5

CHAPTER

Ordinary Differential Equations and Preliminary Notions

5.0 : Introduction

After the discovery of calculus, Newton and Leibnitz studied the differential equations in connection with problems in Physics, especially in the theory of bending of beams, oscillations of mechanical systems and of electric currents, conduction of heat, etc. In Chemistry we come across velocity of chemical reaction and diffusions of solvents etc. These days they are being used in economics also. John Bernoulli and Euler also made their contributions to the theory of differential equations but the name of Leibnitz is very much associated with them. In this chapter we shall present the basic concepts and ideas of differential equations such as their definitions, formation of such equations and solving such equations by the simplest method i.e. by direct integration and discuss their basic physical applications.

5.1 : Significance of Ordinary Differential Equations

An equation involving atleast one derivative and the variables explicitely and implicitely, is called a differential equation. An ordinary differential equation is one into which all the derivatives have reference to only a single variable.

From the history of differential equations we encounter many ordinary differential equation arising from problems specially in geometry and physical science.

Let us cousider some familiar geometric and physical problems that could be expresed in terms of ordinary differential equation.

(*i*) Let us suppose that a curve with cartesion equatian $y = f(x)$ on a plane is such that at any point P(x,y) on the curve, the slope to its tangent is twice the abscissa of the point (x,y). The geometrical statement can be expressed as

$$\frac{dy}{dx} = 2x$$

(*ii*) Let $y = f(x)$ be the equation of a given curve on which we take a point P (x,y). If the radius of curvature of the curve at P varies as y, than we express the situation as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = ky\,\frac{d^2y}{dx^2}$$

$$i.e.,\quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3} = k^2y^2\left(\frac{d^2y}{dx^2}\right)^2$$

where k is a constant and $\frac{d^2y}{dx^2} \neq 0$

(*iii*) Next, let us cousider few physical problems of importance.
Let N be the number of atoms of a given of radioactive decay is proportional to N at an instant t.

$$\therefore \quad \frac{dN}{dt} \infty \mathrm{N}$$

which gives $\frac{dN}{dt} = -\lambda \mathrm{N}$

where λ, the proportionaliyt constant, is known as the decay constant, or disintegration constant that measures the time rate of decay per atom.

(*iv*) Let a particle of mass m, free to move along x-axis [Fig. 1] be attracted towards the origin with a force proportional to its displacement x and in the oposite direction, at an instant t. Then the equation of motion of the particle, by using Newton's second law of motion, can be expressed as

$$m\,\frac{d^2x}{dt^2} = -kx$$

where k, the proportionality constant, is called stiffness or spring constant.

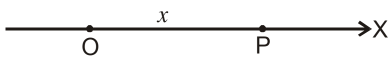

Fig. 1

[$\overrightarrow{\mathrm{OX}}$ gives the positive direction of the x-axis]

5.2 : Order and Degree of an Ordinary Differential Equation

The order of an ordinary differntial equation is the order of the highest derivative that appears in the equation. The order of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + 4y = x \qquad ....(1)$$

is 2, since the order of highest derivative $\left(i.e., \frac{d^2y}{dx^2}\right)$ ocurring in the equation (1) is 2.

A *nth*-order differential equation has the general form

$$F\left(a, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^n y}{dx^n}\right) = 0 \qquad \ldots(2)$$

where x is the indipendent variable, the dependent variable $y = y(x)$ is a function of x and F is a specified function of $(x + 2)$ variable $x, y, \ldots, \frac{d^{n-1}y}{dx^{n-1}}$ and $\frac{d^n y}{dx^n}$.

The degree of an ordinary differential equation is the degree (or positive integral power) of the highest order derivative appearing in the equation after the equation has been made frce of radicals involving derivtives and all fractional indics of the derivatives in the equation, *i.e.,* by the expressing the differential equation as a polynomial in derivatives.

Example – 1 : Find the order and degree of the following differential equations :

(i) $\frac{dy}{dx} + xy = 1$ **(ii)** $(xy^2 + x)\,dx + (y - x^2y)\,dy = 0$

(iii) $\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$ **(iv)** $\left(1+\left(\frac{dy}{dx}\right)^2\right)^{5/2} = 3\left(\frac{d^2y}{dx^2}\right)$

Solution :

(i) The given equation is $\frac{dy}{dx} + xy = 1$.

The order of the differential equation is 1 and its degree is also 1.

(ii) The given equation is

$(xy^2 + x)\,dx + (y - x^2y)\,dy = 0$ or $(xy^2 + x) + (y - x^2y)\frac{dy}{dx} = 0$

Its order is 1 and degree is also 1.

(iii) The given equation is

$\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$, or $\sqrt{1-y^2} + y\sqrt{1-x^2}\frac{dy}{dx} = 0$

Its order is 1 and degree is 1.

(iv) The given equation is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 3\left(\frac{d^2y}{dx^2}\right)$$

squaring, to make the exponents non-fractional.

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = 9\left(\frac{d^2y}{dx^2}\right)^2$$

Its order is 2 is the exponent of the highest order derivative is 2, the degree of the given differential equation is 2.

Example –2 : The order and degree of the differential equation

$$\left(\frac{dy}{dx}\right)^{\frac{2}{3}} = 1 + \frac{d^2y}{dx^2}$$

are 2 and 3 respectively. For,the given equation can be wrilten as

$$\left(\frac{dy}{dx}\right)^{2} = \left(1+\frac{d^2y}{dx^2}\right)^{3}$$

Note : It is not possible to assign **degree** to every **ordinary differential equation.** for example, the degree of the differential equation

$$\text{Sin}\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\right)+e^{\frac{dy}{dx}}-xy=0$$

cannot be defined since $e^{\frac{dy}{dx}}$, sin $\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\right)$ cannot be expressed as polynomials in derivatives.

5.3 : Solution of Ordinary Differential Equation

A continuous function $y = \phi\ (x)$ is called a solution of the ordinary differential equation

$$\text{F}\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^ny}{dx^n}\right) = 0$$

on an open intirval I $\subseteq$ R if $\frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \ldots.., \frac{d^{n-1}\phi}{dx^n}$ exist on I and if

$$\text{F}\left(x, y, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \ldots, \frac{d^{n-1}\phi}{dx^{n-1}}, \frac{d^n\phi}{dx^n}\right) = 0 \text{ for all } x \text{ in I.}$$

In otherwords, we may say that the function $y = \phi\,(x)$ satisfies the above differential equation on I. A solution is also called an integral of the equation.

For example, the function $y = x^3$ is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 \qquad \text{... (1)}$$

in the interveal $-\infty < x < \infty$ because $\frac{d}{dx}(x)^3 = 3x^2$. The function $y = e^{-2x}$ $(-\infty < x < \infty)$ is a solution of the equation

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+ 6y = 0 \qquad \text{...(2)}$$

because $\frac{dy}{dx} = -2\,e^{-2x}$ and $\frac{d^2y}{dx^2} = 4\,e^{-2x}$ and so

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 4e^{-2x}\ 10e^{-2x} + 6e^{-2x} = 0.$$

The solution of a n^{th} order ordinary differential equation containing n independent arbitrary constants is called the **general solution** of the equation. Thus $y = 3x^2 + c$, where c is arbitrary constant, is the general solution of the equation (3). We see that the function $y = c_1\,e^{-2x} + c_2e^{-3x}$, where c_1 and c_z are independent arbitrary constants, is a solution of (2). The solution contains two arbitrary constants. It follows that $y = c_1e^{-2x} + c_2\,e^{-3x}$ is the general solution of (2).

In general solution of an ordinary diffential equation when particular values are specified for arbitrary constants, the specific solution obtained is known as **particular solution.** Thus assigning $c_1 = 4$ and $c_2 = 5$ we get $y = 4e^{-2x} + 5e^{-3x}$ is a particular solution of the equation (2).

In case of some ordinary differential equations, there may exist a solution which is neither general nor a particular one. Such a solution is called a **Singular solution.**

We consider the differential equation

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + 4 = 0 \quad \ldots (3)$$

The Solution $y = cx + \frac{4}{c}$, where c is an arbitrary coustant, is the general solution of (3).

Any choice of c gives a particular solution of (3). The function $y = 4\sqrt{x}$ is also solution, because

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + 4 = x \times \frac{4}{x} - 4\sqrt{x} \times \frac{2}{\sqrt{x}} + 4$$
$$= 4 - 8 + 4 = 0$$

But the solution $y = 4\sqrt{x}$ cannot be obtained from the general solution $y = cx + \frac{4}{c}$ specifying any value to the arobitrary constant c. So $y = 4\sqrt{x}$ is the singular solution of (3).

We have already seen that the *number of arbitrary constants present in the general solution of a differential equation is equal to the order of the equation. The values of the arbitrary constants can be determined if the values of the dependent variable and its derivatives are known for a particular value of the independent variable.*

The condition under which the values of the arbitrary constants can be determined is called **intial condition** and the problem of finding a particular solution of a diffential equation under given intial conditions is called **initial value problem.**

For an example, $y = \text{A} \sin x + \text{B} \cos x$ is a solution of the equation

$$\frac{d^2 y}{dx^2} + y = 0 \qquad \text{... (4)}$$

Here A, B are arbitrary constants. Suppose it is given that $y = 0$ and $\frac{dy}{dx} = 3$, when $x = 0$.

From (4) $\frac{dy}{dx} = \text{A} \cos x - \text{B} \sin x$

Putting $y = 0$, $\frac{dy}{dx} = 3$ and $x = 0$, we get

$$0 = \text{A} \sin 0 + \text{B} \cos 0$$

and $3 = \text{A} \cos 0 - \text{B} \sin 0$, whence we get A = 3, B = 0

Hence the solution of the given differential equation (4) under the given initial conditions is $y = 3 \sin x$.

5.4 : Formation of ordinary Differential Equation by Elimination of Arbitrary Constants

It has already been mentioned in section (5.1) of this chapter that certain geometrical problems lead to to differential equation.

Conversely, from algebraic equations of a family of curves one can form differential equations. As an illustration, let us consider the family of circles with centre at the origin. This family of circles can be represented by the equation,

$$x^2 + y^2 = a^2 \ (a > 0) \ \text{...(1)}$$

Each Positive value of '*a*' will give a circle of radius a, *i.e.,* a particular mamber of the family. By varying a we shall get different members of the family of varying radius, but all of them have their centres at the origin. Thus the equation (1) represents a family of circles, with centre at the origin, '*a*' being called a **parameter.**

Differentiating (1) with respect to *x*, we get

$$y \frac{dy}{dx} + x = 0 \ \text{... (2)}$$

as the differential equation of the family of circles given by (1). Here the single parameter '*a*' has been eliminated by differentiating the given equation once. But when the equation of the family of curves involves more than one parameter, independent of each other, only one differentiation is not sufficient to eliminate the parameters. So elucidate this point, let us consider the family of parabolas, whose vertices lie on the *x*-axis which is the axis of the parabolas. Here, the parabolas may have letera recta of varying lengths, while the vertices may have any co-ordinate of the form $(\alpha, 0)$, where α varies. The equation to the family of parabolas, an indicated above, is given by

$$y^2 = 4a(x - \alpha) \ \text{... (3)}$$

where $a\ (> 0)$ and $\alpha\ (\neq 0)$ are two independent parameters.

Differentiating (3) with respeet to $x,$ we get

$$y \frac{dy}{dx} = 2a \quad \ldots (4)$$

The relation (4) still involves parameter a. Differentiating (4) again with respect to $x,$ we get

$$y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \ldots (5)$$

which is the required differential rquation of the family of curves given by (3).

Here we note that the number of independent arbitrary constants in (3) is two and we get a second-order differential equation by differentiating the given relation twice. In general, for an n-parameter family of curves, we are to differentiate the given relation n-times to eliminate the n mutually independent parameters and get a different equation of order n. We illustrate the above fact through the following examples.

Example – 1 : ***Obtain the differential equation by eliminating the arbitrary constant λ from the equation $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda}$ = 1 where a and b are fixed constants.***

Solution : The given equation is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1. \qquad \ldots (6)$$

Differentiating (6) with respect to x, we get

$$\frac{2x}{a^2+\lambda} + \frac{2y\frac{dy}{dx}}{b^2+\lambda} = 0. \qquad \ldots (7)$$

From (7), we get $\frac{a^2+\lambda}{x} + \frac{b^2+\lambda}{-yy'}$ = K (say), where $y' = \frac{dy}{dx}$.

So, form (6) it follows $\frac{x^2}{kx} - \frac{y^2}{kyy'} = 1$

or, $\quad$ K = $(xy' - y) / y'$

This gives $a^2 + \lambda = \mathrm{K}x = \frac{xy' - y}{y'}\, x$

and $b^2 + \lambda = -\mathrm{K}yy' = \frac{xy' - y}{y'}\,(-yy')$

Eliminating λ from above two relations, we get

$$a^2 - b^2 = \frac{(xy'-y)x}{y'} + (xy' - y)\, y$$

or, $\quad (a^2 - b^2)\, y' = (xy' - y)\,(x + yy')$

which is the required differential equation.

Example – 2 : ***Find the differential equation of all parabolas having their axies parallel to the y-axies represented by the equation $y = Ax^2 + B\,x + C$ where $A\ (\neq 0)$, B, C are arbitrary constants.***

Solution : The differential equation of such a family of parabolas will be obtained by eliminating A,B,C from the equation

$$y \quad = Ax^2 + Bx + C \quad \dots (8)$$

From (8), differentiating suceessively with respect to x, we have

$$\frac{dy}{dx} = 2Ax + B, \quad \frac{d^2y}{dx^2} = 2A \text{ and } \frac{d^3y}{dx^3} = 0.$$

Hence the required differential equation is $\frac{d^3y}{dx^3} = 0.$

Illustrative Examples

Example – 1: ***Find the order and degree of the following differential equations :***

(*i*) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y$ (*ii*) $2\,\frac{dy}{dx}\,\frac{dy}{dx^3} = 5\left(\frac{d^2y}{dx^2}\right)^2$

(*iii*) $(x + y)\,(dx)^2 + 2xy\,dx\,dy - (dy)^2 = 0$ (*iv*) $\frac{d^2x}{dt^2} = \sqrt[4]{\left(\frac{dx}{dt}\right)^2} = 0$

(*v*) $\left(\frac{d^2y}{dx^2}\right)^2 - 2\left(\frac{dy}{dx}\right)^3 + y = 7x^4$

Soluiton :

(*i*) The highest order derivative present in the diferential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y$ is $\frac{d^2y}{dx^2}$, its order is 2, so the order of the differential equation is 2. the power or degree of $\frac{d^2y}{dx^2}$ is 1.

Hence the order and degree of the given differential equation are 2 and 1 respectively.

(*ii*) In the given differential equation, highest order derivative is $\frac{d^3y}{dx^3}$, which is of order 3 and power to this third order derivative is 1. Hence the given differential equation is of order 3 and degree1.

(iii) The given equation can be written as

$$(x + y)^2 + 2xy \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2 = 0$$

The highest order derivative involved is $\frac{dy}{dx}$, whose order is 1 and power of $\frac{dy}{dx}$ is 2. Hence the given differential equation is of order 1 and degree 2.

(iv) Removing the radical, the given differential equation can be written as

$$\left(\frac{d^2x}{dt^2}\right)^4 - \left(\frac{dx}{dt}\right)^2 - x = 0$$

Here, highest order of the derivative is 2 and the power of this highest order derivative is 4. Hence the given differential equation is of order 2 and degree 4.

(v) In the given differential equation, highest order derivative is $\frac{d^2y}{dx^2}$,which is of order 2 and power to this second order derivative is 2.

Hence the differential equation is of order 2 and degree 2.

Example – 2: ***Obtain the differential equation of appropriate order eliminating the arbitrary constants A and B from the equations :***

(i) $y = Ae^{2x} + Be^{-2x}$ ***(ii)*** $y = A + Be^{5x} + Ce^{-7x}$

Solution :

(i) Here $y = \mathrm{A}e^{2x} + \mathrm{B}e^{-2x}$

Differentiating successively twice with respect to x, we get

$$\frac{dy}{dx} = 2\mathrm{A}e^{2x} - 2\mathrm{B}e^{-2x} \text{ and } \frac{d^2y}{dx^2} = 4\mathrm{A}e^{2x} + 4\mathrm{B}e^{-2x} = 4y$$

Hence the required differential equation is $\frac{d^2y}{dx^2} = 4y$.

(ii) Here $y = \mathrm{A} + \mathrm{B}e^{5x} + \mathrm{C}e^{-7x}$

Differentiating suecessively thrice with respect to x, we get

$$\frac{dy}{dx} = 5\,\mathrm{B}e^{5x} - 7\,\mathrm{C}e^{-7x} \qquad \text{... (1)}$$

$$\frac{d^2y}{dx^2} = 25\,\mathrm{B}e^{5x} + 49\,\mathrm{C}e^{-7x} \qquad \text{... (2)}$$

$$\frac{d^3y}{dx^3} = 125\ Be^{5x} - 343\ Ce^{-7x} \qquad \text{... (3)}$$

From (1) amd (2), we get $\quad Be^{5x} = \frac{1}{60}\left(7\frac{dy}{dx} + \frac{d^2y}{dx^2}\right)$

and $Ce^{-7x} = \frac{1}{84}\left(\frac{d^2y}{dx^2} - 5\frac{dy}{dx}\right)$

Putting these values in (3) and simplifying, we get

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 35\frac{dy}{dx} = 0$$

Example – 3 : ***Obtain the differential equation of all circles each of which touches the x–axis at the origin.***

Solution : The equation of circles touching x–axis at the orgin is given by $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant.

Differentiating both sides with respect to x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0 \quad \text{or,} \quad a = \frac{x + yy'}{y'} \quad \text{where } y' = \frac{dy}{dx}$$

Putting the value of a in the equation we get

$$x^2 + y^2 - 2\,\frac{x + yy'}{y'}\,y = 0$$

or, $(x^2 - y^2)\,y' = 2xy$, which is the required differential equation.

Example – 4 : ***Construct a differential equation by the elimination of the arbitrary constants a and b from the equation $ax^2 + by^2 = 1$.***

Solution : $ax^2 + by^2 = 1 \qquad \text{... (1)}$

Differentiating (1) with respect to x, we have

$$2xa + 2by\frac{dy}{dx} = 0 \quad \text{or,} \quad -\frac{a}{b} = \frac{y}{x}\frac{dy}{dx} \qquad \text{... (2)}$$

Differentiating again, we get

$$a + b\left\{\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right\} = 0$$

$$\text{or,} \quad -\frac{a}{b} = \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} \qquad \text{... (3)}$$

From (2) and (3), we get

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation.

Exercise – 5.1

1. Determine the order and degree of the following differential equations :

(*i*) $\left[1+\left(\frac{dy}{dx}\right)\right]^{\frac{1}{3}} = \frac{d^2y}{dx^2}$ (*ii*) $\sqrt{y+\left(\frac{dy}{dx}\right)^2} = 1+x$

(*iii*) $\frac{d^2y}{dx^2} = \frac{x^2}{y(1+\sqrt{x})}$ (*iv*) $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - (\cos x)\sqrt{y} = 2x^2$

2. Find the differential equation of the following :
   (*i*) $y = a + b\cos x$ (a, b arbitrary constants)
   (*ii*) $y = e^x (a\cos x + b\sin x)$ (a, b arbitrary constants)
   (*iii*) $y = A\cos(\log x) + B\sin(\cos x)$ (A, B arbitrary constants)
   (*iv*) $(y-k)^2 = 4a(x-b)$, where k is a fixed constants and a, b are arbitrary constants.
   (*v*) $y = a\sin\theta + b\cos\theta + \theta\cos\theta$, where a, b are arbitrary constants.

3. Show that $y = a\cos x + b$ is a solution of the differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

4. Show that the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f, c are parameters is

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0.$$

5. Show that the solution $y = a\sin x + b\cos x + x\sin x$ satisfies $\frac{d^2y}{dx^2} + y = 2\cos x$ (a, b arbitrary constants)

6. Show that for all values of a and b, the differential equation $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + (k^2 + n^2)x = 0$ is always satisfied by $x = e^{-kt}[a\cos nt + b\sin nt]$

Answers

1. (*i*) order 2, degree 3 (*ii*) order 1, degree 2
   (*iii*) order 2, degree 1 (*iv*) order 3, degree 1
   (*v*) order 2, degree 1

2. (*i*) $\sin x\frac{d^2y}{dx^2} - \cos x\frac{dy}{dx} = 0$ (*ii*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

   (*iii*) $x^2 y^2 + xy_1 + y = 0$ (*iv*) $(y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

   (*v*) $\frac{d^2y}{dx^2} + y + 2\sin\theta = 0.$