6 CHAPTER

Linear Differenitial Equations of Second and Higher Order

6.1 : Introduction

Linear differential equations are of paramount importance in the description of physical phenomena. Many natural processes appear as higher-order linear differential equations, most often as second-order equations. Second-order linear differential equations arise in many areas of physics, electric circuits and vibrations etc.

A linear ordinary differential equation of order n has the form.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots. + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad \ldots (1)$$

where a_0 ($\neq 0$), a_1, a_2, ..., a_{n-1}, a_n and X are functions of x only.

If X = 0, then equation (1) is **called homogeneous** ; otherwise the equation (1) is nonhomogeneous.

If a_0, a_1, a_2 ...., a_{n-1} , a_n are constants, the equation (1) is called **linear differential equation** of order n with constant co-efficients.

Note : When $n = 2$, the equation (1) becomes a second order linear differential equation and it is of the form

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X$$

where a_0 ($\neq 0$), a_1, a_2 and X are functions of x only.

6.2 : Linear Differential Equations with Constant Co-efficients

A linear ordinary differential equation of order n has the form.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots.. + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad \ldots (1)$$

where a_0 ($\neq 0$), a_1, a_2, ..., a_{n-1}, a_n and X are functions of x only.

If X = 0, then equation (1) is **called homogeneous** ; otherwise the **equation (1) is nonhomogeneous.**

If $a_0, a_1, a_2 \ldots\ldots\ldots, a_{n-1}, a_n$ are constants, the equation (1) is called linear differential equation of order n with constant co-efficients.

Note : When $n = 2$, the equation (1) becomes a second order linear differential equation and it is of the form

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X$$

where a_0 ($\neq 0$), a_1, a_2 and X are functions of x only.

A linear ordinary differential equation of order n with constant co-efficients has the form.

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots\ldots + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad \ldots (1)$$

where $a_1, a_2 \ldots\ldots\ldots, a_n$ are real constants and X is a function of x only.

The concept of general solution of the equation (1) requires the idea of linear dependence and linear independence of functions ; so these ideas are introduced first.

Definition :

A set of n real valued functions $y_1, y_2 \ldots, y_n$ are linearly dependent. (L.D.) on the interval I (where they are defined) if there exists real constants $c_1, c_2, \ldots\ldots, c_n$ not all zero such that

$$c_1 y_1(x) + c_2 y_2(x) + \ldots\ldots\ldots + c_n y_n(x) = 0 \text{ for all } x \text{ in I.}$$

These functions are said to be linearly independent (L.I.) on I if the relation

$$c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots + c_n y_n = 0 \text{ for all } x \in I$$

implies $c_1 = c_2 = \ldots\ldots\ldots = c_n = 0$.

We may test the linear independence or dependence of these functions $y_1, y_2, \ldots\ldots\ldots, y_n$ by an elegant tool which is Wronskian determinant.

Definition :

If the n functions $y_1, y_2, \ldots, y_n$ are each $(n-1)$ times differentiable with respect to the independent variable x then the Wronskian of n functions is denoted by W $(y_1, y_2, \ldots\ldots, y_n)$ and is defined by

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 \ldots\ldots\ldots y_n \\ y_1' & y_2' \ldots\ldots\ldots y_n' \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ y_1^{(n-1)} & y_2^{(n-1)} \ldots\ldots y_n^{(n-1)} \end{vmatrix}$$

where primes denotes the differentiation with respect to x. $\left[y_1' = \frac{dy_1}{dx}, y_2'' = \frac{d^2 y_2}{dx^2} \text{ etc.} \right]$

An Example :

Let $y_1 = \sin x$ and $y_2 = \cos x$ where $0 \leq x \leq 2\pi$. Here we see that :

$c_1 \sin x + c_2 \cos x = 0$ for all $x \in [0, 2\pi]$ (c_1, c_2 are real constants) implies $c_1 \cdot 0 + c_2 \cdot 1 = 0$ (taking $x = 0$)

and $c_1 \cdot 1 + c_2 \cdot 0 = 0 \left(\text{taking } x = \frac{\pi}{2}\right)$

from which we get $c_1 = 0, c_2 = 0$.

Hence the set of functions $\{\sin x, \cos x\}$ is linearly independent.

Here we observe that :

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0 \text{ for all } x \in [0, 2\pi]$$

Now we state the theorem for linear independence.

Theorem : The functions $y_1, y_2, \ldots\ldots., y_n$ will be linearly independent if and only if Wronskian $W(y_1, y_2, \ldots.., y_n) \neq 0$ in the corresponding domain.(We omit the proof of the theorem)

In order to solve the equation of the form (1), we first find the general solution of the equation formed by setting X = 0 in the equation (1).

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots. + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad \ldots (2)$$

To find the general solution of (2), we first find n linearly independent functions $y_1, y_2, \ldots., y_n$, that satisfy it. Now we state the following theorem :

Theorem : Let $y_1, y_2, \ldots\ldots. y_n$ be n linearly independent solutions of (1) on an interval I. Every solution of (1) can be written uniquely as :

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots\ldots\ldots + c_n y_n(x) \qquad \ldots\ldots (3)$$

where $c_1, c_2, \ldots\ldots, c_n$ are real constants (we omit proof of the theorem).

Since $c_1, c_2, \ldots\ldots, c_n$ are n arbitrary constants and the equation (2) is of order n, it follows that the general solution of the equation (2) is of the form (3).

Note : The function $y_c(x)$ is called the complementary function (C.F.) of the equatiuon

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots. + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad \ldots (1)$$

where $a_1, a_2, \ldots\ldots.., a_n$ are real constants and X is a function of x.

Now we shall discuss the method of obtaining the general solution of the linear differential equation of order two. We consider the equation.

$$\frac{d^2 y}{dx^2} + a \frac{dy}{dx} + by = 0 \qquad \ldots (4)$$

In previous chapter we have seen that the solution of first order linear equation of the form

$\frac{dy}{dx} - my = 0$ is $y = e^{mx} + c$ (c is an arbitrary constant).

Therefore we try to find a solution [of (4)] of the form $y =- e^{mx}$, where m is a constant. Substituting $y = e^{mx}$ in (4), we get $(m^2 + am + b)\, e^{mx} = 0$.

Since $e^{mx} \neq 0$, we obtain

$$m^2 + am + b = 0 \qquad \text{... (5)}$$

This is a quadratic equation in m. The equation (5) is called the **auxiliary equation** of (4). The auxiliary equation (5) has two roots m_1, m_2 (say). We have the three main cases as follows:

(i) Two roots are real and distinct ($m_1 \neq m_2$) : Let the distinct roots of (5) be m_1, m_2. Then $e^{m_1 x}$ and $e^{m_2 x}$ are two solutions of (4). They are linearly independent, since.

$$W(e^{m_1 x}, e^{m_2 x}) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= (m_2 - m_1)\, e^{(m_1 + m_2)\, x}$$

$$\neq 0 \text{ for all real values of } x$$

Hence the general solution of (4) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2} x$$

where c_1 and c_2 are arbitrary constants.

Note : If the distinct real numbers $m_1, m_2, \ldots\ldots\ldots, m_n$ are the roots of the auxiliary equation corresponding to the homogeneous linear differential equation of order n, then the general solution of the equation is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots\ldots + c_n e^{m_n x}$.

where $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants.

(ii) Two roots are real and equal ($m_1 = m_2$) : If α, α be the repeated roots of (5), then

$$\alpha + \alpha = -a, \ i.e.\ \alpha = -\frac{a}{2}.$$

Here $e^{\alpha x}$ is a solution of (4). Then the solution of (4) can be obtained by setting $y = ve^{\alpha x}$ (v is a function of x). We get

$$y = ve^{\alpha x}, \ \frac{dy}{dx} = \alpha v e^{\alpha x} + \frac{dv}{dx} e^{\alpha x}, \ \frac{d^2 y}{dx^2} = e^{\alpha x}\left[\frac{d^2 v}{dx^2} + 2\alpha \frac{dv}{dx} + \alpha^2 v\right].$$

Then from (3), we get.

$$e^{\alpha x}\left[\frac{d^2 v}{dx^2} + 2\alpha \frac{dv}{dx} + \alpha^2 v\right] + e^{\alpha x} a \left[v\alpha + \frac{dv}{dx}\right] + bve^{\alpha x} = 0$$

$$\text{or, } \frac{d^2 v}{dx^2} + (2\alpha + a)\frac{dv}{dx} + (\alpha^2 + a\alpha + b)^v = 0$$

$$\text{or, } \frac{d^2 v}{dx^2} = 0 \quad [\, 2\alpha + a = 0 \text{ and } \alpha^2 + a\alpha + b = 0, \text{ since } \alpha \text{ is a root of (5)}]$$

On integration, we have $v = c_1 + c_2 x$

where c_1 and c_2 are real constants.

So, if we take $c_1 = 0, c_2 = 1$, we find that $y = x\, e^{\alpha x}$ is a second solution of (3).

$$\text{Now, } W(e^{\alpha x}, xe^{\alpha x}) = \begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ ae^{\alpha x} & e^{\alpha x} + \alpha x e^{\alpha x} \end{vmatrix} = e^{2\alpha x} \neq 0$$

So, $\{e^{ax}, xe^{ax}\}$ is a set of linearly independent solutions of (3).

Hence the general solution of (4) is

$$y = (c_1 + c_2\, x)\, e^{\alpha x}.$$

where c_1 and c_2 are arbitrary constants.

Note : If the order of the equation be three and if the corresponding auxiliary equation has a real double root, say, α_1 and *a* simple real root α_2 then the general solution will be

$$y = (c_1 + c_2 x)\; e^{\alpha_1 x} + c_3 e^{\alpha_2 x}$$

If the roots of the auxiliary equation in this case are all real and equal, say, α then the corresponding general solution will by

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{\alpha x}.$$

where c_1, c_2, c_3 are arbitrary constants.

Similarly we can obtained the general solution of the linear differential equation of order ($>$, 4) when two or more roots of the auxiliary equation are real and equal and the remaining roots (if any) are real and distincit.

CASE (iii) Two roots are imaginary :

If one of the roots of (4) is imaginary, say $\alpha + i\beta$ [α, β ($\neq 0$) are real], then its conjugate $\alpha - i\beta$ is a also a root of (4). Then

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

is a formal solution of (3) where c_1 and c_2 are arbitrary complex constants. Now,

$$c_1\, e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = e^{\alpha x}\; [(c_1 + c_2) \cos \beta x + (ic_1 + ic_2) \sin \beta x]$$

For arbitrary real constants A, B we can find c_1, c_2 such that :

$$c_1 + c_2 = \text{A and } i\,(c_1 - c_2) = \text{B}$$

which yields $c_1 = \dfrac{\text{A} - i\text{B}}{2}$ and $c_2 = \dfrac{\text{A} + i\text{B}}{2}$

Then $y = e^{\alpha x}$ [A cos βx + B sin βx] is a solution of (3) where A and B are real constants and it can be shown that $e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$ are linearly independent solution of (4). So, in this case the general solution is given by

$$y = e^{\alpha x}\, (\text{A} \cos \beta x + \text{B} \sin \beta x)$$

where A and B are arbitrary constants.

Note : If for a linear differential equation of order ($<$, 4) with constant co-efficients, the conjugate imaginary roots $\alpha + i\beta$, $\alpha - i\beta$ of the corresponding auxiliary equation are each of multiplicity k, then the corresponding part of the general solution will be of the form

$$e^{\alpha x}\, [(c_1 + c_2\, x + \ldots\ldots\ldots\ldots. + c_k\, x^{k-1}) \cos \beta\, x + (d_1 + d_2 x + \ldots\ldots.. + d_k x^{k-1}) \sin \beta x]$$

For example, if the roots of the auxiliary equation of a fourth order homogeneous linear differential equation with constant co-efficients be $1 + i$, $1 - i$, $1 + i$, $1 - i$ then the general solution will by

$$y = e^x\,[(c_1 + c_2x)\cos x + (d_1 + d_2x)\sin x]$$

We now give several examples.

Rules for finding the complementary functions

Working Rule to solve the equation :

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots k_n y = X$$

of which the equation is in symbolic form

$D^n y + k_1 D^{n-1} y + \dots\dots + k_n y = X$

$(D^n + k_1 D^{n-1} + \dots\dots + k_n)\, y = X$

$f(D)\, y = X$

Where $f(D) = (D^n + k_1 D^{n-1} + \dots\dots k_n)$

Step –I : To find the complementary function

(i) Write the A.E. $f(D)\, y = 0$ and solve it. Let it's root be $D = m_1, m_2, \dots\dots\dots m_n$.

Write the complete solution

Roots of A.E.

(1) All the roots $m_1, m_2, m_3 \dots\dots\dots m_n$ are real & different.

(2) All the roots $m_1, m_2, m_3 \dots\dots m_n$ are real & $m_1 = m_2$.

(3) All the roots $m_1, m_2, m_3 \dots\dots$ are real & $m_1 = m_2 = m_3$.

(4) Two pairs of the roots are complex i.e. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ & all other roots are real & different.

Complete sol^n :

(i) $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots\dots\dots + c_n e^{m_n x}$

(ii) $y = (c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \dots\dots\dots + c_n e^{m_n x}$

(iii) $y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \dots\dots\dots + c_n e^{m_n x}$

(iv) $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots\dots\dots + c_n e^{m_n x}$

(v) $y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + c_5 e^{m_5 x} + \dots\dots\dots + c_n e^{m_n x}$

Illustrative Examples

Example – 1 : Find the general solution of

$$3\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$$

Solution : Given $3\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} - 8y$ .... (1)

let $y = e^{mx}$ be a trial solution of (1).

To find C. F.

Then the auxiliary equation of (1) is

$3m^2 + 2m - 8 = 0$

i.e., $(3m - 4)(m + 2) = 0$

i.e., $m = \frac{4}{3}, -2$

The general solution of (1) is therefore

$$y = c_1 e^{\frac{4}{3}x} + c_2 e^{-2x},$$

where c_1 and c_2 are arbitrary constants.

Example – 2 : Find the general solution of $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$

Solution : The above equation can be symbolically wirtten as $D^3y - 3D^2y + 4y = 0$

To find C. F.

The auxiliary equation of the given differential equation is

$m^3 - 3m^2 + 4 = 0$

i.e., $(m + 1)(m - 2)^2 = 0$

i.e., $m = -1, 2, 2$

Here a root (*i.e.* 2) is a double root of the auxiliary equation. Thus the general solution of the given differential equation can be written as.

$y = c_1 e^{-x} + (c_2 + c_3 x)\, e^{2x}$

where c_1, c_2, c_3 are arbitrary constants.

Example – 3: Find the general solution of $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

Solution : To find C. F.

The auxiliary equation of the given differential equation is

$m^4 + 8m^2 + 16 = 0$

i.e., $(m^2 + 4)^2 = 0$

i.e., $m = \pm 2i, \pm 2i.$

i.e., the roots of the auxiliary equation are $2i, -2i, 2i\ -2i$

Since each paris of conjugate imaginary roots is double, the general solution of the given differential equation is $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$ where c_1, c_2, c_3, c_4 are arbitrary constants.

Example – 4 : ***Solve the initial value problem,*** $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$ ***, given that*** $x(0) = 0$, $\frac{dx}{dt}(0) = 15$.

Solution : We have, $(D^2 + 5D + 6)x = 0$, where $D = \frac{d}{dt}$

A.E. is $m^2 + 5m + 6$ or $(m+2)(m+3) = 0 \Rightarrow m = -2, -3$

$\therefore\ x = x(t) = c_1 e^{-2t} + c_2 e^{-3t}$ ... (1)

This is the general soultion of the given equation.

Now, consider $x(0) = 0$

(1) becomes, $x(0) = c_1 \cdot 1 + c_2 \cdot 1$

ie., $c_1 + c_2 = 0$ ... (2)

Also we have from (1),

$$\frac{dx}{dt} = -2c_1e^{-2t} - 3c_2e^{-3t}$$

Applying the condition $\frac{dx}{dt}(0) = 15$ we obtain,

$-2c_1 - 3c_2 = 15$ ... (3)

Solving (2) and (3) we get $c_1 = 15$, $c_2 = -15$

Thus $x(t) = 15\left(e^{-2t} - e^{-3t}\right)$ is the particular solution.

We have dealt with the homogeneous linear equation with constant co-efficients and its general solution. Now we consider the nonhomogeneous linear equation (1) with constant co-efficients of order n. The following theorem characterizes the solution of the equation (1), in terms of a general solution of the corresponding homogeneous equation (2).

Example – 5 : ***Find the solution of the differential equation*** $\frac{d^2y}{dt^2} + 4y = 0$ ***given that***

$y(0) = 1, y\left(\frac{\pi}{4}\right) = 1.$ ***[B.P.U.T., June 2005***

Solution : The given equation is

$$\frac{d^2y}{dt^2} + 4y = 0 \quad .... (1)$$

Let $y = e^{mt}$ be a trial solution of (1).

Then from (1) we get, $(m^2 + 4)\, e^{mt} = 0$

Since $e^{mt} \neq 0$, we have the auxiliary equation of (1) $m^2 + 4 = 0$ which gives $m = \pm 2i$.

So, the general solution of (1) is

$y = c_1 \cos 2t + c_2 \sin 2t$ .... (2)

Given conditions are $y(0) = 1, y\left(\frac{\pi}{4}\right) = 1$

Using given conditions we get from (2),

$c_1 = 1$ and $c_2 = 1$.

$\therefore$ The required solution of (1) is

$y = \cos 2t + \sin 2t$.

Example – 6 : Solve $\frac{d^4y}{dx^4}+8\frac{d^2y}{dx^2}+16y=0$

Solution : $\frac{d^4y}{dx^4}+8\frac{d^2y}{dx^2}+16y=0$

The above equation can be symbolically written as

$\Rightarrow$ $D^4y + 8D^2y + 16y = 0$

$\Rightarrow$ $(D^4+8D^2+16)\, y = 0$

Now A.E. = 0, $\Rightarrow D^4 + 8D^2 + 16 = 0$

$\Rightarrow$ $D^4 + 4D^2 + 4D^2 + 16 = 0$

$\Rightarrow$ $D^2 (D^2 + 4) + 4 (D^2 + 4) = 0$ $\Rightarrow (D^2 + 4) \cdot 4(D^2 + 4) = 0$

$\Rightarrow$ $D^2 + 4 = 0$ or $D^2 + 4 = 0$ $\Rightarrow D^2 = -4$, or $D^2 = -4$

$\Rightarrow$ $D^2 = 4i^2$ or $D^2 = 4i^2$ $\Rightarrow D = \pm 2i$ or $D = \pm 2i$

$\therefore$ So the required solution is

$y = e^{0x} \{(c_1 + c_2x) \cos 2x + (c_3 + c_4x) \sin 2x\}$

$= (c_1 + c_2x) \cos 2x + (c_3 + c_4x) \sin 2x$

Theorem : A general solution $y(x)$ of the nonhomogeneous equation (1) is the sum of a general solution $y_c(x)$ of (2) (called the complementary function for (1)) and a particular solution $y_p(x)$ of (1), where

$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + ...+ c_n y_n(x)$ where $y_1, y_2, ..., y_n$

are linearly independent solutions of (2).

Hence $y(x) = y_c(x) + y_p(x)$

Remark : $y_p(x)$ is also called a particular integral of (1).

We rewrite the equation (1) as

$f(D)\,y = X$ .... (1)

Where $f(D) \equiv D^n + a_1D^{n-1} + ... + a_{n-1}D + a_n$ ($a_1, a_2, ..., a_n$ are constants) and D stands for $\frac{d}{dx}$

Here $f(D)$ is a linear operation, *i.e.,*

$f(D)[r_1(x) + r_2(x)] = f(D)[r_1(x)] + f(D)[r_2(x)]$

and $f(D)[kr(x)] = kf(D)[r(x)]$,

where r_1, r_2, r are functions of x and k is a real constant, $f(D)$ is a polynomial in D with constant co-efficients and it behaves just an ordinary algebraic polynomial.

We now present an important method, known as the operator method, for finding particular integral of (1′). Before finding particular integrals of (1′) we define the operator inverse to $f(D)$ which is denoted by $\frac{1}{f(D)}$ or $[f(D)]^{-1}$. It is defined to be an operator such that

$$\frac{1}{f(\mathrm{D})}\mathrm{X}(x) = \psi(x)$$

if and only if $[f(\mathrm{D})]\,\psi(x) = \mathrm{X}(x)$

Now we will show that $\frac{1}{f(\mathrm{D})}\mathrm{X}$ is a particular integral of the equation (1′).

Proof : Substituting $\frac{1}{f(\mathrm{D})}\mathrm{X}$ for y in the equation (1′), we find that

$$f(\mathrm{D})\,y = f(\mathrm{D})\,\frac{1}{f(\mathrm{D})}\mathrm{X} = \mathrm{X}$$

We will find the particular integrals of (1′) when X has the following forms :

(*i*) $\mathrm{X} = x^m$ (m is non-negative integer) or any polynomial in x.

(*ii*) $\mathrm{X} = e^{ax}$, where a is a constant.

(*iii*) $\mathrm{X} = \sin(ax + b)$ or $\cos(ax + b)$, where a and b are constants.

(*iv*) $\mathrm{X} = e^{ax}\,v$, where v is a function of x.

(*i*) To evaluate $\frac{1}{f(\mathrm{D})}x^m$, we can write $f(\mathrm{D})$ as $[f(\mathrm{D})]^{-1}$ and we can expand it in ascending powers of D using Binomial theorem.

The terms of the expansion so obtained, each term operates on x^m and the final result will be a particular integral of the equation (1′) corresponding to x^m. It is obvious that no terms of the expansion beyond the mth power of D need to retained, since the result of their operation on x^m will be zero (*i.e.* $\mathrm{D}^n x^m = 0$ when $n > m$)

The same procedure will have to adopted when X is a polynomial in x of degree m.

We illustrate the above procedure by the following example :

6.3 : Solution of Differential Equation *f(D)y = X or P.I*

We have already discuss from the previous Chapter, that the solution of the equation *f(D)* y = X, consists of two parts, namely **complementary function** and **particular integral**. The complementary function for this equation is same as the complete solution of *f(D)* = 0. The methods to find complementary functions have already been dicussed. We shall discuss the methods of finding the particular Integrals.

Particular Integral

The particular integral of the differential equation *f(D)*y = X is defind as y $= \frac{1}{f(\mathrm{D})}\mathrm{X}$ ....(1)

Where in general 'X' is a function of x, It may of course, be a const. The symbol $\frac{1}{f(\mathrm{D})}\mathrm{X}$, is defined as a function of x, which when separated upon by *f(D)*, gives X, in symbolically $f(\mathrm{D})\left[\frac{1}{f(\mathrm{D})}\mathrm{X}\right] = \mathrm{X}$.

Thus the function $\frac{1}{f(\mathrm{D})}\mathrm{X}$ satisfies the equation (1) and called particular integral.

As already pointed, the polynomial *f(D)*, can be subjected to algebraic operations, such as factorisation, resolutions, in partial fractions and expansions by binomial theorem etc. The following results are quite useful to find particular integrals.

Results :

(1) If X is a function of *x* or a const, then $\frac{1}{\mathbf{D}}\mathbf{X} = \int \mathbf{X}\,\mathbf{dx}$

Solution : Let $y = \frac{1}{\mathrm{D}}\mathrm{X}$ operating both sides D

$$\mathrm{D}y = \mathrm{D}\left(\frac{1}{\mathrm{D}}\mathrm{X}\right) = \mathrm{X}$$

or $\frac{dy}{dx} = \mathrm{X}$, Intergrating both sides w.r.t. x $y = \int x dx$, hence the result.

(II) If X is a function of x or a const, then $\frac{1}{(\mathbf{D}-m)}\mathbf{X} = e^{mx}\int \mathbf{X}e^{-mx}dx$

Solution : Let $\frac{1}{(\mathrm{D}-\mathrm{m})}X = y$

$$(\mathrm{D}-\mathrm{m})y = (\mathrm{D}-\mathrm{m})\left(\frac{1}{(\mathrm{D}-\mathrm{m})}\mathrm{X}\right) = \mathrm{X}$$

or $\frac{dy}{dx} - my = X$, Which is linear

I.F. $= e^{\int -mdx} = e^{-mx}$

Hence sol^{n} is $ye^{-mx} = \int Xe^{-mx}dx$

$$y = e^{mx}\int Xe^{-mx}dx \qquad \therefore \frac{1}{D-m}X = e^{mx}\int Xe^{-mx}dx$$

This formula enables us to evaluate the particular integral $\frac{1}{f(\mathrm{D})}\mathrm{X}$,

where $f(D) = (\mathrm{D}-m_1)(\mathrm{D}-m_2)\,.........(\mathrm{D}-m_n)$

$$\therefore \frac{1}{f(\mathrm{D})}\mathrm{X} = \frac{1}{(D-m_1)(D-m_2).......(D-m_n)}X$$

Resolving R.H.S. into partial fractions

$$\frac{1}{f(D)}X = \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2}.......+\frac{A_n}{D-m_n}\right)X$$

$$= A_1e^{m_1x}\int xe^{-m_1x} + A_2e^{m_2x}\int Xe^{-m_2xdx} + ... + A_ne^{m_nx}\int Xe^{m_nx}dx$$

Particular Integrals in some special cases :

Let us consider the differential equation

$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots\dots\dots + P_n)\, y = X$

where all P_i's are constant.

The above equation can be written as $f(D)\, y = X$

particular integral $= \dfrac{1}{f(D)} X$

Case – 1 :

When $X = e^{ax}$

$D(e^{ax}) = ae^{ax}$

$D^2 (e^{ax}) = a^2 e^{ax}$

....

$D^n(e^{ax}) = a^n e^{ax}$ ....(i)

Multiplying in (i) by $P_n, P_{n-1} \dots P_1$ and respectively and adding,

$(D^n + P_1 D^{n-1} + \dots\dots P_{n-1} D + P_n) e^{ax} = (a_n + P_1 a^{n-1} + \dots\dots P_{n-1} a + P_n) e^{ax}$

$\Rightarrow f(D)\, e^{ax} = f(a) e^{ax}$

Operating both sides of equation (ii) by $\dfrac{1}{f(D)}$

$$\frac{1}{f(D)}\left[f(D)e^{ax}\right] = \frac{1}{f(D)}\left[f(a)e^{ax}\right]$$

or $$\frac{e^{ax}}{f(a)} = \frac{1}{f(D)} e^{ax}$$

Which gives the particular integral of e^{ax}

In case X = K(a const) then

$$\frac{1}{f(D)} k = k \frac{1}{f(D)} e^{0.x} = \frac{k}{f(D)},\ \text{provided } f(0) \pm 0$$

Case of Failure : If $f(a) = 0$, the above method fails and we proceed as under.

Since $f(a) = 0$

$D = a$, is a root of $f(D) = 0$........(i)

$(D - a)$ is a factor of $f(D)$, Suppose $f(D) = (D - a) f'(D)$. where $f'(a) \neq 0$

Then $$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{f'(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{f'(a)} e^{ax}$$

$$= \frac{1}{f'(a)} \cdot \frac{1}{(D-a)} e^{ax} = \frac{1}{f'(a)} e^{ax} \int e^{ax} . e^{-ax} . dx$$

$$= \frac{1}{f'(a)} e^{ax} \int dx = x . \frac{1}{f'(a)} e^{ax}$$

i.e. $\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$ ....(2) $(f'(a) \neq 0)$

$$\begin{bmatrix} \because & f'(D) = (D-a)f''(D) + 1.f(D) \\ \because & f'(a) = O\ f''(a) + f(a) \end{bmatrix}$$

If $f'(a) = 0$, then applying (2) again

we get $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$ provided $f''(a) \neq 0$ ....(3)

and so on,

In general if D = a, is a repeated root of $f(D) = 0$, say (r + 1) times,

we have $\frac{1}{f(D)}e^{ax} = \frac{x^r.e^{ax}}{f^r(a)}$ $\qquad f^r(a) \neq 0$

Rule – 1 :

To find the P.I. $\frac{1}{f(D)}e^{ax}$, replace D by a, provided $f(a) \neq 0$, In case $f(a) = 0$ multiply the nuemerator by x and differentiate the denominator w.r.t. D and apply the above rule, provided $f'(a) \neq 0$ and so on.

Example – 1 : Find the particular integral in the solution of $(D^2 + 16)y = e^{-4x}$.

Solution : P.I. $= \left(\frac{1}{D^2+16}\right)e^{-4x}$

$= \frac{e^{-4x}}{(-4)^2+16}$ (Replacing D by – 4) $= \frac{e^{-4x}}{32}$

Example – 2 : Solve $(D - 2)^2y = e^{2x}$.

Solution : The auxiliary equation is $(D - 2)^2 = 0$. Hence D = 2, 2

$\therefore$ C.F. $= e^{2x}(c_1x + c_2)$

P.I. $= \frac{1}{(D-2)^2}e^{2x} = \frac{x^2e^{2x}}{2!}$

$\therefore$ y = C.F. + P.I. $= e^{2x}(c_1x + c_2) + \frac{x^2e^{2x}}{2}$.

Example – 3 : Solve $(D^2 - 4)y = e^{2x} + e^{-4x}$.

Solution : The auxiliary equation is $D^2 - 4 = 0$.

Hence D = 2, –2.

$\therefore$ C.F. $= c_1e^{2x} + c_2e^{-2x}$

$$\therefore \text{ P.I. } = \left(\frac{1}{D^2-4}\right)\left(e^{2x}+e^{-4x}\right) = \left(\frac{1}{D^2-4}\right)e^{2x} + \left(\frac{1}{D^2-4}\right)e^{-4x}$$

$$= \left(\frac{1}{(D+2)(D-2)}\right)e^{2x} + \frac{e^{-4x}}{4^2-4} = \frac{1}{4}\left(\frac{1}{D-2}\right)e^{2x} + \frac{e^{-4x}}{12} = \frac{1}{4}xe^{2x} + \frac{1}{12}e^{-4x}.$$

The solution is y = C.F. + P.I.

$$\therefore y = c_1e^{2x} + c_2e^{-2x} + \frac{1}{4}xe^{2x} + \frac{1}{12}e^{-4x}.$$

Case – II :

When X = sin ($ax + b$) or cos ($ax + b$)

D sin ($ax + b$) = a cos ($ax + b$)

D^2 sin ($ax + b$) = – a^2 sin ($ax + b$)

D^3 sin ($ax + b$) = $-a^3$ cos ($ax + b$)

D^4 sin ($ax + b$) = $(-a^2)^2$ sin ($ax + b$)

....

$\therefore$ $(D^2)^n$ sin ($ax + b$) = $(-a^2)^n$ sin ($ax + b$)

$\Rightarrow$ $f(D^2)$ sin ($ax + b$) = $f(-a^2)$ sin ($ax + b$)

operating both sides by $\frac{1}{f(D^2)}$

$$\frac{1}{f(D^2)}\cdot f(D^2) \sin(ax+b) = \frac{1}{f(D^2)}\cdot f(-a^2)\sin(ax+b)$$

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b) \text{ provided } f(-a^2) \neq 0$$

Case of failure :

In case $f(-a^2) = 0$, the above method fails and we proceed as under we know

$e^{i(ax+b)} = \cos(ax+b) + i\sin(ax+b)$, since $f(-a^2) = 0$

$$\frac{1}{f(D^2)}[\cos(ax+b)+i\sin(ax+b)] = \frac{x\,e^{i(ax+b)}}{f'(-a^2)}, \text{ provided } f'(-a^2) \neq 0,$$

$$\frac{1}{f'(-a^2)}x[\cos(ax+b)+i\sin(ax+b)]$$

Equating imaginary parts

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{x}{f'(-a^2)}\sin(ax+b)$$

If real parts are equated then $\frac{1}{f(D^2)}\cos(ax+b) = \frac{x}{f'(-a^2)}\cos(ax+b)$

In case $f'(-a^2) = 0$, the P.I. is obtained by repeated application of above rule as follow.

$$\frac{1}{f(D^2)}\sin(ax+b)=\frac{x}{f'(-a^2)}\sin(ax+b)$$

$$\frac{1}{f(D^2)}\cos(ax+b)=\frac{x}{f'(-a^2)}\cos(ax+b) \text{ and so on}$$

Rule – 2 :

To find the P.I., $\frac{1}{f(D^2)}\sin(ax+b)$ or $\frac{1}{f(D^2)}\cos(ax+b)$, replace D^2 by $(-a^2)$ provided $f(-a^2)\neq 0$, In case $f(-a^2)=0$, multiply the numerator by x and differentiate the denominator w.r.t. D and apply the above rule provided $f'(-a^2)\neq 0$, and so on.

Note : In case *f(D)* contain terms such as D^3 or D^5 etc. replace by $-a^2D$ and a^4D respectively.

Example – 1 : Solve $(D^2+D+1)y=\sin 2x$.

Solution : The auxiliary equation is $D^2+D+1=0$.

$$\therefore D=\frac{-1\pm i\sqrt{3}}{2}$$

$$\therefore C.F.=e^{-x/2}\left[c_1\cos\left(\frac{\sqrt{3}x}{2}\right)+c_2\sin\left(\frac{\sqrt{3}x}{2}\right)\right].$$

$$P.I.=\left(\frac{1}{D^2+D+1}\right)\sin 2x=\left(\frac{1}{-4+D+1}\right)\sin 2x$$

$$=\left(\frac{1}{D-3}\right)\sin 2x=\left(\frac{(D+3)}{(D^2-9)}\right)\sin 2x$$

$$=\frac{(D+3)}{-13}\sin 2x=-\frac{2\cos 2x}{13}-\frac{3\sin 2x}{13}.$$

$\therefore$ The general solution is y = C.F. + P.I.

Example – 2 : Evaluate the P.I. $=\frac{1}{D^2+9}(4\sin 3x)$

Solution : $P.I.=\left(\frac{4}{D^2+9}\right)\sin 3x$

$$\left[f(-a^2)=0, i.e\ D^2=-3^2 \text{ cases failure}\right]$$

$$=\frac{x.4}{2D}\sin 3x$$

$$=2x\int \sin 3x dx=\frac{-2x}{3}\cos 3x$$

Example – 3 : Solve $(D^2 + 9)y = \cos 3x$.

Solution : The auxiliary equation is $D^2 + 9 = 0$.

Hence $D = \pm 3i$.

$\therefore$ C.F. $= c_1\cos 3x + c_2\sin 3x$

$$P.I. = \left(\frac{1}{D^2+9}\right)\cos 3x$$

[Here $f(-a^2) = f(-3^2) = 0$, cases failure]

$$= \frac{x}{2D}\cos 3x$$

$$= \frac{x}{2}\cdot\frac{\sin 3x}{3} = \frac{x}{6}\sin 3x$$

Case – III

When $X = x^m$, where $m > 0$,

Here P.I.$= \dfrac{1}{f(\mathrm{D})} x^m = [f(D)]^{-1} x^m$

Expand $[f(D)]^{-1}$ in assending powers of D as far as the terms in D^m and operate on x^m terms by term. We need not retain terms D^{m+1}, D^{m+2} etc. because $D^{m+1}(x^m) = 0 = D^{m+2}(x^n)$ and so on.

Binomial expansion of any order

When x is (–ve) or a rational number $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{1.2}x^2 + ... + x^n + ...$

$$T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r\,!}x^r$$

$(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...$

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...$

$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...$

Example – 1 : Solve $(D^2 + 1)y = x$.

Solution : The auxiliary equation $D^2 + 1 = 0$

$\therefore$ $D = \pm i$. Hence C.F. $= c_1 \cos x + c_2 \sin x$.

$$\therefore P.I. = \left(\frac{1}{D^2+1}\right)x$$

$= (1 + D^2)^{-1}x$

$= (1 - D^2)x$ (using binomial expansion)

$= x.$

$\therefore$ The general solution is y = C.F. + P.I.

$\therefore$ $y = c_1 \cos x + c_2 \sin x + x$.

Example – 2 : Solve $(D^2 + 3D + 2)y = x^2$.

Solution : The auxiliary equation is $D^2 + 3D + 2 = 0$

$\therefore$ $(D + 2)(D + 1) = 0$. Hence $D = -2, -1$.

$\therefore$ C.F. $= c_1e^{-2x} + c_2e^{-x}$.

$$\text{P.I.} = \left(\frac{1}{D^2+3D+2}\right)x^2$$

$$= \frac{1}{2}\left(1+\frac{3D+D^2}{2}\right)^{-1}x^2 = \frac{1}{2}\left(1-\frac{3D+D^2}{2}+\left(\frac{3D+D^2}{2}\right)^2\right)x^2$$

$$= \frac{1}{2}\left(1-\frac{3D}{2}+\frac{7D^2}{4}\right)x^2 = \frac{1}{2}\left(x^2-3x+\frac{7}{2}\right).$$

$\therefore$ The general solution is y = C.F. + P.I.

Example – 3 : Find a particular solution of $y'' + y = x^3$. *[B.P.U.T., 2006*

Solution : The given equation is

$(D^2 + 1)y \quad = x^3$

where D stands for $\dfrac{d}{dx}$.

$\therefore$ A particular solution of the given equation is $\dfrac{1}{D^2+1}x^3$

$= (1+D^2)^{-1}x^3$

$= (1 - D^2 + D^4 - \ldots\ldots..)\,x^3$

$= x^3 - 6x$

Case – IV :

When $X = e^{ax}V$, where V is a function of x.

$D(e^{ax}V) = e^{ax}DV + ae^{ax}V$

$= e^{ax}(D + a)V$

$D^2(e^{ax}V) = D[e^{ax}(D + a)]V \qquad = e^{ax}(D^2 + aD)V + ae^{ax}(D + a)V$

$= e^{ax}(D^2 + 2aD + a^2)V \quad = e^{ax}(D + a)^2V$

Thus we see that D on the L.H.S. is replaced by $(D + a)$ on the R.H.S. and e^{ax} proceeds $(D + a)$

$\therefore$ $f(D)\,[e^{ax}V)] = e^{ax}f(D + a)V$

Operating both sides by $\dfrac{1}{f(D)}$

$$\frac{1}{f(D)}\,.f(D)\,[e^{ax}V] = \frac{1}{f(D)}[e^{ax}f(D+a)V]$$

or $e^{ax}V = e^{ax}\dfrac{1}{f(D)}f(D+a)V$ ....(1)

Let $f(D+a)\,V = V_1$

or $V = \dfrac{1}{f(D+a)}V_1$

substituting the value of V in eq^n (i)

$$e^{ax}\frac{1}{f(D+a)}V_1 = \frac{1}{f(D)}(e^{ax}V_1)$$

Now $V_1 = f(D+a)\,V$, is a function of x, replacing V_1 by V,

We get $\dfrac{1}{f(D)}[e^{ax}V] = e^{ax}\dfrac{1}{f(D+a)}V$

Hence P.I. $= \dfrac{1}{f(D)}[e^{ax}V] = e^{ax}\dfrac{1}{f(D+a)}V$

Rule – 3 :

To find P.I. $\dfrac{1}{f(D)}[e^{ax}V]$ replace D by (D + a) and write it as $e^{ax}\dfrac{1}{f(D+a)}V$, and apply the rules stated above, depending upon the nature of the function V.

Example – 1 : Find the P.I. of (D^2 – 4D + 3) y = e^x cos 2x.

Solution : P.I. $= \left(\dfrac{1}{D^2-4D+3}\right)e^x\,cos\,2x$

$$= \left(\frac{e^x}{(D+1)^2-4(D+1)+3}\right)cos\,2x$$

$$= \left(\frac{e^x}{D^2-2D}\right)cos\,2x = \left(\frac{e^x}{-4-2D}\right)cos\,2x$$

$$= -\frac{1}{2}\left(\frac{e^x}{D+2}\right)cos\,2x = -\frac{e^x}{2}\left(\frac{D-2}{D^2-4}\right)cos\,2x$$

$$= -\frac{e^x}{2}\left[\frac{(D-2)\,cos\,2x}{-8}\right]$$

$$= \frac{e^x}{16}(-2\,sin\,2x - 2\,cos\,2x)$$

$$= -\frac{e^x}{8}(sin\,2x + cos\,2x).$$

Example – 2 : Solve (D^2 – 2D + 2) y = e^x sin *x*.

Solution : The auxiliary equation is $D^2 - 2D + 2 = 0$. Hence D = 1 ± i.

$\therefore$ C.F. = $e^x(c_1 \cos x + c_2 \sin x)$

$$\text{P.I.} = \left[\frac{1}{D^2 - 2D + 2}\right]\left(e^x \sin x\right) = \left[\frac{e^x}{(D+1)^2 - 2(D+1) + 2}\right]\sin x$$

$$= \left[\frac{e^x}{D^2 + 1}\right]\sin x = \left[\frac{e^x}{(D+i)(D-i)}\right]\sin x$$

$$= e^x \text{ Imaginary part of } \left[\frac{1}{(D+i)(D-i)}\right]e^{ix}$$

$$= e^x \text{ Imaginary part of } \left[\frac{1}{2i}xe^{ix}\right]$$

$$= e^x \text{ Imaginary part of } \left[-\frac{1}{2}ix(\cos x + i \sin x)\right]$$

$$= -\frac{1}{2}xe^x \cos x$$

Alternatives $P.I. = \dfrac{e^x}{\left(D^2 + 1\right)}\sin x$

[Here $f(-1)^2 = 0$ cases failure.]

$$= \frac{xe^x}{2D}\sin x = \frac{-x}{2}e^x \cos x$$

$\therefore$ The general solution is y = C.F. + P.I.

Case – V :

When X is any other function of x.

Here P.I. = $\dfrac{1}{f(\text{D}+\text{a})}\text{X}$

If $f(D) = (D - m_1)$ ....$(D - m_n)$, resolving into partial fractions

$$\frac{1}{f(D)} = \frac{A_1}{(D - m_1)} + \frac{A_2}{(D - m_2)} + \dots\dots.. + \frac{A_n}{(D - m_n)}$$

$$\text{P.I.} = \frac{A_1}{(D - m_1)} + \frac{A_2}{(D - m_2)} + \dots\dots.. + \frac{A_n}{(D - m_n)}\text{X}$$

$$= A_1\frac{1}{(D - m_1)}X + A_2\frac{1}{(D - m_2)}X + \dots\dots.. + A_n\frac{1}{(D - m_n)}X$$

$$= A_1 e^{m_1 x}\int Xe^{-m_1 x}dx + A_2 e^{m_2 x}\int X e^{-m_2 x} + \dots\dots. + A_n e^{m_n x}\int Xe^{-m_n x}dx$$

Theorem, If V is a function of x

then $\frac{1}{f(D)}xV = x\frac{1}{f(D)}V + \left[\frac{d}{dD}.\frac{1}{f(D)}\right]V$

Example – 3 : Solve ($D^3 – 3D^2 + 3D – 1$) y = x^2e^x.

Solution : The auxiliary equation is $D^3 – 3D^2 + 3D – 1 = 0$

(i.e.) $(D – 1)^3 = 0$

$\therefore$ D = –1 (thrice).

$\therefore$ C.F. = $e^{-x}(c_1 + c_2x + c_3x^2)$.

$\therefore$ P.I. $= \left(\frac{1}{D^3 - 3D^2 + 3D - 1}\right)x^2e^x$

$= \left(\frac{e^x}{(D+1)^3 - 3(D+1)^2 + 3(D+1) - 1}\right)x^2$

$= e^x\left(\frac{1}{D^3}\right)x^2$

$= \frac{e^x x^5}{60}$ (By integrating x^2 thrice w.r.t. x)

$\therefore$ The general solution is y = C.F. + P.I.

$\therefore y = e^{-x}(c_1 + c_2x + c_3x^2) + \frac{x^5e^x}{60}$.

Illustrative Examples

Example –1 : *Find the general solution by using substitution. $y'' – 8y = 0$*

Solution: Here $y'' – 8y = 0$ which is a 2nd order homogeneous differential equation

Let $y = e^{mx}$ be the required solution.

$\Rightarrow y' = m\cdot^{mx}$ and $y'' = m^2\cdot e^{mx}$

$\Rightarrow m^2\cdot e^{mx} – 8e^{mx} = 0$

$\Rightarrow (m^2 – 8)e^{mx} = 0$

$\therefore$ The auxiliary eqution is

$m^2 – 8 = 0 \qquad (\because e^{mx} \neq 0)$

$\Rightarrow m^2 = 8$

$\Rightarrow m = \pm 2\sqrt{2}$ are two real and distinct roots.

$\therefore$ The general solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$

$\Rightarrow y = c_1e^{2\sqrt{2}x} + c_2e^{-2\sqrt{2}x}$

Example – 2 : *Solve the initial value problems $y'' – y = 0$, $y(0) = 3$, $y'(0) = –3$*

Solution : Here $y'' – y = 0$, $y(0) = 3$, $y'(0) = –3$

which is a 2nd order homogeneous differential equation

$\Rightarrow z^2e^{mx} - e^{mx} = 0$ Let $y = e^{mx} \neq 0$ be the required solution.

$\Rightarrow (m^2-1)e^{mx} = 0$ $\Rightarrow y' = m \cdot e^{mx}$ and $y'' = m^2 \cdot e^{mx}$

So the auxiliary equation is

$m^2 - 1 = 0$

$\Rightarrow m = \pm 1$ are two real and distinct roots.

$\therefore$ The general solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$

$$\Rightarrow y = c_1e^{x} + c_2e^{-x} \quad \text{.......... (1)}$$

$$\Rightarrow y' = c_1e^{x} - c_2e^{-x} \quad \text{.......... (2)}$$

Given $y(0) = 3$ i.e. for $x = 0, y = 3$

$\therefore$ equation (1) $c_1 + c_2 = 3$ .... (3)

$y'(0) = -3$ i.e. for $x = 0, y' = -3$

$\therefore$ equation (2) $c_1 - c_2 = -3$ .... (4)

$\therefore$ equation (3) + equation (4) $\Rightarrow 2c_1 = 0$ $\Rightarrow \boxed{c_1 = 0}$ $\Rightarrow \boxed{c_2 = 3}$

$\therefore$ equation (1) $\Rightarrow y = -3e^{-x}$ which is required particular solution.

Are the following functions be linearly independent of dependent on the given interval of the following examples.

Example – 3 : (a) x^2, $x^2 lnx\ (x \geq 1)$ **(b)** $\sin 2x, \cos x \sin x\ (x < 0)$

Solution : (a) *Let,* $y_1 = x^2, y_2 = x^2\ lnx\ (x \geq 1)$

Since $\dfrac{y_2}{y_1} = lnx$ (except for $x = 1$) $\Rightarrow \dfrac{y_2}{y_1} \neq$ constant.

Hence y_1, y_2 are linearly independent.

Note : The concepts of linear dependence and independence of functions refer to an infinite set (the points of I) but not merely to a single point.

(b) *Let,* $y_1 = \sin 2x, y_2 = \cos x \sin x\ (x < 0)$

Since $\dfrac{y_1}{y_2} = -2$ for all $x < 0$.

Hence y_1, y_2 are linearly dependent function.

Verify that the given function is a solution and derive the corresponding real general solution of the following examples.

Example – 4 : (a) $y = c_1e^{2\pi ix} + c_2e^{-2\pi ix}, y'' + 4\pi^2 y = 0$ **(b)** $y = c_1e^{(-k+2i)x} + c_2e^{(-k-2i)x}, y'' + 2ky' + (k^2+4)y = 0$

Solution : (a) *Let,* $y = c_1e^{2\pi ix} + c_2e^{-2\pi ix}, y'' + 4\pi^2 y = 0$............ (1)

which is a 2nd order homogeneous differential equation

Let $y = e^{mx} \neq 0$ be the required solution.

$\Rightarrow y' = me^{mx} \Rightarrow y'' = m^2e^{mx}$

$\therefore$ equation (1) becomes $m^2e^{mx} + 4\pi^2e^{mx} = 0$

$\Rightarrow (m^2 + 4\pi^2)e^{mx} = 0$

So the auxillary equation is

$m^2 + 4\pi^2 = 0$

$\Rightarrow m = \pm 2\pi i$ are two complex and distinct roots.

$\therefore$ The general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$\Rightarrow \left(c_1 e^{2\pi i x} + c_2 e^{-2\pi i x}\right)$$

(b) $y = c_1 e^{(-k+2i)x} + c_2 e^{(-k-2i)x}$, $y'' + 2ky' + (k^2+4)y = 0$

which is a 2nd order homogeneous differential equation

$\Rightarrow m^2 \cdot e^{mx} + 2\,kme^{mx} + (k^2+4)e^{mx} = 0$ Let $y = e^{mx} > 0$ be the required solution.

$\Rightarrow (m^2 + 2\,km + k^2 + 4)e^{mx} = 0$ $\Rightarrow y' = me^{mx}$

$\Rightarrow y'' = m^2 e^{mx}$

So the auxillary equation is

$$m^2 + 2km + (k^2+4) = 0$$

$$\Rightarrow m = \frac{-2k \pm \sqrt{4k^2 - 4k^2 - 16}}{2}$$

$$= \frac{-2k \pm 4i}{2} = -k \pm 2i$$

$\Rightarrow m = -k \pm 2i$ are two complex and distinct roots.

$\therefore$ The general solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$\Rightarrow y = c_1 e^{(-k+2i)x} + c_2 e^{(-k-2i)x}$$

Find a real general solutions of the following examples.

Example – 5 : ***$D-4;\ 3x^2+4x,\ 4e^{4x},\ \cos 2x - \sin 2x$***

Solution : $D-4;\ 3x^2+4x,\ 4e^{4x},\ \cos 2x - \sin 2x$

(i) Le $y_1 = 3x^2 + 4x$

$\Rightarrow Dy_1 = 6x + 4$

$\therefore (D-4)y_1 = Dy_1 - 4y_1 = 6x + 4 - 4(3x^2 + 4x)$

$= 6x + 4 - 12x^2 - 16x = -12x^2 - 10x + 4.$

$\Rightarrow (D-4)(3x^2+4x) = 4 - 10x - 12x^2.$

(ii) Le $y_2 = 4e^{4x}$

$\Rightarrow Dy_2 = 16e^{4x}$

$\therefore (D-4)y_2 = Dy_2 - 4y_2 = 16e^{4x} - 16e^{4x} = 0$

$\Rightarrow (D-4)4e^{4x} = 0.$

(iii) Le $y_3 = \cos 2x - \sin 2x$

$\Rightarrow Dy_3 = -2\sin 2x - \cos 2x$

$\therefore (D-4)y_3 = Dy_3 - 4y_3$

$= -2\sin 2x - 2\cos 2x - 4\cos 2x + 4\sin 2x$

$= 2\sin 2x - 6\cos 2x$

$\Rightarrow (D-4)(\cos 2x - \sin 2x) = 2\sin 2x - 6\cos 2x.$

Example – 6 : Solve, $(D^2 - 4D + 3)\,y = \sin 3x \cdot \cos 2x$

Solution : $(D^2 - 4D + 3)\,y = \sin 3x \cdot \cos 2x$

For finding out C.F.

It's A.E., $D^2 - 4D + 3 = 0$

$\Rightarrow \quad D^2 - 3D - D + 3 = 0 \Rightarrow D(D-3) - 1(D-3) = 0$

$\Rightarrow \quad (D-3)(D-1) = 0 \quad \Rightarrow D = 3, 1$

Hence, C.F. = $C_1 e^{3x} + C_2 e^x$

$$\text{P.I.} = \frac{\sin 3x . \cos 2x}{D^2 - 4D + 3} = \frac{1}{2}\frac{(\sin 5x + \sin x)}{D^2 - 4D + 3} = \frac{1}{2}\left[\frac{\sin 5x}{D^2 - 4D + 3} + \frac{\sin x}{D^2 - 4D + 3}\right]$$

$$\therefore \quad \text{P.I.} = \frac{1}{2}(\text{P.I.}_1 + \text{P.I.}_2)$$

$$P.I._1 = \frac{\sin 5x}{D^2 - 4D + 3} = \frac{\sin 5x}{-22 - 4D} (\because D^2 = -25)$$

$$= \frac{(-4D + 22)\sin 5x}{16D^2 - 484} = \frac{(-4D + 22)\sin 5x}{-884}$$

$$= -\frac{1}{884}(-4 \times 5\cos 5x + 22\sin 5x) = \frac{2}{884} \times (10\cos 5x - 11\sin 5x)$$

$$= \frac{1}{442}(10\cos 5x - 11\sin 5x)$$

$$\text{P.I.}_2 = \frac{\sin x}{D^2 - 4D + 3} = \frac{\sin x}{-1 - 4D + 3} = \frac{(2 + 4D)\sin x}{(2 + 4D)(2 - 4D)}$$

$$= \frac{(2 + 4D)\sin x}{4 - 16D^2} = \frac{(2 + 4D)\sin x}{20}$$

$$= \frac{2}{20}\sin x + \frac{4}{20}\cos x = \frac{1}{10}\sin x + \frac{1}{5}\cos x = \frac{1}{10}(\sin x + 2\cos x)$$

$$\text{Hence P.I.} = \frac{1}{2}\left[\frac{1}{442}(10\cos 5x - 11\sin 5x) + \frac{1}{10}(\sin x + 2\cos x)\right]$$

$$= \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

$\therefore \quad$ C.S. is y = C.F. + P.I.

$$\Rightarrow \quad y = C_1 e^{3x} + C_2 e^x + \frac{1}{884}(10\cos x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

Example – 7 : Solve, $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{-2x} + \sin 2x$

Solution : The given equation is in symbolic form

$(D^2 + 5D + 6)y = e^{-2x} + \sin x$

It's A.E. $D^2 + 5D + 6 = 0 \Rightarrow (D + 2)(D + 3) = 0$

$\Rightarrow D = -2, D = -3$

C.F. = $C_1 e^{-2x} + C_2 e^{-3x}$

P.I. $= \frac{1}{D^2+5D+6}\left(e^{-2x}+\sin x\right) = \frac{1}{D^2+5D+6}e^{-2x} + \frac{1}{D^2+5D+6}\sin x$

$= \frac{1}{4-10+6}e^{-2x} + \frac{1}{-1+5D+6}\sin x \;= x\cdot\frac{1}{2D+5}e^{-2x} + \frac{2}{5+5D}\sin x$

(cases failure)

$= \frac{x}{-4+5}e^{-2x} + \frac{5-5D}{25-25D^2}\sin x$

$= xe^{-2x} + \frac{5-5D}{25+25}\sin x = xe^{-2x} + \frac{5(\sin x-\cos x)}{50} = xe^{-2x} + \frac{1}{10}(\sin x - \cos x)$

C.S. i.e. $y = C_1 \;\; e^{-2x} + C_2e^{-3x} + xe^{-2x} + \frac{1}{10}(\sin x - \cos x)$

Example – 8 : Solve, $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$

Solution : $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$

For finding out C.F. it's A.E. is, $D^3 - D = 0 \Rightarrow D(D^2 - 1) = 0$

$\Rightarrow \quad D(D + 1)(D - 1) = 0 \Rightarrow D = 0, -1, 1$

Hence C.F. $= C_1 + C_2e^{-x} + C_3e^x$

P.I. $= \frac{2x+1+4\cos x+2e^x}{D^3-D}$

$= \frac{2x}{D^3-D} + \frac{1}{D^3-D} + \frac{4\cos x}{D^3-D} + \frac{2e^x}{D^3-D}$

$=$ P.I.$_1$ + P.I.$_2$+P.I.$_3$+P.I.$_4$

P.I.$_1 = \frac{2x}{D^3-D} = \frac{2x}{D(D^2-1)} = \frac{-2}{D}\frac{x}{(1-D^2)} = -\frac{2}{D}(1-D^2)^{-1}x$

$= -\frac{2}{D}(1+D^2+D^4+.....)x = -\frac{2}{D}(x) = -2\int x\,dx \;= -2\left(\frac{x^2}{2}\right) = -x^2$

P.I.$_2 = \frac{1}{(D^3-D)} = \frac{1}{D(D^2-1)} = \frac{x^0}{D(D^2-1)} = \frac{1}{D}\frac{x^0}{(D^2-1)}$

$= -\frac{1}{D}(1-D^2)^{-1}x^0 = -\frac{1}{D}\;(1 + D^2 + D^4 + ...)\,x^0$

$= -\frac{1}{D}x^0 = -\frac{1}{D}\times 1 = -\int dx = -x$

P.I.$_3 = \frac{4\cos x}{D^3-D} = \frac{4\cos x}{D(D^2-1)} = 4.\frac{\cos x}{-2D} = -2\sin x$

P.I.$_4 = \frac{2e^x}{D^3-D} = 2.\frac{e^x}{D^3-D}$

Hence case of failure occures

So, $\text{P.I.}_4 = 2x \cdot \dfrac{e^x}{3D^2-1} = 2x\dfrac{e^x}{2} = xe^x$

$\text{P.I.} = \text{P.I.}_1 + \text{P.I.}_2 + \text{P.I.}_3 + \text{P.I.}_4$

$= -x^2 - x - 2\sin x + xe^x$

$= -(x^2 + x) - 2\sin x + xe^x$

Hence is y = C.F. + P.I.

$\Rightarrow y = C_1 + C_2 e^{-x} + C_3 e^x + xe^x - (x^2 + x) - 2\sin x$

Example – 9 : Solve $\dfrac{d^4y}{dx^4} - y = \cos x . \cosh x$

Solution : $\dfrac{d^4y}{dx^4} - y = \cos x . \cosh x$

The above equation can be symbolically written by

$(D^4 - 1)y = \cos x \cdot \cosh x$

For finding out C.F. it's A.E. is, $D^4 - 1 = 0$

$\Rightarrow \quad (D^2 + 1)(D^2 - 1) = 0 \Rightarrow \quad D = \pm i, \pm 1$

Hence C.F. $= C_1 \cos x + C_2 \sin x + C_3 e^x + C_4 e^{-x}$

$$\text{P.I.} = \frac{\cos x . \cosh x}{D^4 - 1} = \frac{\cos x . \left(\frac{e^x + e^{-x}}{2}\right)}{D^4 - 1}$$

$$= \frac{1}{2}\left[\frac{e^x \cos x}{D^4 - 1} + \frac{e^{-x}\cos x}{D^4 - 1}\right] = \frac{1}{2}[P.I._1 + P.I._2]$$

$$\text{P.I.}_1 = \frac{e^x \cos x}{D^4 - 1} = e^x . \frac{\cos x}{(D+1)^4 - 1} = e^x . \frac{\cos x}{(D^2 + 2D + 2)(D^2 + 2D)}$$

$$= e^x . \frac{\cos x}{(2D+1)(2D-1)} = e^x . \frac{\cos x}{4D^2 - 1} = e^x . \frac{\cos x}{-5} = \frac{e^x}{-5}\cos x$$

$$\text{P.I.}_2 = \frac{e^{-x}\cos x}{D^4 - 1} = e^{-x} . \frac{\cos x}{(D-1)^4 - 1} = \frac{e^{-x}\cos x}{\left(D^2 - 2D\right)\left(D^2 - 2D + 2\right)} e^{-x} . \frac{\cos x}{(-2D+1)(-2D-1)}$$

$$= e^{-x} \frac{\cos x}{(1-2D)(-2D-1)} = e^{-x} \frac{\cos x}{4D^2 - 1} = \frac{e^{-x}}{-5}\cos x$$

$$\therefore \text{ P.I.} = \frac{1}{2}\left[\frac{e^x}{-5}\cos x + \frac{e^{-x}}{-5}\cos x\right] = \frac{\cos x}{-5}\left(\frac{e^x + e^{-x}}{2}\right) = -\frac{1}{5}\cos x . \cosh x$$

$\therefore$ Hence C.S. is y = C.F. + P.I.

$\Rightarrow y = C_1 \cos x + C_2 \sin x + C_3 e^x + C_4 e^{-x} - \dfrac{1}{5}\cos x . \cosh x$

Example – 10 : Solve $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=xe^x\sin x$ **.**

Solution : Given equation in symbolic form is $(D^2-2D+1)\,y=xe^x\sin x$

A.E. is $D^2-2D+1=0$ or $(D-1)^2=0$ so that D = 1, 1

$\therefore$ C.F. $=(c_1+c_2x)\,e^x$

$$P.I.=\frac{1}{(D-1)^2}e^x\cdot x\sin x=e^x\cdot\frac{1}{(D+1-1)^2}x\sin x=e^x\frac{1}{D^2}x\sin x=e^x\frac{1}{D}\int x\sin x\,dx$$

Integrating by parts $=e^x\dfrac{1}{D}\left[x(-\cos x)-\int 1(-\cos x)dx\right]=e^x\dfrac{1}{D}(-x\cos x+\sin x)$

$$=e^x\int(-x\cos x+\sin x)dx=e^x\left[-\left\{x\sin x-\int 1\cdot\sin x\,dx\right\}-\cos x\right]$$

$=e^x\,[-x\sin x-\cos x-\cos x]=-\,e^x\,(x\sin x+2\cos x)$

Hence the C.S. is $y=(c_1+c_2x)\,e^x-e^x\,(x\sin x+2\cos x)$.

Example – 11 : ***Solve the differential equation*** $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=e^x\cos x.$ ***[B.P.U.T. – 2004***

Solution : The given equation is

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^x\cos x \qquad \text{.... (1)}$$

The corresponding homogeneous equation of (1) is

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0 \qquad \text{.... (2)}$$

The auxiliary equation of (2) is

$m^2-4m+4=0$

or, $(m-2)^2=0$ or, $m=2,2$

So, the complementary function of (1) is

$$y_c=(c_1+c_2\,x)\,e^{2x} \qquad \text{.... (3)}$$

where c_1 and c_2 are arbitrary constants.

The nonhomogeneous part of the equation (1) is $e^x\cos x$ which is not in the complementary function y_c. We also note that the successive derivatives of $e^x\cos x$ are linear combination of $e^x\cos x$ and $e^x\sin x$.

Thus we assume a particular integral of (1) as.

$y_p\ =\mathrm{A}\,e^x\cos x+\mathrm{B}\,e^x\sin x.$

where A, B are arbitrary constants to be determined , such that y_p satisfies the given eq^n (1).

Now, $y_p'=\dfrac{dy_p}{dx}=\mathrm{A}e^x\cos x-\mathrm{A}e^x\sin x+\mathrm{B}\,e^x\sin x+\mathrm{B}\,e^x\cos x.$

and $y_p'' = \dfrac{d^2 y_p}{dx^2}$
$= -2A\, e^x \sin x + 2B\, e^x \cos x.$

Substituting the values of y_p, y_p' and y_p'' in (1), we get

$$-2A\, e^x \sin x + 2B\, e^x \cos x - 4\,(A\, e^x \cos x - A\, e^x \sin x + B\, e^x \sin x + B\, e^x \cos x) + 4\,(A\, e^x \cos x + B\, e^x \sin x) = e^x \cos x.$$

or, $2A\, e^x \sin x - 2\,B\, e^x \cos x = e^x \cos x.$

Equating the co-efficients of similar terms from both sides, we get

$2A = 0$ and $-2B = 1$ $\therefore$ $A = 0$ and $B = -\dfrac{1}{2}$

So, $y_p = -\dfrac{1}{2} e^x \sin x$

Hence the general solution of (1) is

$y = y_c + y_p$

i.e., $y = (c_1 + c_2\, x)\, e^{2x} - \dfrac{1}{2} e^x \sin x$.

6.4 : Solution of Linear Equations by Reducing the order

We consider the general nth order linear differential equation :

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + a_2(x)\frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\, y = X(x) \quad \dots (1)$$

where $a_i(x)$ $(i = 1, 2, \dots\dots, n)$ and X are functions of x. In the preceeding sections of this chapter we have developed a theory for the solutions of the linear equation (1) when $a_1, a_2, \dots\dots, a_n$ are constants. But the method depends on the knowledge of n linearly independent solutions of the corresponding homogeneous equation.

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + a_2(x)\frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\, y = 0 \quad \dots. (2)$$

But, if a_i's are arbitrary functions of x, then no general method can be suggested for finding general solution of (1). In this section, if one particular solution of (2) is given, then there is a technique for finding a second independent solution. This technique reduces the given equation to a linear equation that is one order lower than the original. This method is especially useful when the given equation is of second order. For, in that case, the reduced equation is a linear equation of order one which is outrightly solvable.

Let us consider the second order linear equation.

$$\frac{d^2 y}{dx^2} + a(x)\frac{dy}{dx} + b(x) y = X(x) \quad \dots. (3)$$

We suppose that $y = u$ (where u is some function of x) be a solution of the reduced equation.

$$\frac{d^2 y}{dx^2} + a\frac{dy}{dx} + by = 0 \quad \dots. (4)$$

We set $y = uv$ in the equation (1) where v is a function of x.

Then $\dfrac{dy}{dx} = u\dfrac{d^2v}{dx^2} + v\dfrac{d^2u}{dx^2} + 2\dfrac{du}{dx}\cdot\dfrac{dv}{dx}$

Substituting the expressions for y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$v\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + au\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu\right)v = \mathrm{X}$$

or, $\dfrac{d^2v}{dx^2} + a_1\dfrac{dv}{dx} = \mathrm{X}_1$ ... (4)

where $a_1 = \dfrac{2}{u}\dfrac{du}{dx} + a$ and $\mathrm{X}_1 = \dfrac{\mathrm{X}}{u}$

and $\dfrac{d^2u}{dx^2} + a\dfrac{du}{dx} + bu = 0$, since u is a solution of (2).

Putting $\dfrac{dv}{dx} = w$ in (4), we have

$$\frac{dw}{dx} + a_1 w = \mathrm{X}_1 \quad \text{... (5)}$$

which is a linear equation in w of first order. An integrating factor of (5) is

$$e^{\int\left(\frac{2}{u}\frac{du}{dx}+a\right)dx} = u^2e^{\int a\,dx}$$

Multiplying the equation (5) by $u^2e^{\int a\,dx}$, we get

$$\frac{d}{dx}\left\{wu^2e^{\int a\,dx}\right\} = \frac{\mathrm{X}}{u}\,u^2e^{\int a\,dx}$$

Integrating, $wu^2e^{\int a\,dx} = \mathrm{A} + \int u \times e^{\int a\,dx}\,dx$

Since $v = \dfrac{dw}{dx}$, we have

$$v = \mathrm{B} + \mathrm{A}\int\frac{1}{u^2}e^{-\int a\,dx}dx + \int\left\{\frac{1}{u^2}e^{-\int a\,dx}\int u \times e^{\int a\,dx}dx\right\}dx$$

where A and B are arbitrary constants.

Hence the general solution of (3) is

$y = uv$.

$$= \mathrm{B}u + \mathrm{A}u\int\frac{1}{u^2}e^{-\int a\,dx}\,dx + u\int\left\{\frac{1}{u^2}e^{-\int a\,dx}\int u \times e^{\int a\,dx}\,dx\right\}dx$$

where A and B are arbitrary constants.

Note : Some solutions of the homogeneous linear equation.

$$\frac{d^2 y}{dx^2} + a\frac{dy}{dx} + by = 0 \ (a, b \text{ are arbitrary functions})$$

Can be determined by inspection. They are :

(i) $y = x$ is a particular solution if $a + bx = 0$.

(ii) $y = e^x$ is a particular solution if $1 + a + b = 0$

(iii) $y = e^{-x}$ is a particular solution if $1 - a + b = 0$.

(iv) $y = e^{mx}$ is a particular solution if $m^2 + am + b = 0$.

Example – 1 : Using the method of reduction of order, solve the following differential equation $(1 - x^2) y'' - 2xy' + 2y = 0$. Given that $y_1 = x$ is a solution.

Solution : $$(1-x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \quad \text{.... (1)}$$

We first observe that $y_1 = x$ satisfies the equation (1). We set $y = vx$ in the equation (1) where v is a function of x.

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $\frac{d^2 y}{dx^2} = x\frac{d^2 v}{dx^2} + 2\frac{dv}{dx}$ substituting the expressions for y, $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in (1), we get,

$$(1-x^2)\left(x\frac{d^2 v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + 2vx = 0$$

or, $$x(1-x^2)\frac{d^2 v}{dx^2} + 2(1-2x^2)\frac{dv}{dx} = 0$$

$$\frac{d^2 v}{dx^2} + \frac{2(1-2x^2)}{x(1-x^2)}\frac{dv}{dx} = 0 \quad \text{... (2)}$$

We put $\frac{dv}{dx} = u$

Then the equation (2) becomes

$$\frac{du}{dx} + \frac{2(1-2x^2)}{x(1-x^2)}u = 0 \ \ (x \neq 0, 1)$$

which is a linear equation in u of order one.

or, $$\frac{du}{u} + \frac{2(1-x^2-x^2)}{x(1-x^2)}dx = 0 \quad \text{or,} \quad \frac{du}{u} + \left(\frac{2}{x} - \frac{2x}{1-x^2}\right)dx = 0$$

Integrating,

$$\log u + \log x^2 + \log(1 - x^2) = \log c_1,$$

where c_1 is a positive arbitrary constant.

or, $$u = \frac{c_1}{x^2(1-x^2)} \quad \text{or,} \quad \frac{dv}{dx} = \frac{c_1}{x^2(1-x^2)} \quad \left[\because u = \frac{dv}{dx}\right]$$

or, $dv = c_1\left(\frac{1}{x^2}+\frac{1}{1-x^2}\right)dx$

Integrating, $v = c_1\left(-\frac{1}{x}+\frac{1}{2}\log\left|\frac{1-x}{1+x}\right|\right)+c_2$, (where c_2 is an arbitrary constant)

Hence the general solution of the equation (1) is

$$y = c_1\left(-1+\frac{1}{2}x\log\left|\frac{1-x}{1+x}\right|\right)+c_2 x$$

Example – 2 : ***Solve the equation (x – 1) y″ – xy′ + y = 0 by reducing the order using y = e^x as one of the solution.*** ***[B.P.U.T., 2007***

Solution : The given equation is

$$(x-1)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0 \qquad .... (1)$$

We note that $y = e^x$ satisfies the equation (1). We set $y = e^x v$ in the equation (1) where v is a function of x.

Now, $\frac{dy}{dx} = e^x v + e^x \frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = e^x v + 2e^x\frac{dv}{dx} + e^x\frac{d^2v}{dx^2}$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(x-1)\left(e^x v+2e^x\frac{dv}{dx}+e^x\frac{d^2v}{dx^2}\right)-x\left(e^x v+e^x\frac{dv}{dx}\right)-e^x v = 0$$

or, $(x-1)\,e^x\frac{d^2v}{dx^2}+(2xe^x-2e^x-x\,e^x)\frac{dv}{dx}+(x\,e^x v-e^x v-xe^x v+e^x v) = 0$

or, $(x-1)\,e^x\frac{d^2v}{dx^2}+(x-2)e^x\frac{dv}{dx} = 0$

or, $\frac{d^2v}{dx^2}+\frac{x-2}{x-1}\frac{dv}{dx} = 0$ or, $\frac{dw}{dx}+\frac{x-2}{x-1}w = 0$, where $w = \frac{dv}{dx}$

or, $\frac{dw}{w}+\frac{x-2}{x-1}dx = 0$

or, $\frac{dw}{w}+\left(1-\frac{1}{x-1}\right)dx = 0$

Integrating, $\log w + x - \log(x-1) = \log c_1$
where c_1 is a positive arbitrary constant.

or, $\log \dfrac{w}{c_1(x-1)} = -x$ or, $w = c_1 (x-1) e^{-x}$.

or, $\dfrac{dv}{dx} = c_1 (x-1) e^{-x}$

or, $dv = c_1 (x-1) e^{-x} dx$.

Integrating, $v = c_1 (-xe^{-x} + e^{-x} - e^{-x}) + c_2$

where c_2 is an arbitrary constant.

Hence the general solution of (1) is $y = ve^x$.

i.e., $y = -c_1 x + c_2 e^x$.

where c_1 and c_2 are arbitrary constants.

6.5 : Cauchy-Euler Homogeneous Equations

Equation Reducible to linear equations with constant co-efficients

A. Cauchy's homogeneous linear equations :

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n y = X \qquad \ldots.(1)$$

i.e. $(x^n D^n + a_1 x^{n-1} D^{n-1} + \ldots\ldots\ldots + a_n)\, y = X$ ....(2)

Where $a_1, a_2 \ldots\ldots\ldots a_n$ are constants and X is either a constant or a function of x only is called a homogeneous linear differential equation. Note that the index of x and the order of derivative is same in each term of such equations .These are also known as Cauchy Euler equations.

Where D stands for $\dfrac{d}{dx}$, D^2 for $\dfrac{d^2}{dx^2}$ ..... D^n for $\dfrac{d^n}{dx^n}$

Working Rule for finding solution of (1) or (2).

To solve (1) or (2) , we generally change the variable x to t, by putting $x = e^t$, *i.e.*

$t = \log x$, *i.e.* $e^{\log x} = x$

$$t = \log x, \quad \frac{dt}{dx} = \frac{1}{x} \qquad \ldots(3)$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{x}.\frac{dy}{dt} \qquad \ldots(4)$$

$$x\frac{dy}{dx} = \frac{dy}{dt}, \left(\frac{d}{dt} = D\right) \text{ or, } x\frac{dy}{dx} = Dy \qquad \ldots.(5)$$

$$\text{Again } \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right).\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}.\frac{d^2y}{dt^2}.\frac{1}{x} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}.\frac{d^2y}{dt^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

or, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} = (D^2 - D)y = D(D-1)y$ .... (6)

and so on. Thus we have the following formulas.

$x\frac{dy}{dx} = Dy$, $\quad x^2\frac{d^2y}{dx^2} = D(D-1)y$ or $x^2D^2 = D(D-1)$

$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ or $x^3D^3 = D(D-1)(D-2)$

....

....

$x^n\frac{d^ny}{dx^n} = D(D-1).........(D-n+1)y$

Using (3) and (6), the given equation (2) becomes f(D) y = T, where T is now a function of t only.

Now D, D^2........etc will stand for diffferentiation once, with respect to t and so on; similarly $\frac{1}{D}, \frac{1}{D^2}$ will mean integration once twice with respcet to t and so on.

B. Legender's Linear Differential Equation:

An equation of the form $(ax+b)^n\frac{d^ny}{dx^n} + K_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... k_ny = X$

Where k's are constant and 'X' is a function of x, is called Legendre's linear equation. such equations can be reduced to linear equations with constant cefficients by putting $(ax+b) = e^t$, $t = \log(ax+b)$, $ax = e^t - b \Rightarrow x = \frac{e^t - b}{a}$

i.e. $(ax+b)\frac{dy}{dx} = a\,Dy$, $(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y$ and so on.

$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ and so on.

$$t^n\frac{d^ny}{dt^n} + \frac{k_1}{a}t^{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + \frac{k_{n-1}}{a^{n-1}}t\frac{dy}{dt} + \frac{k}{a^n}y = \frac{a}{a^n}F\left(\frac{t-b}{a}\right)$$

For example, we consider the equation

$$(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$$

Let us put $3x + 2 = t$

Then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = 3\frac{dy}{dt}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3\frac{dy}{dt}\right)$

$$= 3\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$= 9\frac{d^2y}{dt^2}$$

$\therefore$ The given equation reduces to

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - 4y = \frac{1}{9}\left\{\left(\frac{t-2}{3}\right)^2 + 4\left(\frac{t-2}{3}\right) + 1\right\}$$

which is an equation of Cauchy-Euler type and so it can be solved by the method given in section

Considering a Cauchy-Euler equation of order two in the following example we illustrates the above method.

Illustrative Examples

Example – 1 : Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 4y = x^4$

Solution : $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

Put x = e^t, $D = \frac{d}{dt}$, $x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$ in (1), we get

D (D – 1)y –2 Dy – 4y = e^{4t} or (D^2 – 3D – 4)y = e^{4t}

A.E. $\quad D^2 - 3D - 4 = 0 \Rightarrow (D-4)(D+1) = 0 \Rightarrow D = -1, 4$

C.F. $\quad$ C.F. = $C_1e^{-t} + C_2e^{4t}$

P.I. $\quad P.I. = \frac{1}{D^2 - 3D - 4}e^{4t}$

$$= t\cdot\frac{1}{2D-3}e^{4t} = t\frac{1}{2(4)-3}e^{4t} = \frac{te^{4t}}{5}$$

Thus, the complete solution is given by

$$y = C_1\frac{1}{x} + C_2x^4 + \frac{1}{5}x^4\log x.$$

Example – 2 : Solve, $x^2\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2\log x$

Solution : $x^3\frac{x^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3\log x$

Let $x = e^z$ so that $z = \log x,\ D \equiv \frac{d}{dz}$

The equation becomes after substitution

[D (D – 1) (D – 2) + 3D (D – 1) + D] y = ze^{3z}

or $D^3y = ze^{3z}$

Auxiliary equation is $D^3 = 0$, or D = 0, 0, 0]

$$\text{C.F} = C_1 + C_2 z + C_3 z^2 = C_1 + C_2 \log x + C_3(\log x)^2$$

$$\text{P.I.} = \frac{1}{D^3}\cdot ze^{3z} = e^{3z}\cdot\frac{1}{(D+3)^3}\cdot z$$

$$= e^{3z}\frac{1}{27}\left(1+\frac{D}{3}\right)^{-3}\cdot z = \frac{e^{3z}}{27}(1-D)z$$

$$= \frac{e^{3z}}{27}(z-1) = \frac{x^3}{27}(log\, x - 1)$$

Complete solution is $y = C_1 + C_2 \log x + C_3(\log x)^2 + \frac{x^3}{27}(\log x - 1)$

Example – 3 : Solve $\boldsymbol{(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x}$

Solution : $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Let $2x + 3 = e^t \Rightarrow 2x = e^t - 3 \Rightarrow x = \frac{e^t - 3}{2}$ then $\log(2x + 3) = t$

we know that $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} \qquad \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\times\frac{2}{2x+3}$

$\Rightarrow\ (2x + 3)\frac{dy}{dx} = 2Dy$ and $(2x + 3)^2\frac{d^2y}{dx^2} = 2^2D(D - 1)\, y$

Hence the above eq^n can be written as,

$[4D (D - 1) - 2D - 12]\, y = 6\left(\frac{e^t - 3}{2}\right)$

For finding out C.F. it's A.E. is, $4D^2 - 6D - 12 = 0$

$$D = \frac{6\pm\sqrt{228}}{8} = \frac{6\pm 2\sqrt{57}}{8} = \frac{3\pm\sqrt{57}}{4}$$

Hence C.F. = $c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)t}$

$$\text{P.I} = \frac{1}{4D^2 - 6D - 12} 3(e^t - 3)$$

$$\text{P.I.}_1 = 3 \,.\, \frac{e^t}{4D^2 - 6D - 12} = 3.\frac{e^t}{-14} = \frac{-3}{14} e^t$$

$$\text{P.I.}_2 = \frac{-9}{4\text{D}^2 - 6\text{D} - 12} = \frac{9}{12} = \frac{3}{4}$$

Hence C.S. is y = C.F. + P.I.

$$\Rightarrow \quad y = c_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)t} + c_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)t} - \frac{3}{14} e^t + \frac{3}{4}$$

$$\Rightarrow \quad y = c_1 (2x+3)\left(\frac{3+\sqrt{57}}{4}\right) + c_2 (2x+3)\left(\frac{3-\sqrt{57}}{4}\right) - \frac{3}{14}(2x+3) + \frac{3}{4}$$

Example – 4 : Solve $\boldsymbol{(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\{\log(1+x)\}}$

Solution : $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\{\log(2+x)\}$

Let $(1 + x) = e^t \Rightarrow t = \log(1 + x)$

Hence the above equation can be symbollically written as

[D(D – 1) + D + 1] y = 4 cos t

For finding out C.F., it's A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

C.F. = $c_1 \cos t + c_2 \sin t$

$$\text{P.I.} = 4 \frac{\cos t}{\text{D}^2 + 1} = 4.\text{t}.\frac{\cos t}{2\text{D}} = 2t \sin t$$

Hence C.S. is y = C.F. + P.I.

$\Rightarrow \quad y = c_1 \cos t + c_2 \sin t + 2t \sin t$

$\Rightarrow \quad y = c_1 \cos\{\log(1 + x)] + c_2 \sin\{\log(1 + x)\}] + 2\log(1 + x) \sin\{\log(1 + x)\}]$

6.6 : The Method of Undermined Co-efficients

The method of undetermined co-efficients is a procedure for finding particular solution y_p of the differential equation of the form.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_{n-1} \frac{dy}{dx} + a_n y = \text{X} \quad .... (1)$$

Where a_0, a_1, a_2, ..., a_n are constants and X either (1) a function defined by one of the following :

(*i*) x^n , where $n = 0, 1, 2,$ ....

(*ii*) e^{ax}, where a ($\neq 0$) is a constants.

(*iii*) sin ($bx + c$), where b ($\neq 0$) and c are constants.

(*iv*) cos ($bx + c$), where b ($\neq 0$) and c are constants.

or (2) a function defined as a finite product (or sum) of two or more functions of types given in (*i*), (*ii*), (*iii*) and (*iv*).

This method is useful only when X contains terms in some special forms. The form of a particular integral y_p can be inferred from the form of X. The following table suggests the form of the trial solution y_p (for particular integral) to be used corresponding to a special form of X.

TABLE

S.No.	**Special form of X**	**Trial solution y for P.I.**
1	x^n or $a_n x^n$ or $a_0 + a_1 x + a_2 x^2 + ...+a_n x^n$	$A_0 + A_1 x + ...+ A_n x^n$
2.	e^{ax} or pe^{ax}	Ae^{ax}
3.	$a_n x^n e^{ax}$ or $e^{ax}(a_0 + a_1 x + a_2 x^2 + ... L\ a_n x^n)$	$e^{ax}(A_0 + A_1 x + A_2 x^2 + ...+A_n x^n)$
4.	p sin ax or q cos ax or p sin ax + q cos ax	A sin ax + B cos ax
5.	pe^{bx} sin ax or qe^{bx} cos ax or e^{bx} (p sin ax + q cos ax)	e^{bx} (A sin ax + B cos ax)
6.	x^n sin ax or $a_n x^n$ sin ax or x^n cos ax or $a_n x^n$ cos ax or $(a_0 + a_1 x + ...+ a_n x^n)$ sin ax or $(a_0 + a_1 x + ...+ a_n x^n)$ cos ax	$(A_0 + A_1 x + ... + A_n x^n)$ sin ax + $(A_0' + A_1' x + ...+A_n' x^n)$ cos ax

Remark - 1 : In the above table, n is a positive integer and $a_0, a_1,......,a_n$, p, q, a, b, A_1, A_1, ...A_0', A_1', ...A_n' are constants. The constants occuring in second column are known and the constants occuring in third column are determined by subsituting the trial solution in given equation i.e., they are found from the resulting identity $f(D)y = X$.

Remark - 2 : If R.H.S. X of given equation $f(D)y = X$ is a linear combination of more than one special forms of the above table, then the trial solution must be taken as the sum of the corresponding trial solutions with appropriate constant coefficients to be evaluated later on.

For example, the function X may be defined as $X = x^3 + x^2 \sin 2x + e^{4x}$.

We consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = e^{4x} \quad .... (2)$$

We proceed to find a particular solution (y_p) of the equation (2). We observe that the differential equation (2) requires a solution which is such that its second derivaties, minimum twice its first derivative, minimum three times the solution itself add up to the exponential function e^{4x}. Since the successive derivatives of e^{4x} are $4e^{4x}$, $16e^{4x}$, $64e^{4x}$,........., it seems reasonable that the desire particular solution might be a constant multiple of e^{4x}.

Thus we assume $y_p = Ae^{4x}$..... (3)

where A is a constant (undetermined co-effcient) to be determined such that (3) is a solution of (2). Differentiating (3) with respect to x, we obtain

$$y_p' = 4Ae^{4x},\ y_p'' = 16Ae^{4x}$$

Then from (2), we get

$$16Ae^{4x} - 8Ae^{4x} - 3Ae^{4z} \qquad = e^{4x}.$$

or $\quad 5Ae^{4x} = e^{4x}$ .... (4)

Since the solution (3) is to satisfy the differential equation (2) identically for all x. On some real interval, relation (4) must be an identity in x and hence the co-efficient of e^{4x} on bothsides of (4) must be equal. Equating the co-efficient we get $A = \frac{1}{5}$. So, a particular solution of (2) will be $y_p = \frac{1}{5}e^{4x}$.

Now we consider the differential equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = e^{3x} \qquad \text{.... (5)}$$

which is exactly the same as equation (2) except that e^{4x} in right member has been replaced by e^{3x}. As before if we now assume a particular solution of the form

$$y_p = Ae^{3x} \qquad \text{.... (6)}$$

We find that

$$9Ae^{3x} - 6Ae^{3x} - 3Ae^{3x} = e^{3x}$$

or, $\quad 0 = e^{3x}$.

which does not hold for any real x. This impossible situation tells us that there is no particular solution of the assumed from (6).

We have already mentioned that the equations (2) and (5) are almost the same, only difference between them being the constant multiple of x in the exponents of their respective nonhomogeneous terms e^{4x} and e^{3x}. The equation (2) involving e^{4x} has a particular solution of the assumed form Ae^{4x} where the equation (5) involving e^{3x} has not any solution of the assumed form Ae^{3x}. The reason for the difference in the two cases is found by examining the solution of the homogeneous differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0 \qquad \text{.... (7)}$$

The auxiliary equation corresponding to (7) is $m^2 - 2m - 3 = 0$ whose roots are 3 and –1 and so e^{3x}, e^{-x} are linearly independent solutions of (7).

The indicates that the failure to obtain a solution of the form $y_p = Ae^{3x}$ for the equation (5) is due to the fact that the function e^{3x} in this assumed solution is a solution of the homogeneous equation (7) corresponding to (5).

Then for a particular solution of (5) we can assume a solution of the form $y_p = Axe^{3x}$.

We now proceed to present the method of undetermined co-efficients for finding a particular solutions through the following examples.

Illustrative Examples

Example – 1 : ***Solve the initial value problem $y'' + 4y = 4 \cos 2x$ with $y(0) = 0$, $y'(0) = 2$, using the method of undetermined co-efficients.*** *[B.P.U.T.– 2007*

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + 4y = 4 \cos 2x \;.... (1)$$

The auxiliary equation of the corresponding homogeneous equation.

$$\frac{d^2y}{dx^2} + 4y = 0 \qquad .... (2)$$

is $m^2 + 4 = 0$, which gives $m = \pm 2i$.

So, the complementary function of (1) is

$$y_c = c_1 \cos 2x + c_2 \sin 2x \qquad .... (3)$$

where c_1 and c_2 are arbitrary constants.

The nonhomogeneous part of the equation (1) is 4 cos $2x$ and cos $2x$ appears in the complementary function y_c. We replace the function cos $2x$ and sin $2x$ x cos $2x$ and x sin $2x$ respectively by asitis So, we assume a particular integral of (1) as

$$y_p = x\,(A \cos 2x + B \sin 2x)$$

where A and B are arbitrary constants to be determined such that y_p satisfies the given equation.

Now, $y_p' = (-2A \sin 2x + 2B \cos 2x)\, x + A \cos 2x + B \sin 2x$

and $y_p'' = (-4\,A \cos 2x - 4B \sin 2x)\, x + (-2A \sin 2x + 2B \cos 2x) - 2A \sin 2x + 2B \cos 2x$

Substituting the expressions for y_p and y_p'' in the equation (1), we have

$$(-4A \cos 2x - 4B \sin 2x)\, x - 4A \sin 2x + 4B \cos 2x + 4x\,(A \cos 2x + B \sin 2x) = 4 \cos 2x$$

or $4B \cos 2x - 4A \sin 2x = 4 \cos 2x$

Equating the co-efficients of similar terms from bothsides, we get

$4B = 4$ and $-4A = 0$ or, $B = 1$ and $A = 0$

so, $y_p = x \sin 2x$

Hence the general solution of (1) is $y = y_c + y_p$.

i.e., $y = c_1 \cos 2x + c_2 \sin 2x + x \sin 2x \qquad .... (4)$

Now, $y' = -2\, c_2 \sin 2x + 2c_2 \cos 2x + \sin 2x + 2x \cos 2x \qquad .... (5)$

Given conditions are $y(0) = 0$, $y'(0) = 2$

Using given conditions we get from (4) and (5),

$0 = c_1$ and $2 = 2c_2$

i.e., $c_1 = 0$ and $c_2 = 1$

$\therefore$ The required solution of (1) is

$y = \sin 2x + x \sin 2x$

Example – 2 : ***Solve the differential equation*** $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$ ***using the method of undetermined co-efficients.***

Solution : The given equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y \quad = 2x^2 + e^x + 2xe^x + 4e^{3x} \text{ .... (1)}$$

The auxiliary equation of the corresponding homogensous equation.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \text{ .... (2)}$$

is $m^2 - 3m + 2 = 0$

i.e., $(m - 1)(m - 2) = 0$

i.e., $m = 1, 2$

so, the complementary function of (1) is

$y_c \quad = c_1 e^x + c_2 e^{2x}$.

Where c_1 and c_2 are arbitrary constants.

The non-homogeneous part of (1) is $2x^2 + e^x + 2xe^x + 4e^{3x}$ which is the linear combination of x^2, e^x, xe^x and e^{3x}.

Now, the function x^2 and its e^x successive derivatives of x^2 are multiplies of x^2, x, 1. The function and its successive derivatives of e^x are e^x. The function xe^x and the successive derivatives of xe^x are expression involving xe^x and e^x. The function e^{3x} and the derivatives of e^{3x} are multiple of e^{3x}.

We now observe that the set $\{x^2, x, 1\}$ includes x^2 which is included in the complementary function y_c, e^x appears in y_c, xe^x of the set $\{xe^x, e^x\}$ occurs in y_c and e^{3x} also appears in y_c.

Thus we take a particular integral of (1) as

$y_p \quad = \mathrm{A}\,x^2 + \mathrm{B}\,x + \mathrm{C} + \mathrm{D}e^{3x} + \mathrm{E}\,x^2 e^x + \mathrm{F}xe^x$.

where A, B, C, D, E and F are arbitrary constants.

$\therefore \ y_p' = 2\mathrm{A}\,x\,\mathrm{B} + 3\mathrm{D}\,e^{3x} + \mathrm{E}x^2 e^x + 2\mathrm{E}\,xe^x + \mathrm{F}\,xe^x + \mathrm{F}e^x$.

and $y_p'' = 2\mathrm{A} + 9\mathrm{D}e^{3x} + \mathrm{E}\,x^2 e^x + 2\mathrm{E}\,xe^x + 2\mathrm{E}e^x + 2\mathrm{E}\,xe^x + \mathrm{F}e^x + \mathrm{F}\,xe^x + \mathrm{F}e^x$.

$\quad = 2\mathrm{A} + 9\mathrm{D}e^{3x} + \mathrm{E}x^2 e^x + 4\mathrm{E}xe^x + 2\mathrm{E}e^x + \mathrm{F}xe^x + 2\mathrm{F}e^x$.

Substituting y_p, y_p' and y_p'' in the equation (1), we get

$$(2\mathrm{A} - 3\mathrm{B} + 2\mathrm{C}) + (2\mathrm{B} - 6\mathrm{A})\,x + 2\mathrm{A}\,x^2 + 2\mathrm{D}\,e^{3x} - 2\mathrm{E}\,xe^x + (2\mathrm{E} - \mathrm{F})\,e^x = 2x^2 + e^x + 2xe^x + 4e^{3x}.$$

Equating co-efficients of like terms from bothsides, we have

$2A - 3B + 2C = 0,\ 2B - 6A = 0,$

$2A = 2,\ 2D = 4,$

$-2E = 2, 2E - F = 1,$

From the above relations we have

$A = 1, B = 3, C = \frac{7}{2}, D = 2, E = -1, F = -3$

So, the particular integral is

$y_p = x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2 e^x - 3xe^x.$

Hence the general solution of (1) is

$y = y_c + y_p.$

i.e., $y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$

Example – 3 : Solve $(D^3 + 2D^2 - D - 2)\, y = e^x + x^2$.

Solution : Given $(D^3 + 2D^2 - D - 2)\, y = e^x + x^2$ ....(1)

Hence the auxiliary equation is $D^3 + 2D^2 - D - 2 = 0$

or $D^2(D + 2) - 1\,(D + 2) = 0$ so that $D = 1, -1, -2$.

$\therefore$ C.F. $= c_1e^x + c_2e^{-x} + c_3e^{-2x}$. ....(2)

Corresponding to special form x^2 of R.H.S. of (9), we choose trial solution for P.I. as $A_0 + A_1x + A_2x^2$. Since e^x occurs in R.H.S. of (1) and it also occurs in the C.F. (2) corresponding to a root of multiplicity one, so we choose trial solution for P.I. as $A_3x\, e^x$ (note that term e^x is not included, since it already appears in C.F. (2) with arbitrary coefficient c_1). Combining the above two trial solutions, we attempt a trial solution for P.I. of the form.

$y_p = A_0 + A_1x + A_2x^2 + A_3x\, e^x$. ....(3)

Since y_p must satisfy (1), we have $(D^3 + 2D^2 - D - 2)\, y_p = e^x + x^2$

or $D^3y_p + 2D^2y_p - Dy_p - 2y_p = e^x + x^2$ ....(4)

$(3) \Rightarrow Dy_p = A_1 + 2A_2x + A_3e^x(x + 1)$ ....(5)

$(5) \Rightarrow D^2y_p = 2A_2 + A_3e^x\,(x + 2)$ ....(6)

$(6) \Rightarrow D^3y_p = A_3e^x(x + 3)$. ....(7)

Using (3), (5), (6) and (7), (4) reduces to

$A_3e^x(x + 3) + 4A_2 + 2A_3e^x(x + 2) - A_1 - 2A_2x - A_3e^x(x + 1)$

$- 2A_0 - 2A_1x - 2A_2x^2 - 2A_3xe^x = e^x + x^2$

or $-2A_2x^2 - 2(A_1 + A_2)x + 4A_2 - A_1 - 2A_0 + 6A_3e^x = e^x + x^2,$

which is an identity and so equating coefficients of like terms,

$-2A_2 = 1, -(A_1 + A_2) = 0, 4A_2 - A_1 - 2A_0 = 0, 6A_3 = 1$; hence

$A_2 = -\left(\frac{1}{2}\right), A_1 = \frac{1}{2}, A_0 = -\left(\frac{5}{4}\right), A_3 = \frac{1}{6}$. So from (3), P.I. = $-\left(\frac{5}{4}\right)+\left(\frac{1}{2}\right)x-\left(\frac{1}{2}\right)x^2+\left(\frac{1}{6}\right)xe^x$ and general solution is

$$y = c_1e^x + c_2e^{-x} + c_3e^{-2x} - \left(\frac{5}{4}\right)+\left(\frac{1}{2}\right)x-\left(\frac{1}{2}\right)x^2+\left(\frac{1}{6}\right)xe^x.$$

Example – 4 : Solve $\frac{d^2y}{dx^2} - 9y = x + e^{2x} - \sin 2x$. Using method of undetermined coefficients for finding particular integral.

Solution : $\frac{d^2y}{dx^2} - 9y = x + e^{2x} - \sin\ 2x$ .....(1)

$\therefore$ S.F. is $(D^2 - 9)\ y = x + e^{2x} - \sin 2x$

A.E. is $D^2 - 9 = 0 \therefore D = \pm 3$

C.F. = $c_1e^{3x} + c_2e^{-3x}$

Now, none of the terms in X (= $x + e^{2x} - \sin 2x$) is present in in C.F.

$\therefore$ We write the trial solutions corresponding to x, e^{2x}, $\sin 2x$ and sum them up.

$\therefore$ Trial solution for P.I.

$$y_p = (A_0 + A_1x) + (A_2e^{2x}) + (A_3 \sin 2x + A_4 \cos 2x)$$

Differentiate (2) w.r.t.x twice, we get

$$Dy_p = A_1 + 2A_2e^{2x} + 2A_3\cos 2x - 2A_4\sin 2x$$

$$D^2y_p = 4A_2e^{2x} - 4A_3\sin 2x - 4A_4\cos 2x$$

Substituting in (1) the values of y^p and D^2y^p

$4A_2e^{2x} - 4A_3\sin 2x - 4\ A_4\cos 2x - 9A_0 - 9A_1x - 9A_2e^{2x} - 9A_3\sin 2x - 9A_4 \cos 2x$

$= x + e^{2x} - \sin 2x - 9A_0 - 9A_1x - 5A_2e^{2x} - 13\ A_3\sin 2x - 13\ A_4\cos 2x = x + e^{2x} - \sin 2x$

Equating coefficients of like terms on the both sides

$-9A_0 = 0 \quad \therefore A_0 = 0, \quad -9A_1 = 1 \quad \therefore A_1 = -\frac{1}{9}$

$-5A_2 = 1 \quad \therefore A_2 = -\frac{1}{5}, \quad -13A_3 = -1 \quad \therefore A_3 = \frac{1}{13}$

$-13A_4 = 0 \quad \therefore A_4 = 0$

$\therefore A_0 = 0, A_1 = -\frac{1}{9}, A_2 = -\frac{1}{5}, A_3 = \frac{1}{13}, A_4 = 0$

$$\text{P.I.} = -\frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$$

$\therefore$ C.S. is y = C.F. + P.I. = $c_1e^{3x} + c_2e^{-3x} - \frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$