,(n-1)\, a_n\, x^{n-2} + ...]\,(1-x^2) - 2x\,(a_0 + a_1\, x + a_2\, x^2 + ...) = 0$

or, $2a_2 + 3{\cdot}2\, a_3\, x + ... + n\,(n-1)\, a_n\, x^{n-2} + ...$
$- 2a_2\, x^2 - 3{\cdot}2\, a_3\, x^3\; .... - n\,(n-1)\, a_n\, x^{n+1}\; ....$
$-(2a_0\, x + 2a_1\, x^2 + 2a_2\, x^3 + ...) = 0$

or, $(1-x^2)\sum\limits_{n=2}^{\infty} n\,(n-1)\, a_n x^{n-2} - 2x\sum\limits_{n=0}^{\infty} a_n x^n = 0$

or, $\sum\limits_{n=2}^{\infty} n\,(n-1)\, a_n x^{n-2} - \sum\limits_{n=2}^{\infty} n\,(n-1)\, a_n x^n - 2\sum\limits_{n=0}^{\infty} a_n\, x^{n+1} = 0$ .... (4)

Now (4) being an identity in x, successive powers of x are equated to zero and we get

$2a_2 = 0$
$3{\cdot}2\, a_3 - 2{\cdot}a_0 = 0$
$4{\cdot}3\, a_4 - 2{\cdot}1\, a_2 - 2\, a_1 = 0$
....

$$n\,(n-1)\, a_n - (n-2)\,(n-3)\, a_{n-2} - 2\, a_{n-3} = 0,\ n \geq 3.$$

[The last relation is obtained equating the coefficient of x^{n-2} ($n \geq 3$) to zero]

From the above relations we get

$a_2 = 0,\ a_3 = \dfrac{2}{3{\cdot}2}a_0,\ \ a_4 = \dfrac{2}{4{\cdot}3}a_1,$

$5{\cdot}4\, a_5 - 3{\cdot}2\, a_3 - 2a_2 = 0$

i.e., $5{\cdot}4\, a_5 = 3{\cdot}2\, a_3 (\because\ a_2 = 0)$

or, $a_5 = \dfrac{3{\cdot}2}{5{\cdot}4}a_3 = \dfrac{3{\cdot}2}{5{\cdot}4}\cdot\dfrac{2}{3{\cdot}2}\, a_0$

and so on.

So the formal solution is

$$y = a_0 + a_1\, x + \frac{1}{3}\, a_0\, x^3 + \frac{2}{4{\cdot}3}\, a_1\, x^4 + \frac{3{\cdot}2}{5{\cdot}4}\cdot\frac{2}{3{\cdot}2}\, a_0 x^5 + ...$$

or, $y = a_0\left(1+\frac{1}{3}x^3+\frac{2}{5\cdot 4}x^5+........\right)+a_1\left(x+\frac{2}{4\cdot 3}x^4+.........\right)$,

where a_0, a_1 are arbitrary constants. The power series

$$1+\frac{1}{3}x^3+\frac{2}{5\cdot 4}x^5+....., \; x+\frac{2}{4\cdot 3}x^4+......$$

are convergent within the respective intervals of convergenee say ($-R_1$, R_1).and ($-R_2$, R_2).
So if min = $\{R_1, R_2\}$ = R min $\{R_1, R_2\}$
the required power series solution is

$$y = a_0\left(1+\frac{1}{3}x^3+\frac{2}{5\cdot 4}x^5+......\right)+a_1\left(x+\frac{2}{4\cdot 3}x^4+........\right),$$

valid for $|x| < R$.

Example – 6 : ***Solve the equation*** $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y$ ***in power series, near the ordinary point x = 0.***

Solution : Let $y = \sum_{n=0}^{\infty} a_n x^n$ .... (1)

be solution of the given differential equation.

Then $\frac{dy}{dx} = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$ .... (2)

$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2}$ .... (3)

Substituting (1), (2), (3) in the given differential equation we get

$$(1-x^2)\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - x\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} + 4\sum_{n=0}^{\infty} a_n\,x^n = 0$$

or, $\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n - \sum_{n=1}^{\infty} n\,a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$ .... (4)

Now (4) being an identity in x, successive powers of x can be equated to zero.
Equating the coefficient of x^{n-2} $(n \geq 2)$ to zero we find that

$n\,(n-1)\,a_n - (n-2)\,(n-3)\,a_{n-2} - (n-2)\,a_{n-2} + 4a_{n-2} = 0$, for $n \geq 2$.

or, $n\,(n-1)\,a_n - [(n-2)\,(n-3) + (n-2) - 4]\,a_{n-2} = 0$

or, $n\,(n-1)\,a_n - (n^2 - 5n + 6 + n - 2 - 4)\,a_{n-2} = 0$

or, $n\,(n-1)\,a_n - (n^2 - 4n)\,a_{n-2} = 0$

or, $(n-1)\,a_n - (n-4)\,a_{n-2} = 0$ for $n \geq 2$.

or, $a_n = \dfrac{n-4}{n-1}a_{n-2}$, for $n \geq 2$.

So, $a_2 = -\dfrac{2}{1}a_0,\ a_3 = -\dfrac{1}{2}a_1,$

$a_4 = 0, \qquad a_5 = \dfrac{1}{4}a_3 = \dfrac{1}{4}\cdot\left(-\dfrac{1}{2}\right)a_1$

$a_6 = \dfrac{2}{5}\cdot a_4 = 0,\ a_7 = \dfrac{3}{6}\cdot a_5 = \dfrac{3}{6}\cdot\dfrac{1}{4}\cdot\left(-\dfrac{1}{2}\right)a_1$

$a_8 = \dfrac{4}{7}\cdot a_6 = 0,\ a_9 = \dfrac{5}{8}a_7 = \dfrac{5}{8}\cdot\dfrac{3}{6}\cdot\dfrac{1}{4}\cdot\left(-\dfrac{1}{2}\right)a_1$

and so on.

So the formal solution is

$$y = a_0 + a_1\,x + \left(-\frac{2}{1}\right)a_0\,x^2 - \frac{1}{2}\,a_1\,x^3 + 6\,x^4 + a_1\,\frac{1}{4}\left(-\frac{1}{2}\right)x^5$$
$$+\ 0x^6 + a_1\frac{3}{6}\cdot\frac{1}{4}\cdot\left(-\frac{1}{2}\right)x^7 + 0x^8 + a_1\cdot\frac{5}{8}\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\left(-\frac{1}{2}\right)x^9 + \ldots\ldots\ldots.$$

or, $$y = a_0(1-2x^2) + a_1\left[x - \frac{1}{2}x^3 - \frac{1}{4}\cdot\frac{1}{2}x^5 - \frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}x^7 - \frac{5}{8}\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}x^9 - \ldots\ldots\right]$$

It can be shown that the power series

$$x - \frac{1}{2}x^3 - \frac{1}{4}\cdot\frac{1}{2}x^5 - \frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}x^7\ldots\ldots..$$

is convergent for $|x|<|$ and non-convergent for $|x|>|$.

Hence the required series solution is

$$y = a_0\,(1-2x^2) + a_1\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 - \frac{1}{16}x^7 - \frac{5}{128}x^9 - \ldots\ldots\right)$$

valid for $|x|<|$.

Exercise – 7.1

1. What is the radius of convergence of the power series $\sum_{n=0}^{\infty}\dfrac{x^n}{(n+1)^a}$, $(a \in \mathbb{R})$?

[B.P.U.T. – 2003

2. Find the radius of convergence of the power series.

(*a*) $x + \dfrac{2^2\,x^2}{2!} + \dfrac{3^3\,x^3}{3!} + \ldots\ldots$

(*b*) $\sum_{n=0}^{\infty} a_n x^n$ where $a_0 = 0$, $a_1 = 1$ and $a_n = \frac{(n!)^2}{(2n)!}, n \geq 2$

(*c*) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n+2)} (x-2)^n$

(*d*) $1 + 3x + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \ldots\ldots..$

(*e*) $1 + 3x + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \ldots\ldots.$

(*f*) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} (x+1)^{n+1}$

(*g*) $1 + \frac{1}{2} x + \frac{1 \cdot 3}{2 \cdot 4} x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^3 + \ldots\ldots.$

3. Solve the equation $\frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 3y = 0$ by power series method.

4. Show that $x = 1$ is a regular singularity of the equation $(x-1)^2 \frac{d^2 y}{dx^2} + 2x(x-1) \frac{dy}{dx} + 3$ $(x + 1) y = 0$; but $x = 0$ is not a regular singularity of the equation $x^3 \frac{d^2 y}{dx^2} + x^4 \frac{dy}{dx} + y = 0$.

5. Show that the equation $2\frac{d^2 y}{dx^2} + 5x\frac{dy}{dx} + 2y = 0$ has no singular point while $x = 0$ is an ordinary point of the equation $\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + 2y = 0$.

6. Locate and classify the singular points of the equations :

(*a*) $x^3(x-1) \frac{d^2 y}{dx^2} - 3(x-1) \frac{dy}{dx} + 5xy = 0$

(*b*) $x^2(x^2 - 4) \frac{d^2 y}{dx^2} + 3x^3 \frac{dy}{dx} + 4y = 0$

(*c*) $(3x+1) x \frac{d^2 y}{dx^2} - (x+1) \frac{dy}{dx} + 3y = 0$

(*d*) $x^2 \frac{d^2 y}{dx^2} + y \sin x = 0.$

7. Solve the equation $\frac{d^2 y}{dx^2} + x^2 \frac{dy}{dx} - 4xy = o$ in series near the point $x = o$

8. Solve $\frac{d^2 y}{dx^2} + (x-2) y = 0$ by power series method about $x = 2$.

$$\left[\text{Hint : Assume } y = \sum_{n=0}^{\infty} a_n (x-2)^n\right]$$

9. Solve the equation $x(1-x)\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$ about the regular singularity $x = 1$.

10. Apply Frobenius method to solve $(x + 1)\, y' - (x + 2)\, y = 0$ about regular singularity -1.

[$x = a$ is called a regular singularity of $\dfrac{dy}{dx}$ + P $(x)\, y = 0$ if $(x - a)$ P (x) is analytic at $x = a$.

Here P $(x) = -\dfrac{x+2}{x+1}$. Then $(x + 1)$ P $(x) = -(x + 2)$ which is analytic at $x = -1$. So -1 is a regular singularity of $(x + 1\, y' - (x + 2)\, y = 0$]

11. Solve $\dfrac{dy}{dx} = -2xy$ by power series method. ***[B.P.U.T. – 2004***

Answers

1. 1 ; **2.** (*a*) $\dfrac{1}{e}$; (*b*) 4 ; (*c*) 1 (*d*) $\dfrac{1}{3}$; (*e*) $\dfrac{1}{2}$; (*f*) 1 ;

3. $y = a_0\left[1 + \sum_{k=1}^{\infty} \dfrac{(-3)^k\, x^{2k}}{2^k \cdot k!}\right] + a_1\left[x + \sum_{k=1}^{\infty} \dfrac{(-3)^k\, 2^k \cdot k!\; x^{2k+1}}{(2k+1)!}\right]$.

6. (a) 1 is a regular singularity, 0 is an irregular singularity and no other singularity ;
(b) 0, 2, –2 are regular singularities and no other singularity.
(c) 0, $-\dfrac{1}{3}$ are regular singularities and no other singularity ;
(d) $x = 0$ is a regular singularity and no other singularity ;

7. $y = a_0\left[1 + \sum_{n=1}^{\infty} \dfrac{4(-1)^n \cdot x^{3n}}{3^n \cdot n!\,(3n-1)(3n-4)}\right] + a_1\left(x + \dfrac{1}{4}x^4\right)$

8. $y = a_0\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n (x-2)^{3n}}{3^n\, n!\, 2\cdot5\cdot8 \ldots\ldots (3n-1)}\right] + a_1\left[(x-2) + \sum_{n=1}^{\infty} \dfrac{(-1)^n (x-2)^{3n+1}}{3^n \cdot n!\; 4\cdot7\cdot10 \ldots\ldots (3n+1)}\right]$

9. $y = a_0\,[(x-1)^{-2} + 4(x-1)^{-1}] + a_2\left[1 + \dfrac{2}{3}(x-1) + \dfrac{1}{6}(x-1)^2\right]$;

10. $y = c\,(x+1)\left[1 + (x+1) + \dfrac{(x+1)^2}{2!} + \dfrac{(x+1)^3}{3!} + \ldots\ldots\right]$ where c is arbitrary constant.

11. $y = a_0\left(1 - x^2 + \dfrac{x^4}{2!} + \ldots\ldots\right)$, where a_0 is arbitrary constant.