1

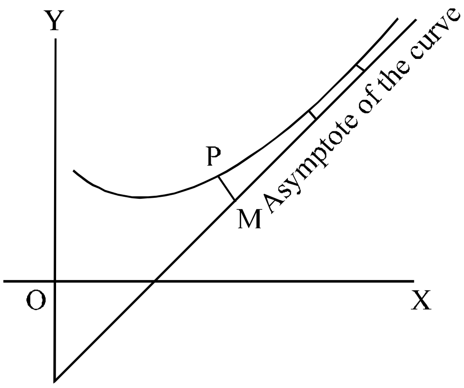
CHAPTER

Asymptotes

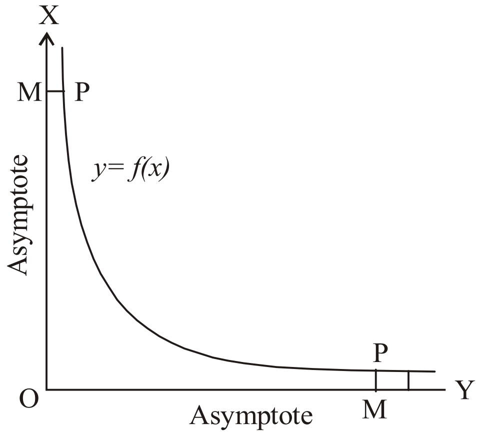
1.0 : Introduction

The curves like circles, ellipse etc. are finite in length i.e. those are limited extent , while the curves like parabola , hyperbola etc extend to infinity. If we draw a tangent at a point to any of the curves the tangent may further and farther away from the origin or it may keep oscillating or it may tend to a definite straight line.The straight line to which the tangent tends is known as the asymptote. Thus an asymptote is a straight line at a finite distance from origin, to which a tangent to a curve tends as the distance from the origin of the point of contact tends to infinity.

1. **Definition** – A straight line, at a finite distance from origin, is said to be an asymptote of the curve $y = f(x)$, if the perpendicular distance of the point P on the curve from the line tends to zero when x or y both tends to infinity.*(fig.1.1)*



(Fig.1.1)



(Fig.1.2)

Let $y = f(x)$ be a curve and (x,y) be a point on it.Tangent at (X,Y) is given by *(fig.1.2)*

$$Y - y = \frac{dy}{dx}(X - x)$$

or , $$Y = \frac{dy}{dx}.X + \left(y - x\frac{dy}{dx}\right) \qquad ....(1)$$

Now, if asymptote exists, then as x $\to \infty$, $\frac{dy}{dx}$ and $\left(y - x\frac{dy}{dx}\right)$ both will tend to finite limits, say 'm' and 'c' respectively. So that an asymptote must exist.Thus the tangent (I) tends to the asymptote $y = mx + c$ *if* $\frac{dy}{dx} = m$ and $\lim_{x\to\infty}\left(y - x\frac{dy}{dx}\right) = c$.

also $c = \lim_{x\to\infty}\left(y - x\frac{dy}{dx}\right)$ ....(2)

$$= \lim_{x\to\infty} y - \lim_{x\to\infty}(x)\cdot \lim_{x\to\infty}\left(\frac{dy}{dx}\right) \quad ....(3)$$

$$= \lim_{x\to\infty} y - m\lim_{x\to\infty}(x) = \lim_{x\to\infty}(y - mx) \quad ....(4)$$

$$\therefore \lim_{x\to\infty}\left(\frac{y}{x} - \frac{dy}{dx}\right) = c = \frac{\lim_{x\to\infty} y - x\left(\frac{dy}{dx}\right)}{x} = 0 \quad (\because \text{c is finite}).....(5)$$

i.e. $\lim_{x\to\infty}\frac{y}{x} = \lim_{x\to\infty}\frac{dy}{dx} = m$ ....(6)

i.e. $\lim_{x\to\infty}\left(\frac{y}{x} - \frac{dy}{dx}\right) = 0$ *i.e.* $\lim_{x\to\infty}\frac{y}{x} - \lim_{x\to\infty}\frac{dy}{dx} = 0$.

Note : A curve may have more than asymptotes.
Thus, if $y = mx + c$ is an asymptote to the curve $y = f(x)$ or $f(x, y) = 0$.

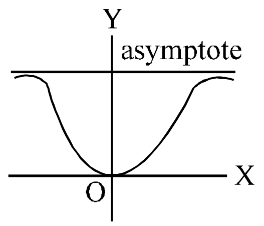
Then $m = \lim_{x\to\infty}\left(\frac{y}{x}\right)$ and $c = \lim_{x\to\infty}(y - mx)$

Rule – An equation of n[th] degree has one root infinite , if the coefficient of x^n is zero. It has two roots infinite, if the coefficients of x^n and x^{n-1} are both zero.

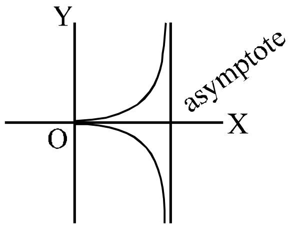
1.1 : Types of Asymptotes

Rectangular Asymptotes : If an asymptote to a curve is either parallel to x-axis or y-axis then it is called a rectangular asymptote. Asymptote parallel to x-axis is called horizontal asymptote and asymptote parallel to y-axis is called vertical asymptote.

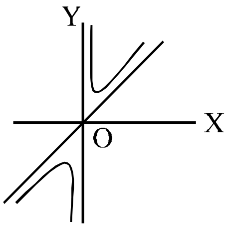
Oblique Asymptote : An asymptote which is neither parallel to x-axis nor parallel to *y-axis* is called an oblique asymptotes.



(Fig.1.3)



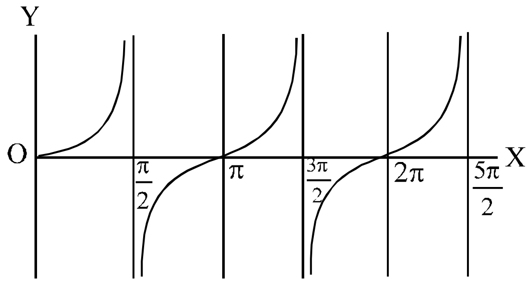
(Fig.1.4)



(Fig.1.5)

Example : What are the asymptotes of the curves y = tan$x$, y = cot$x$, y = sec$x$, y = cosec$x$.

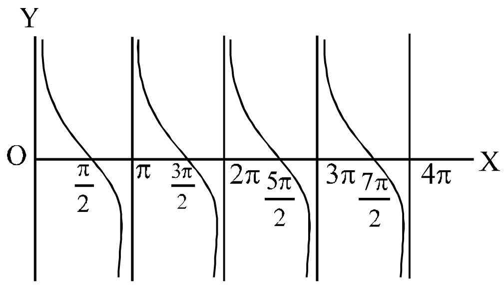
(i)



Graph of *y = tanx(Fig.1.6)*

From the above figure *(fig.1.6)* of the curve of the function y = tan$x$ it follows that the equation of the asymptotes of the curve $y = tanx$, are $x = \frac{\pi}{2}, x = \frac{3\pi}{2}, x = \frac{5\pi}{2}$ .... etc; In general $x = (2n+1)\frac{\pi}{2}$, for all $n \in Z$.

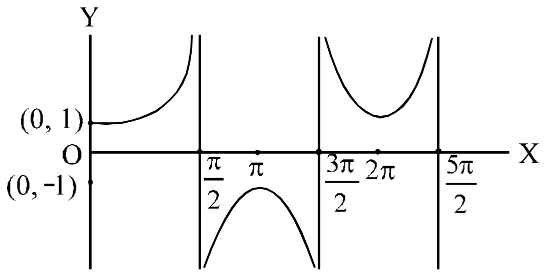
(ii)



Graph of *y = cot x (Fig.1.7)*

From the graph of the function y = cotx, it follows that the equations of the asymptotes are x=0, x=π, x=2π,.........etc, In general the equation of the asymptotes are given by x=nπ, for all n ∈Z.

(iii)

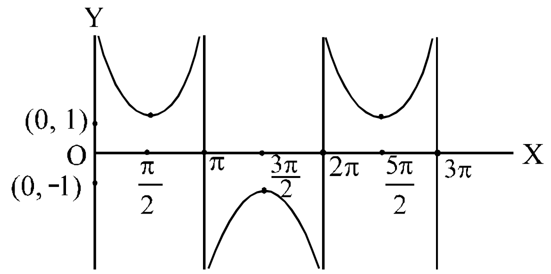


Graph of *y = sec x(Fig.1.8)*

So their equations of the asymptotes of the curve y = secx *(fig.1.8)* are $x = \frac{\pi}{2}, x = \frac{3\pi}{2}, x = \frac{5\pi}{2}$.................

etc. In general equation of the asymptotes is given by $x = (2n+1)\frac{\pi}{2}, \forall n \in Z$

(iv)

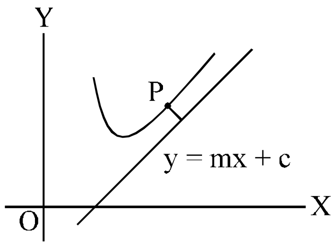


Graph of y = cosec x (Fig.1.9)

So the equations of the asymptotes of the curve y= cosecx *(fig.1.9)* are x=0, $x = \pi$, x= 2π,................etc. In general the equation of the asymptotes are given by $x = n\pi, \forall n \in Z$.

1.2 : Determination of Asmyptote of cartesian curve not parallel to co-ordinate axes.

Any straight line not parallel to y-axis is of the form $y = mx + c$ ....(i)

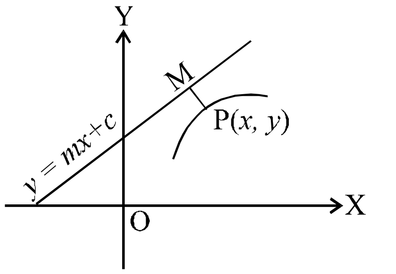


(Fig.1.10)

The abscissa x tends to infinity as the point P moves to infinity on the curve. We shall determine the values of m and c so that straight line (1) may be an asymptote of the curve.*(fig.1.10)*
If p=PM be the perpendicular distance of the point P(x, y) on the st.line (1) *(fig.1.11)*.

then $p = \dfrac{mx - y + c}{\sqrt{1 + m^2}}$

According to definition of asymptote $\lim\limits_{x \to \infty} p = 0 \Rightarrow \lim\limits_{x \to \infty} \dfrac{mx - y + c}{\sqrt{1 + m^2}} = 0$



(Fig.1.11)

i.e. $\lim\limits_{x \to \infty}(y - mx) = c$

Again $\left(\dfrac{y}{x} - m\right) = \left(\dfrac{y - mx}{x}\right)$

When $\lim\limits_{x\to\infty}\left(\frac{y}{x}-m\right)=\lim\limits_{x\to\infty}(y-mx).\frac{1}{x}=c.0=0 \Rightarrow \lim\limits_{x\to\infty}\frac{y}{x}=m$

Hence $\lim\limits_{x\to\infty}\frac{y}{x}=m$ and $\lim\limits_{x\to\infty}(y-mx)=c$

So we have the following method to determine the asymptotes which are not parallel to the y-axis.

Find $\lim\limits_{x\to\infty}\left(\frac{y}{x}\right), \lim\limits_{x\to\infty}(y-mx)$

Let $\lim\limits_{x\to\infty}\frac{y}{x}=m$ and let $\lim\limits_{x\to\infty} y-mx=c$

Then the straight line $y = mx + c$ is an asymptote.

Similarly we have the following method to determine the asymptote not parallel to x-axis.To determine such asymptote, we start with the equation $x= my+d$ where m is defined. then show that $\lim\limits_{x\to\infty}\left(\frac{x}{y}\right)$ and $\lim\limits_{x\to\infty}(x-my)=d$ then $x= my + d$ will be an asymptote.

1.3 : Asymptotes of algbraic curves

An asymptote which is not paralled to y-axis is called an oblique asymptote . Let $y= mx + c$ be an asymptote to the curve of $y= f(x)$, then

$$m=\lim_{\substack{x\to\infty \\ or \\ x\to-\infty}}\frac{y}{x},\quad c=\lim_{\substack{x\to\infty \\ or \\ x\to-\infty}}(y-mx)$$

Working Rule :

Method (I) – Suppose $y = mx + c$ is an asymptote of the curve. Put y = mx + c in the equation of the curve and arrange it in descending powers of x. Equate to zero the coefficients of two highest degree terms. Solve these two equations find m and c. Put them in $y = mx + c$ to get asymptotes. Notice must be taken that in this way we will find non parallel or non repeated asymptotes only. Notice must also be taken that all imaginary value of 'm' must be rejected.

Example – 1: ***Find the asymptotes of the given curve*** $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$.

Solution : Put $y = mx + c$ in the curves equation.

$x^3 + 2x^2(mx + c) - x(mx + c)^2 - 2(mx + c)^3 + 4(mx + c)^2 + 2x(mx + c) + (mx + c) - 1 = 0$

or , $x^3(1+2m - m^2 - 2m^3) + x^2(2c - 2mc - 6m^2c + 4m^2 + 2m) + ... = 0$

Equating to zero the coefficient of two highest degree terms in x,

we have $1 + 2m - m^2 - 2m^3 = 0$ ....(1)

and $2c(1 - m - 3m^2) + 2m^2 + m = 0$ ....(2)

(1) give $m = 1, -1, -\frac{1}{2}$, From (2), $c = 1, 1, 0$

Hence the asymptotes are $y = x + 1$, $y = -x + 1$, $y = -\frac{x}{2} + 0$

Example – 2: ***Find the asymptotes of the given curve $x^2 + y^3 = 3ax^2$*** ***[B.P.U.T. - 2007, 2010***

Solution: $x^3 + y^3 = 3ax^2$ which is a 3rd degree equation in x and y.

Let $\lim\limits_{x \to \infty} \left(\dfrac{y}{x} \right) = m$.

On dividing the given equation by x^3, we obtain $1 + \left(\dfrac{y}{x} \right)^3 = \dfrac{3a}{x}$

Let $x \to \infty$, then $m^3 + 1 = 0 \Rightarrow m = -1$

Again to find $\lim\limits_{x \to \infty} (y - mx)$ i.e when $m = -1$, $\lim\limits_{x \to \infty} (y + x)$.

We are to put $y + x = p$ so that p is a variable which tends to c when $x \to \infty$.

Putting $y = p - x$ in the given equation.

We obtain $x^3 + (p - x^3) = 3ax^2 \Rightarrow p^3 - 3p^2x + 3px^2 = 3ax^2$

On dividing by x^2, we get $\dfrac{p^3}{x^2} - \dfrac{3p^2}{x} + 3p = 3a$

When $x \to \infty$, we have $3p = 3a \Rightarrow p = a \Rightarrow c = a$.

Thus on putting m –1 and $c = a$ in the equation $y = mx + c$, we get the required asymptote as $x + y = a$.

1.4 : Asympototes Parallel to the Coordinate Axes

Now we shall discuss the rules for the determination of the asymptotes parallel to the coordinate axes of a rational algebraic curve.

Determination of asymptotes parallel to the Y–axis.

Let $x = K$............... (1) be an asymptote of the curve so that we are to determine K.

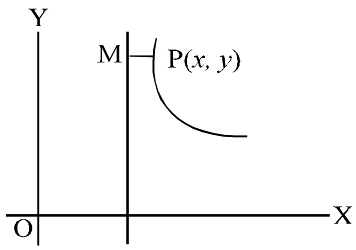
Here y alone tends to infinity as the point $P(x, y)$ moves infinity along the curve.

The distance PM of any point $P(x, y)$ on the curve from the line (1) is equal to $(x - K)$

$\therefore \lim\limits_{x \to \infty} (x - K) = 0$ when $y \to \infty$.

$\lim\limits_{x \to \infty} x = K$ when $y \to \infty$, Which gives K.

Thus to find the asymptotes parallel to Y-axis,
We find the definite value or values K_1, K_2 etc
to which x tends as y tends to infinity. *(fig.1.11)*
Then $x = K_1$ and $x = K_2$ are the requird asymptotes .



(fig.1.11)

Simple Rule to find Asymptotes of Rational Algebraic curve parallel to Y-axis.

We arrange the equation of the curve in descending powers of y. So that it takes the form

$$y^m \phi(x) + y^{m-1}\phi_1(x) + y^{m-2}\phi_2(x) + \ldots\ldots\ldots\ldots = 0 \qquad \ldots(1)$$

Where $\phi(x)$, $\phi_1(x)$, $\phi_2(x)$ etc are polynomials in x.

On dividing equation (1) by 'y^m' we get

$$\phi(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \ldots\ldots\ldots\ldots\ldots\ldots = 0 \qquad \ldots(2)$$

Let $y \to \infty$. We write $\lim x \to K$

$\therefore$ The equation (2) gives $\phi(K)=0$

So that K is a root of the equation $\phi(x) = 0$

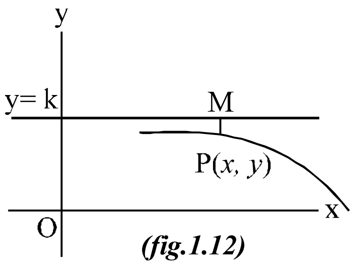
Let K_1, K_2 etc be the roots of $\phi(K) = 0$

Then the asymptotes parallel to Y-axis are $x = K_1$ and $x = K_2$.

Asymptotes parallel to x-axis

Similarly asymptotes parallel to the x-axis can be obtained by equating to zero the real linear factors in the coefficient of the highest power of x, in the equation of the curve.

Note : If the coefficient of highest power of x in the equation of the curve is a constant or it's linear factors are all imaginary there will be no asymptotes parallel to x-axis. *(fig.1.12)*



(fig.1.12)

We arrange the equation of the curve in descending powers of x. So that it takes the form $x^m\phi(y) + x^{m-1}\phi_1(y) + x^{m-2}\phi_2(y) + \ldots\ldots = 0$ ....(1)

Where $\phi(y)$, $\phi_1(y)$, $\phi_2(y)$ etc are polynomials in y on dividing equation (1) by x^m, we get

$$\phi(y) + \frac{1}{x}\phi_1(y) + \frac{1}{x^2}\phi_2(y) \ldots\ldots\ldots = 0 \qquad \ldots.(2)$$

Let $x \to \infty$, we write lim $y \to k$.

$\therefore$ The equation (2) gives $\phi(k) = 0$

So that k is a root of the equation, $\phi(y) = 0$

Let K_1, K_2 etc be the roots of $\phi(K) = 0$

Then the asymptotes parallel to X-axis are $y = K_1, K_2$.

The asymptotes parallel to the axis of y are obtained by equating to zero the real linear factors in the coefficient of the highest power of y in the equation of the curve.

Note. The coefficient of the heighest powers of y, in the equation of the curve is a constant or if it's linear factors are imaginary. There will be no asymptotes parallel to y-axis.

Example – 1: ***Find the asymptotes parallel to the x-axis of the curve $y^3 + x^2y + 2xy^2 - y + 1 = 0$***

Solution: The given curve is $y^3 + x^2y + 2xy^2 - y + 1 = 0$

Equating to zero the coefficient of the highest power of x(i.e., of x^2), the asymptote parallel to x-axis is given by $y = 0$.

Example –2: ***Obtain those asymptotes of the curve $x^4 + x^2y^2 = a^2(a^2 + y^2)$ which are parallel to the y-axis.***

Solution: The given curve is $x^4 + x^2y^2 - a^2(a^2 + y^2) = 0$

$\Rightarrow x^4 + (x^2 - a^2)y^2 - a^4 = 0$

Equating to zero the coefficient of the highest power of y(i.e., of y^2) , the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ i.e., $x = \pm a$.

Exapmle – 3: ***Find the asymptotes parallel to the coordinate axes of the curve $x^2(x - y)^2 + a^2(x^2 - y^2) - a^2xy = 0$.***

Solution: Equating to zero the coefficient of the highest power of y (i.e, of y^2) the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ i.e., $x = \pm a$. The coefficient of the highest power of x(i.e., of x^4) is merely a constant.Hence there is no asmyptote parallel to x-axis.

Example – 4 : ***Find the asymptotes of the curve*** $x^2y^2 - a^2(x^2+y^2) - a(x+y) + a^4 = 0$

Solution: The given curve is $x^2y^2 - a^2(x^2+y^2) - a(x+y) + a^4 = 0$

or, $(x^2 - a^2)y^2 - a^2x^2 - a(x+y) + a^4 = 0$

Equating to zero the coefficient of y^2, we get $x^2 - a^2 = 0 \Rightarrow x = \pm a$

$\therefore$ The vertical asymptotes are $x = \pm a$

The given equation can also be written as $(y^2 - a^2)x^2 - a^2y^2 - a(x+y) + a^4 = 0$

Equating to zero the coefficient of x^2, we get $y^2 - a^2 = 0 \Rightarrow y = \pm a$

$\therefore$ Horizontal asymptotes are $y = a, y = -a$.

Example – 5: ***Find the asymptotes for the curve,*** $y = \dfrac{2x^2 - 5x + 8}{5x^2 + 3x - 2}$

Solution: The given curve is $y(5x^2 + 3x - 2) = 2x^2 - 5x + 8$

or $x^2(5y - 2) + ... = 0$

Equating to zero the coefficient of x^2, we get $\therefore\ 5y - 2 = 0 \Rightarrow y = \dfrac{2}{5}$

$\therefore\ y = \dfrac{2}{5}$ is a horizontal asymptote.

Now, from the given equation $y(5x-2)(x-1) - (2x^2 - 5x + 8) = 0$

Equating to zero the coefficient of y, we get $5x - 2 = 0, x + 1 = 0$

$\Rightarrow\ x = \dfrac{2}{5}, x = -1$

$\therefore$ Vertical asymptotes are $x = \dfrac{2}{5}, x = -1$

Note.

- A curve of degree n has in general n asymptotes. This is because the equation $\phi_n(m) = 0$ (having degree n) has n roots and each value of m gives the corresponding value of c.
- In case some of the roots of $\phi_n(m) = 0$, are imaginary, the corresponding asymptotes are said to be imaginary.Thus the circle $x^2 + y^2 = k^2$ has imaginary asymptotes.
- There may be no asymptote corresponding to even a real root.Thus the parabola $y^2 = 4ax$ has no asymptotes even though the roots of $\phi_n(m) = 0$ are real. This is because the value of c becomes infinite.
- The number of asymptotes real or imaginary, of an algebraic curve of the nth degree cannot exceed n. This is because if the equation for determing c is quadratic $\phi_n(m) = 0$ and hence $\phi_n(m) = 0$ has a double roots (two equal roots).So the two values of c correspond to the the equal roots and there will be atmost $(n - 2)$ other sysmptotes corresponding to the remaining $(n - 2)$ roots. In case the equation for dertermining c is a cubic , the equation $\phi_n(m) = 0$ has three equal roots and so on. Hence a curve of degree n can not have more than n asymptotes.

1.5 : The Asymptopes of the General Rational Algebraic Curve

(Oblique Asymptotes)

Let $f(x, y) = 0$ be the equation of any rational algebraic curve of n^{th} degree.

Let the equation of the curve on being arranged in groups of homogeneous terms, be

$a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + ... + a_{n-1}yx^{n-1} + a_nx^n + b_1y^{n-1} + b_2y^{n-2}x + b_{n-1}yx^{n-2}$
$+ b_nx^{n-1} + c_2y^{n-2} + ... + = 0$ .....(1)

So we write equation (1) in form

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}.\phi_{n-2}\left(\frac{y}{x}\right) + ...x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0.......(2)$$

Dividing by x^n, we get $\phi_n\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + ...+ \frac{1}{x^n}\phi_0\left(\frac{y}{x}\right) = 0$

Let $x \to \infty$, $\lim\limits_{x\to\infty}\left(\frac{y}{x}\right) = m$

So the above equation reduces to $\phi_n(m) = 0$ which determines the slopes of the asymptotes.
Let m_1 be one of the roots of the equation $\phi_n(m) = 0$
So that $\phi_n(m_1) = 0$

We write $y - m_1x = c_1$ i.e. $\frac{y}{x} = m_1 + \frac{c_1}{x}$

Putting the valuse of (y/x) in equation (2) we get

$$x^n\phi_n\left(m_1 + \frac{c_1}{x}\right) + x^{n-1}.\phi_{n-1}\left(m_1 + \frac{c_1}{x}\right) + x^{n-2}\phi_{n-2}\left(m_1 + \frac{c_1}{x}\right) + ...= 0$$

Expanding each term by Taylors theorem we get

$$\Rightarrow x_n\left\{\phi_n(m_1) + \left(\frac{c_1}{x}\right)\phi_n'(m_1) + \frac{\left(\frac{c_1}{x}\right)^2}{2!}\phi_n''(m_1) + \frac{\left(\frac{c_1}{x}\right)^3}{3!}\phi_n''(m_1) + ...\right\}$$

$$+x^{n-1}\left\{\phi_{n-1}(m_1) + \left(\frac{c_1}{x}\right)\phi'_{n-1}(m_1) + ...\right\}$$

$$+x^{n-2}\left\{\phi_{n-2}(m_1) + \left(\frac{c_1}{x}\right)\phi_{n-2}'(m_1) + \frac{\left(\frac{c_1}{x}\right)^2}{2!}\phi_{n-2}''(m_1) + ...\right\} + ...0 \qquad .....(3)$$

Since $\phi_n(m_1) = 0$ so on dividing the above equation by x^{n-1},

We get $[c_1\,\phi_n'(m_1) + \phi_{n-1}(m_1)] + \left[\frac{c_1^2}{2}\phi_n''(m_1) + c_1\phi_{n-1}(m_1) + \phi_{n-2}(m_1)\right]\frac{1}{x} + ... = 0$

and taking the limit $x \to \infty$. $c_1\phi_n'(m_1) + \phi_{n-1}(m_1) = 0$

$$\Rightarrow C_1 = \frac{-\phi_{n-1}(m_1)}{\phi_n'(m_1)}$$

Let $x \to \infty$, There for we get $C_1 = \frac{-\phi_{n-1}(m_1)}{\phi_n'(m_1)}$

where $\phi'_n(m_1) \neq 0$

So $y = m_1 x - \dfrac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}$ is the asymptote corresponding to the slope m_1 where $\phi'_n(m_1) \neq 0$.

Simillarly $y = m_2 x - \dfrac{\phi_{n-1}(m_2)}{\phi'_n(m_2)}$: $y = m_3 x = \dfrac{\phi_{n-1}(m_3)}{\phi'_n(m_3)}$ etc. are asymptotes of the curve corresponding to the slopes m_2, m_3 etc.

If $\phi'_n(m_1)=0$, then on dividng the equation (3) by x^{n-2}

we get $\dfrac{C_1^2}{2}\phi''_n(m_1) + C_1\phi'_{n-1}(m_1) + \phi_{n-2}(m_1) = 0$

Let $x\to\infty$, which gives two values of c_1 say c'_1 and c''_1 provided that $\phi''_n(m_1)\neq 0$.

Then $y = m_1 x + c'_1$ and $y = m_1 x + c''_1$ are two parallel asymptotes corresponding to the slope m_1. This is known as the case of parallel asymptotes.

An rule for finding the Asymptotes

- An asymptote is a straight line which cuts a curve in two points at an infinite distance from the origin and yet is not itself wholly at infinity.
- Substitue $mx_1 + c_1$ for y in the equation of the curve and arrange it in descending powers of x.
- By equating the coefficients of the two highest powers of x to zero, determine m and c.
- Substitute these values of m_1 and c_1 in $y = mx_1 + c_1$.
- If a value of m_1 makes the coefficient of x^{n-1} identically zero, find c_1 from the equation obtained by equating to zero the cofficient of x^{n-2} and so on.

Working Method – To find an oblique asymptote of an algebraic curve , If the curve is an n^{th} degree equation, first compute $\phi_n(m)$, $\phi_{n-1}(m)$ ....by putting $x=1$, $y=m$ in the respective terms of degree $n,(n-1)$........... in the given equation of curve . Equate to zero $\phi_n(m)$ and solve it, this gives us n values of m say m_1, m_2...........m_n... Thus, lines with slopes m_1, m_2.... may give equation of asymptotes.

Case - I : If $\phi'_n(m_i)\neq 0$, $i = 1,2,3$ .... Then c_i corresponding to this values m_i is given by

$$c = -\frac{\phi_{n-1}(m_i)}{\phi'_n(m_i)}, i = 1,2,3...........$$

Thus, the corresponding asymptote is $y = m_i x + c_i$

Case - II : If $\phi'_n(m_i) = 0$ and $\phi_{n-1}(m_i)\neq 0$ for some i.

In this case there is no asymptote corresponding to slope m_i.

Case - III : If $\phi'_n(m_i) = 0$ and $\phi_{n-1}(m_i) = 0$ for some i, then c given by roots of the quadratic.

$$\frac{c^2}{2}\phi''_n(m_i) + c\phi'_{n-1}(m_i) + \phi_{n-2}(m_i) = 0$$

If c_1 and c_2 are the roots of this equation, then the equations of two parallel asymptotes are given by

$$y = m_1 x + c_1 \text{ and } y = m_2 x + c_2$$

Illustrative Examples

Example – 1: ***Find the asymptotes of the curve $x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0$***

Solution: The given curve is $(x^3 + 2x^2y - xy^2 - 2y^3) + (x^2 - y^2) - (2x + 3y) = 0$

Putting $x = 1, y = m$, we get $\phi_3(m) = 1 + 2.1.m - 1.m^2 - 2.m^3 = 1 + 2m - m^2 - 2m^3$

$\phi_2(m) = 1 - m^2$

$\phi_1(m) = -(2 + 3m)$

Now $\phi_3(m) = 0 \Rightarrow 1 + 2m - m^2 - 2m^3 = 0 \Rightarrow$ m = ± 1, m = $-\dfrac{1}{2}$

Now , $\phi'_3(m) = 2 - 2m - 6m^2$

$$c = -\frac{\phi_2(m)}{\phi'(m)} = -\left(\frac{1-m^2}{2-2m-6m^2}\right)$$

When $m = 1$, $c = -\dfrac{1-(1)^2}{2-2(1)-6.(1)^2} = 0$, When $m = -1$, $c = -\dfrac{1-(-1)^2}{2-2(-1)-6(-1)^2} = 0$

When $m = -\dfrac{1}{2}$, $c = \dfrac{-\left[1-\left(\dfrac{-1}{2}\right)^2\right]}{2-2\left(\dfrac{-1}{2}\right)-6\left(\dfrac{-1}{2}\right)^2} = \dfrac{-1}{2}$

$\therefore$ The asymptotes are $y = x$, $y = -x$ and $y = -\dfrac{1}{2}x - \dfrac{1}{2}$

i.e. $y = \pm x$, $2y + x + 1 = 0$

Example – 2 : ***Find the asymptote of the curve $x^3 + y^3 = 3ax^2$*** ***[B.P.U.T. - 2007***

Solution: Putting $x = 1, y = m$ in the 3rd degree and 2nd degree terms seperately, we get

$\phi_3(m) = 1 + m^3$

$\phi_2(m) = -3a$

$\phi'_3(m) = 3m^2$

One equating $\phi_3(m)$ to zero we obtain $1 + m^3 = 0$

$\Rightarrow$ $(1 + m)(1 - m + m^2) = 0$

$\Rightarrow$ $(m + 1) = 0$ or $(m^2 - m + 1) = 0$

In 2nd case the value of m is imaginary.

The corresponding values of c is given by $c = \dfrac{-\phi_2(-1)}{\phi'_3(-1)} = \dfrac{3a}{3} = a$

On putting the values of m and c in the equation $y = mx + c$ we get the required asymptote as $y = -x + a \Rightarrow x + y = a$.

Example – 3 : ***(a) Examine the Folium $x^3 + y^3 - 3axy = 0$ for asymptotes***

or Find the asymptotes to the curve $x^3 + y^3 = 3axy$ ***[B.P.U.T. - 2010***

Solution: The given curve is $x^3 + y^3 - 3axy = 0$ ....(1)

Putting $x = 1$ and $y = m$ in the third and second degree terms separately in eq^n (1) we get

$\phi_3(m) = 1 + m^3$ and $\phi_2(m) = -3am$

Solving the equation $\phi_3(m) = 0$, we get $1+m^3 = 0 \Rightarrow (1+m)(1-m+m^2) = 0$

$\Rightarrow m = -1$ or $1 - m + m^2 = 0$

But $1 - m + m^2 = 0$ give the imaginary values of m and hence rejected. So $m = -1$ is the only real root.

We have the formula $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = -\left\{-\dfrac{3am}{3m^2}\right\} = \dfrac{a}{m}$

Putting $m = -1$, we get $c = -a$

Thus the only real asymptote to the curve is $y = -x - a$ i.e., $x + y + a = 0$

(b) ***How many asymptotes does a straight line have ?***

Solution : As a straight line is of degree 1, it can have atmost 1 asymptote.

(c) ***Find the asymptotes of the curve whose parametric equations are given by x= cosh t, y= sinh t where t is a parameter.***

Solution: The given curve is given by $x = \cosh t$, $y = \sinh t$

$\therefore y^2 = \sinh^2 t = \cosh^2 t - 1 = x^2 - 1$ or $x^2 - y^2 - 1 = 0$

$\therefore \phi_2(m) = 1 - m^2$ and $\phi_1(m) = 0$, Also $\phi'_2(m) = -2m$. Now $\phi_2(m) = 0 \Rightarrow 1 - m^2 = 0 \Rightarrow m = \pm 1$.

So $c = -\dfrac{\phi_1(m)}{\phi_2'(m)} = -0/-2m = 0/2m$. For $m = \pm 1$, $c = 0$.

Hence the corresponding asymptotes are $y = \pm x$.

1.6 : Non-Existence of Asymptopes

If $\phi_n(m) = 0$, gives one or more values of m, such that they make $\phi_n(m) = 0$, whereas $\phi_{n-1}(m) \neq 0$, then from the equation $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$ we get $c = +\infty$ or $-\infty$. This corresponds to the case when the tangent goes further and further away from the origin as $x \to \infty$ .Thus for such values of m, we shall get no asymptotes.

Example – 4: ***Find the asymptotes of the curve $y^3 = x^2 + 3x$.***

Solution : The given curve is $y^3 = x^2 + 3x$

Putting $x = 1$, $y = m$ in the third and second degree terms separately, we have $\phi_3(m) = m^3$ and $\phi_2(m) = -1$.

So $\phi'_3(m) = 3m^2$

Now $\phi_3(m) = 0$, i.e., $m^3 = 0 \Rightarrow m = 0$, Also $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = \dfrac{1}{3m^2} = \infty$ for $m = 0$

Hence $y^3 = x^2 + 3x$ has no asymptotes

1.7 : Parallel Asymptotes

Now we shall discuss the method for the determination of two parallel asymptotes and three parallel asymptotes.

Two Parallel asymptotes

Suppose the equation $\phi_n(m) = 0$ gives two equal values of m.This repeated values of m makes $\phi'_n(m) = 0$.

If it does not make $\phi_{n-1}(m)$equal to zero, the value of c determind by the equation $c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}$ will infinite and hence the asymptotes corresponding to this values of m do not exist. Thus for the existence of the asymptotes corresponding to this value of m, it is necessary that it must make $\phi_{n-1}(m)$ equal to zero. This will happen when $\phi_n(m)=0$ has multiple roots. Then the equation $c\phi'_n(m)+\phi_{n-1}(m)=0$ from which c is usually determind reduces to the identity $0.c + 0 = 0$ and hence we can not find the value of c in this process. To determine c, in this case the coefficient of x^{n-2} in the eq^n will be equated to zero and we get the equation.

$$\frac{c^2}{2!}\phi''_n(m) + \frac{c}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

This equation is a quadratic in c. It gives two values of c, say c_1 and c_2, corresponding to that repeated value of m. The corresponding asymptotes are $y = mx + c_1$ and $y = mx + c_2$ which are parallel.

Three parallel asymptotes

If the equation $\phi_n(m) = 0$ gives three equal values of m, then this repeated value of m makes $\phi'_n(m)$ and $\phi''_n(m)$ equal to zero. For the exsistence of the corresponding asymptotes $\phi_{n-1}(m)$ must be equal to zero. If $\phi'_{n-1}(m)$ and $\phi_{n-2}(m)$ equal to zero, then the equation determining 'c' reduces to identity $0.\,c^2 + 0\cdot c + 0 = 0$ and hence we cannot find the values of c in this process. To determine 'c' in this case, the coefficient of x^{n-3} in the equation will be equal to zero and we get the equation

$$\frac{c^3}{3!}\phi''_n(m) + \frac{c^2}{2!}\phi''_{n-1}(m) + \frac{c}{1!}\phi'_{n-1}(m) + \phi_{n-3}(m) = 0$$

This equation gives three values of 'c' corresponding to that repeated value of 'm' and accordingly we get three parallel asymptotes.

Note : Suppose $\phi(m)c = 0$ *i.e.* $c = 0$. Then this inference would be correct only when the coefficient of c is non-zero otherwise 'c' cannot be defined.

The equation of the tangent to equation (1) at (x,y) is $Y - y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \ldots\ldots\right)(X - x)$

or $Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \right)X + y - \left(m - \frac{A}{x^2} - \frac{2B}{x^3}\ldots\right)x$

Substituting the value of 'y' from eq^n (1) in above equation we get

$$Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \ldots\right)X + c + \frac{A}{x} + \frac{2B}{x^2} + \ldots.. \quad \ldots.(2)$$

Now when $x \to \infty$, equation (2) tends to the equation Y =mX +c.

Hence y =mx +c is the asyptote of the curve $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} +$

1.8 : Position of the curve with respect to the asymptote

If the equation of a curve is of the form $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \ldots$

Then the following cases arise

Case - I : Let A ≠ 0. Let y_1 be the ordinate of the curved and y_2 that of the asymptote when the abscissa is x_1.

Then $y_1 = mx_1 + c + \frac{A}{x_1} + \frac{B}{x_1^2} + \frac{C}{x_1^3} + \ldots..$ & $y_2 = mx_1 + c$.

Now $(y_1 - y_2) = \frac{A}{x_1} + \frac{B}{x_1^2} + \frac{C}{x_1^3} + \ldots = \left(\frac{1}{x_1}\right)\left(A + \frac{B}{x_1} + \frac{C}{x_1^2} + \ldots\right)$

For sufficiently large values of x, the expression $A + \frac{B}{x_1} + \frac{C}{x_1^2} + \ldots$ has the sign of A.

If x_1 and A are of the same sign, then $y_1 > y_2$, *i.e.*, the curve is above the asymptote. If x_1 and A are of opposite sign, then $y_1 < y_2$ *i.e.,* the curve lies below the asymptote.

Case - II : Let A =0 B ≠ 0 we have $y_1 - y_2 = \left(\frac{1}{x_1^2}\right)\left(B + \frac{C}{x_1} + \frac{D}{x_1^2} + \ldots\ldots\right)$

For numerically sufficeintly large values of x, the expression. $B + \frac{C}{x_1} + \frac{D}{x^2{}_1} + \ldots..$

has the sign of B.

Hence the curve lies on the same side of the asymptote both the positive and negative values of x. It will be above or below the asymptote according as B is positive or negative.

Case - III : If B = 0 & C ≠ 0, then we will have a situation similar to that of case I.

1.9 : Asymptotes by Expansion

To show that $y = mx + c$ is an asymptote of the curve $y = mx + c + A/x + B/x^2 + C/x^3$ ... (1)

Let the equation of the curve be

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \ldots\ldots \qquad \ldots\ldots(1)$$

Where the series $\frac{A}{x} + \frac{B}{x^2} + \frac{c}{x^3} + \ldots\ldots$ is convergent for sufficiently large value of x.

Differentiating *w.r.t* x we get $\frac{dy}{dx} = m - \frac{A}{x^2} - \frac{2B}{x^3} - \ldots\ldots$.

Example – 1: ***Find the asymptotes of the hyperbola*** $\left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right) = 1$.

Solution : The given curve may be written as $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$

$$\therefore y = \pm\frac{b}{a}x\left(1 - \frac{a^2}{x^2}\right)^{1/2} = \pm\frac{b}{a}x\left[x - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} \ldots..\right]$$

$$= \pm\frac{b}{a}\left[x - \frac{a^2}{2x} - \frac{a^4}{8x^3} - .....\right]$$

Hence the asymptotes of the curve $y = \pm\left(\frac{b}{a}\right)x$.

Example – 2: ***Find the asymptote of the curve $y^3 = x^2(x - a)$.***

Solution: The curve is

$$y^3 = x^2(x-a) = x^3\left(1 - \frac{a}{x}\right)$$

or, $y = x\left(1 - \frac{a}{x}\right)^{1/3}$

$$= x\left(1 - \frac{1}{3}\cdot\frac{a}{x} - \frac{1}{9}\cdot\frac{a^2}{x^2}...\right)$$

or, $\quad y = x - \frac{a}{3} - \frac{1}{9}\cdot\frac{a^2}{x}...$ which is of the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + ...$$

Hence, $y = x - \frac{a}{3}$ is an asymptote of the given curve.

1.10 : Asymptotes by Inspection

If the equation of a curve is of the form $F_n + F_{n-2} = 0$, where F_n is of degree n (i.e., contains terms of degree n and may also contain terms of lower degree) and F_{n-2} is of degree (n – 2) at the most, and if $F_n = 0$ can be broken up into n linear factors so as to represent n straight lines no two of which are parallel or coincident, then $F_n = 0$ gives all the asymptotes of the curve.

Example – 1 : ***Find the asymptotes of the curve***

$$\boldsymbol{xy(x^2 - y^2)(x^2 - 4y^2) + 3xy(x^2 - y^2) + x^2 + y^2 - 7 = 0.}$$

Solution : The given equation can be put in the form

$$[xy(x^2 - y^2)(x^2 - 4y^2)] + [3xy(x^2 - y^2) + x^2 + y^2 - 7] = 0$$

This equation is of the form $F_n + F_{n-2} = 0$, where F_n can be broken up into n linear factors so as to represent n straight lines no two of which are parallel or coincident.

All the asymptotes are given by $F_n = 0$ or $xy(x - y)(x + y)(x - 2y)(x + 2y) = 0$

Then by inspection the asymptotes are $x = 0$, $y = 0$, $x - y = 0$, $x+y = 0$, $x - 2y = 0$ and $x + 2y = 0$.

Example – 2 : ***Find the asymptotes of the curve $x^2y + xy^2 = a^3$.***

Solution: The given curve is $x^2y + xy^2 - a^3 = 0$

This equation is of the form $F_n + F_{n-2} = 0$.

Here $F_3 = x^2y + xy^2$

$F_0 = -a^3$.

$\therefore$ By insepection, the asymptotes are given by

$x^2y + xy^2 = 0 \quad$ or, $\quad xy(x + y) = 0$

$\therefore$ The required asymptotes are $x = 0$ $y = 0$, $x+y = 0$.

1.11 : Intersection of a Curve and its Asymptotes

An asymptote of curve of n[th] degree cut the curve in (n – 2) points provided the asymptote is not parallel to any asymptote.

Hence if there be *'n'* asymptote of the curve, then they cut the curve in n (n – 2) points.

Note. The number of asymptotes of an algebraic curve of n[th] degree can not be more than n.

A straight line y = mx + c, cuts the curve of the nth degree. ....(1)

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+....=0 \qquad ....(2)$$

in n points real or imaginary.

Eliminating y from eq[n](1) and (2), we get

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+....=0$$

Expanding each term by Taylor's theorem and arranging in descending powders of x, we get

$$x^n\phi_n(m)+\left[c\phi_n'(m)+\phi_{n-1}(m)\right]x^{n-1}+\left[\frac{c^2}{2!}\phi_n''(m)+\frac{c}{1!}\phi_{n-1}'(m)+\phi_{n-2}(m)\right]x^{n-2}+....=0 \qquad (3)$$

The eqution (3) gives the abscissa of the points of intersection of eqution (1) and (2)

If y = mx + c is an asymptote of eqution (2), then $\phi_n(m) = 0$ and $c\phi'_n(m) + \phi_{n-1}(m) = 0$

So eqution (3) reduces to an equation of (n – 2)[th] degree in x and therefore the asymptote (1) cuts the curve (2) in (n – 2) points.

Hence any asymptote of an algebraic curve of the n[th] degree cuts the curve in (n – 2) points.

Note.
- A straight line cuts a curve of nth degree in general, in n points
- Any asymptote of an algebraic curve of the nth degree cuts the curve in (n – 2) points.
- The, n, aysmptotes of a curve of degree n cut it n(n –2) points.
- In general, a curve of degree n – 2 or less, can be made to pass through the n(n –2) points of intersection of a curve of degree n and it's asymptotes.
- If the equation of the curve of degree n can be put in the form $P_n + P_{n-2} = 0$ where P_{n-2} is of degree (n –2) at most and it consists of n non-repeated linear factors then the n(n–2) points of intersection of the curve and it's asymptotes lie on the curve $P_{n-2} = 0$.

  For example, for a cubic, n = 3 and so the asymptotes cut the curve in 3(3–2) = 3 points which lie on a curve of degree 3 – 2 = 1 i.e., on a straight line.

  For a quandrartic, n = 4 and so the asymptotes cut the curve in 4(4 – 2) = 8 points which lie on a curve of degree 4 – 2 = 2 i.e., on a conic such as cubic, parabola, ellipse, hyperbola etc.

Example – 1 : ***Prove that the asymptotes of the curve*** $xy(x^2-y^2)+x^2+y^2-1=0$ ***cut the curve in 8 points.***

Solution: The equation of the curve is $xy(x^2 - y^2) + x^2 + y^2 - 1 = 0$ ...(i)

Here, n = 4

This equation is of the type $F_n + F_{n-2} = 0$

Here, $F_n = xy\,(x^2 - y^2) = xy\,(x - y)\,(x + y)$ and $F_{n-2} = x^2 + y^2 - 1 \therefore F_n = 0$.

$\Rightarrow x = 0,\ y = 0,\ x - y = 0$ and $x + y = 0$ are the equations of asymptotes.

The combined equation of the asymptotes is

$$xy\,(x - y)\,(x + y) = 0 \qquad .....(ii)$$

Subtracting (ii) from (i), we get

$$x^2 + y^2 - 1 = 0$$

Thus, intersection of curve and asymptotes lie on this curve.

Since, there are 4 asymptotes i.e. n = 4

$\therefore$ Point of intersection of curve and asymptotes = 4(4 – 2) = 8.

Example – 2 : *Show that asymptotes of the cubic $x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$ cut the curve in three points which lie on the straight line $x - y + 1 = 0$.*

Solution: The given curve is $x^3 - 2y^3 + xy\,(2x - y) + y\,(x - y) + 1 = 0$ ....(1)

Putting $x = 1$ & $y = m$ in third and second degree terms of eq^n(1), we get

$\phi_3(m) = 1 - 2m^3 + 2m - m^2$ and $\phi_2(m) = m - m^2$. Also $\phi'_3(m) = -6m^2 + 2 - 2m$

$\therefore\ \phi_3(m) = 0 \Rightarrow 1 - 2m^3 + 2m - m^2 = 0$

$\Rightarrow 1 - m + 3m - 3m^2 + 2m^2 - 2m^3 = 0$

$\Rightarrow (1 - m) + 3m\,(1 - m) + 2m^2\,(1 - m) = 0$

$\Rightarrow (1 - m)\,(1 + 3m + 2m^2) = 0$

$\Rightarrow (1 - m)\,(1 + m)\,(1 + 2m) = 0$

$\Rightarrow m = 1, -1, -1/2.$

$$\therefore c = -\frac{\phi_2(m)}{\phi'_3(m)} = -\frac{(m - m^2)}{(2 - 2m - 6m^2)}$$

When m = 1, c = 0; when m = –1, c = –1; and when $m = -\dfrac{1}{2}, c = \dfrac{1}{2}$

Hence the asymptotes of (1) are $y = x$, $y = -x - 1$ and $y = -\dfrac{1}{2}x + \dfrac{1}{2}$.

Hence the combined equation of the asymptotes of (1) is $(x - y)\,(x + y + 1)\,(x + 2y - 1) = 0$

or $x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 - x + y = 0$ ....(2)

Subtracting eq^n(2) from eq^n(1), we get $x - y + 1 = 0$

which shows that the points of intersection of the curve and its asymptotes lie on the straight line $x - y + 1 = 0$. Also the three asymptotes cut the cubic $n(n - 2)$, i.e. 3(3 – 2) = 3 points and these three points lie on the straight line $x - y + 1 = 0$.

1.12 : Curvilinear asymptotes

If the equation of a curve is of the form

$$y = \alpha_0 x^n + \alpha_1 x^{n-1} + \ldots + \alpha_n + \frac{A}{x} + \frac{B}{x^2} + \ldots \qquad \ldots\ldots\ldots\ldots(1)$$

then $y = \alpha_0 x^n + \alpha_1 x^{n-1} + \ldots + \alpha_n$

is said to be asymptote to (1)

For example, if the equation to the curve is of the form $y = \alpha x^2 + \beta x + y + \dfrac{A}{x} + \dfrac{B}{x^2} + \ldots\ldots, \alpha \neq 0$

then it has the parabolic asymptote $y = \alpha^2 x + \beta x + y$

Example – 1 : *Find the parabolic asymptote of the curve $x^3 + aby - axy = 0$.*

Solution: The given curve is $x^3 + aby - axy = 0$.

or $y = \dfrac{x^3}{a(x-b)} = \dfrac{x^3}{ax\left(1-\dfrac{b}{x}\right)} = \dfrac{x^2}{a}\left(1-\dfrac{b}{x}\right)^{-1}$

$= \dfrac{x^2}{a}\left\{1+\dfrac{b}{x}+\dfrac{b^2}{x^2}+\dfrac{b^3}{x^3}+\ldots\right\}$, by expanding by Binomial theorem.

or $y = \dfrac{1}{a}\left(x^2+bx+b^2+\dfrac{b^3}{x^3}+\ldots\right)$. So the parabolic asymptote is $y=\dfrac{1}{a}\left(x^2+bx+b^2\right)$ or $ay = x^2 + bx + b^2$

Asymptotes to non-algebraic curves

In the case of non-algebraic curves, the asymptotes can be determined in simple cases by applying the definition or by expansion of y in negative power of x.

Examples – 1 : ***Find the asymptote of*** $\mathbf{y = e^{-x^2}}$.

Solution : The given curve is $y = e^{-x^2} \Rightarrow \dfrac{dy}{dx} = -2x.e^{-x^2}$

Tangent at (x, y) to the given curve is

$Y - y = \dfrac{dy}{dx}(X-x)$ or $Y - e^{-x^2} = -2x\cdot e^{-x^2}(X-x)$

or $Y - e^{-x^2} = -2x\ (X e^{-x^2}) - 2x^2\cdot e^{-x^2}$ ....(1)

When $x \to \infty$, then $e^{-x^2} \to 0$

$\therefore \lim\limits_{x\to\infty} xe^{-x^2} = \lim\limits_{x\to\infty}\dfrac{x}{e^{x^2}}\left(form\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\dfrac{1}{2xe^{x^2}}$ (by L.H. rule) = 0.

Also $\lim\limits_{x\to\infty} x^2\cdot e^{-x^2} = \lim\limits_{x\to\infty}\dfrac{x^2}{e^{x^2}} \qquad \left(form\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty}\dfrac{2x}{2xe^{x^2}} = \lim\limits_{x\to\infty}\dfrac{1}{e^{x^2}} = 0$

Taking limit as $x \to \infty$ on both sides of (1) . We get y = 0 when $x \to \infty$ the tangent tends to the asymptote. Hence the equation of the asymptote is y = 0.

Illustrative Examples

Example – 1 : ***Find the asymptotes of the given curve $(a + x)^2 (b^2 + x^2) = x^2y^2$.*** ***[B.P.U.T. - 2014***

Solution : Here $(a + x)^2 (b^2 + x^2) = x^2y^2$

$\Rightarrow (a^2 + x^2 + 2ax)(b^2 + x^2) = x^2y^2$

$\Rightarrow a^2b^2 + a^2x^2 + b^2x^2 + x^4 + 2ab^2x + 2ax^3 - x^2y^2 = 0$

$\Rightarrow x^4 - x^2y^2 + 2ax^3 + (a^2 + b^2)x^2 + 2ab^2x + a^2b^2 = 0$ ....(1)

Inspection for asymptote parallel to x-axis.

Here on dividing both sides of equation (1) by x^4, we get

$$\Rightarrow 1-\left(\frac{y^2}{x^2}\right)+\frac{2a}{x}+\left(\frac{a^2+b^2}{x^2}\right)+\frac{2ab^2}{x^3}+\frac{a^2b^2}{x^4}=0$$

$\therefore$ When $x \to \infty,\ y \to k,\ \boxed{1 \neq 0}$.

Which shows that there exists no asymptote parallel to x-axis.

Inspection for asymptote parallel to y-axis.

Here on dividing both sides of equation (1) by y^2, we get

$$\frac{x^4}{y^2}-x^2+\frac{2ax^3}{y^2}+\left(a^2+b^2\right)\cdot\frac{x^2}{y^2}+2ab^2\cdot\frac{x}{y^2}+\frac{a^2b^2}{y^2}=0$$

$\therefore$ When $y \to \infty,\ x \to k,\ -k^2 = 0 \Rightarrow \boxed{k=0}$.

Hence the equation of asymptote parallel to y-axis is $x = \text{k}$, $\Rightarrow \boxed{x=0}$.

Oblique asymptotes.

Here on putting $y = m$ and $x = 1$ in homogeneous terms of 4th degree, 3rd degree, 2nd degree and 1st degree terms separately, we get,

$$\phi_4(m) = 1 - m^2 \Rightarrow \phi'_4(m) = -2m$$
$$\phi_3(m) = 2a$$
$$\phi_2(m) = a^2 + b^2$$
$$\phi_1(m) = 2ab^2$$
$$\phi_0(m) = a^2b^2$$

Now on equating $\phi_4(m) = 0 \quad \Rightarrow 1 - m^2 = 0$

$$\Rightarrow m = \pm 1$$

We know $\boxed{c=\dfrac{-\phi_3(m)}{\phi'_4(m)}} \Rightarrow c=\dfrac{-2a}{-2m}=\dfrac{a}{m}$

For $m = 1$, $c = a$.

$\therefore$ The equation of asymptote is $y = mx + c \Rightarrow \boxed{y=x+a}$

For $m = -1$, $c = -a$

$\therefore$ The equation of asymptote is $y = mx + c$

$$\Rightarrow y=-x-a$$

$$\Rightarrow \boxed{-y=x+a}$$

Hence the equations of asymptote are $\boxed{\begin{array}{l} x=0 \\ \pm y=x+a \end{array}}$.

Example – 2 : ***Find the asymptote*** $(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0.$

Solution : Here $(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$

$\Rightarrow x^2y^2 - 4x^4 - y^4 + 4x^2y^2 - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$

$\Rightarrow 5x^2y^2 - 4x^4 - y^4 - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$ ....(1)

Inspection for asymptote parallel to x-axis.

Here on dividing both sides of equation (1) by x^4, we get

$$\frac{5y^2}{x^2} - 4 - \frac{y^4}{x^4} - \frac{6}{x} + \frac{5y}{x^2} + \frac{3y^2}{x^3} - \frac{2y^3}{x^4} - \frac{1}{x^2} + \frac{3y}{x^3} - \frac{1}{x^4} = 0$$

$\therefore$ When $x \to \infty, y \to k$, $\boxed{-4 \neq 0}$

Which shows that there exists no asymptote parallel to y-axis.

Oblique asymptotes.

Here on putting $y = m$ and $x = 1$ in homogeneous terms of 4th degree, 3rd degree, 2nd degree and 1st degree terms separately, we get

$$\phi_4(m) = 5m^2 - 4 - m^4 \Rightarrow \phi'_4(m) = 10m - 4m^3$$
$$\phi_3(m) = -6 + 5m + 3m^2 - 2m^3$$
$$\phi_2(m) = -1 + 3m$$
$$\phi_1(m) = 0$$
$$\phi_0(m) = -1$$

Now on equating $\phi_4(m) = 0 \Rightarrow 5m^2 - 4 - m^4 = 0$

$$\Rightarrow m^4 - 5m^2 + 4 = 0$$
$$\Rightarrow m^4 - m^3 + m^3 - m^2 - 4m^2 + 4 = 0$$
$$\Rightarrow m^3(m-1) + m^2(m-1) - 4(m^2 - 1) = 0$$
$$\Rightarrow m^3(m-1) + m^2(m-1) - 4(m+1)(m-1) = 0$$
$$\Rightarrow (m-1)(m^3 + m^2 - 4m - 4) = 0$$
$$\Rightarrow (m-1)(m^3 + 2m^2 - m^2 - 2m - 2m - 4) = 0$$
$$\Rightarrow (m-1)\{m^2 + (m+2) - m(m+2) - 2(m+2)\} = 0$$
$$\Rightarrow (m-1)(m+2)(m^2 - m - 2) = 0$$
$$\Rightarrow (m-1)(m+2)(m(m-2) + 1(m-2)) = 0$$
$$\Rightarrow (m-1)(m+2)(m-2)(m+1) = 0$$
$$\Rightarrow m = \pm 1, \pm 2.$$

We know $c = \dfrac{-\phi_3(m)}{\phi'_4(m)} = \dfrac{6 - 5m - 3m^2 + 2m^3}{10m - 4m^3}$

For $m = 1, c = 0$. $\therefore$ The equation of asymptote is $y = mx + c \Rightarrow \boxed{y = x}$

For $m = -1, c = -1$ $\therefore$ The equation of asymptote is $y = mx + c \Rightarrow \boxed{y = -x - 1}$

For $m = 2, c = 0$ $\therefore$ The equation of asymptote is $y = mx + c \Rightarrow \boxed{y = 2x}$

For $m = -2, c = -1$ $\therefore$ The equation of asymptote is $y = mx + c \Rightarrow \boxed{y = -2x - 1}$

Hence the equation of asymptotes are $y = x, y = 2x$

$y = -x - 1$ and $y = -2x - 1$.

Example – 3 : ***Find the asymptotes $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$.***

Solution : Here $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$

$\Rightarrow x^3 + 2x^2y + x^2 - y^2x - 2y^3 - y^2 + x + y + 1 = 0$

$\Rightarrow x^3 + 2x^2y - 2y^3 - y^2x + x^2 - y^2 + x + y + 1 = 0$ ....(1)

Inspection for asymptote parallel to x-axis

Here on dividing both sides of equation (1) by x^3, we get

$$1 + \frac{2y}{x} - \frac{2y^3}{x^3} - \frac{y^2}{x^2} + \frac{1}{x} - \frac{y^2}{x^3} + \frac{1}{x^2} + \frac{y}{x^3} + \frac{1}{x^3} = 0$$

$\therefore$ When $x \to \infty$, $y \to k$, $\boxed{1 \neq 0}$

Which shows that there exists no asymptote parallel to x-axis.

Inspection for asymptote parallel to y-axis

Here on dividing both sides of equation (1) by y^3, we get

$$\frac{x^3}{y^3} + \frac{2x^2}{y^2} - 2 - \frac{x}{y} + \frac{x^2}{y^3} - \frac{1}{y} + \frac{x}{y^3} + \frac{1}{y^2} + \frac{1}{y^3} = 0$$

$\therefore$ When $y \to \infty$, $x \to k$, $\boxed{-2 \neq 0}$

Which shows that there exists no asymptotes parallel to y-axis.

Oblique asymptotes

Here on putting $y = m$ and $x = 1$ in homogenous terms of 3rd degree, 2nd degree and 1st degree terms separately, we get

$\phi_3(m) = 1 + 2m - 2m^3 - m^2 \qquad \Rightarrow \phi'_3(m) = 2 - 2m - 6m^2$

$\phi_2(m) = 1 - m^2$, $\phi_1(m) = 1 + m$, $\phi_0(m) = 1$

Now on equating $\phi_3(m) = 0$

$\Rightarrow 1 + 2m - m^2 - 2m^3 = 0$

$\Rightarrow 2m^3 + m^2 - 2m - 1 = 0$

$\Rightarrow 2m^3 + 2m^2 - m^2 - m - m - 1 = 0$

$\Rightarrow 2m^2(m + 1) - m(m + 1) - 1(m + 1) = 0$

$\Rightarrow (m + 1)(2m^2 - m - 1) = 0$

$\Rightarrow (m + 1)(m^2 - m + m^2 - 1) = 0$

$\Rightarrow (m + 1) + \{m(m - 1) + (m + 1)(m - 1)\} = 0$

$\Rightarrow (m + 1)(m - 1)(2m + 1) = 0$

$\Rightarrow m = \pm 1, \frac{-1}{2}$

We know $c = \frac{-\phi_2(m)}{\phi'_3(m)} = \frac{m^2 - 1}{2 - 2m - 6m^2}$

From $m = 1$, $c = 0$ $\therefore$ The equation of asymptote is $\boxed{y = x}$

For $m = -1$, $c = 0$ $\therefore$ The equation of asymptote is $\boxed{y = -x}$

For $m = \frac{-1}{2}, c = \frac{-1}{2}$

$\therefore$ The equation of asymptote is $\boxed{y = \frac{-1}{2}x - \frac{1}{2}}$

Hence the equation of asymptotes are $y = x, y = -x$ and $y = \frac{-1}{2}x - \frac{1}{2}$.

Example – 4 : ***Find out the asymptote $y^2(x^2 - a^2) = x$.***

Solution : Here $y^2(x^2 - a^2) = x$

$\Rightarrow x^2y^2 - a^2y^2 - x = 0$ ....(1)

Inspection for asymptote parallel to Y-axis

Here on dividing both sides of equation (1) by y^2, we get

$$x^2 - a^2 - \frac{x}{y^2} = 0$$

$\therefore$ When $y \to \infty, x \to k$

$k^2 - a^2 = 0$

$\Rightarrow k = \pm a.$

Hence the equation of asymptote is $x = k, \Rightarrow \boxed{x = \pm a}$

Inspection for asymptote parallel to x-axis

Here on dividing both sides of equation (1) by x^2, we get

$$y^2 - a^2 \cdot \frac{y^2}{x^2} - \frac{1}{x} = 0$$

$\therefore$ When $x \to \infty, y \to k$

$\therefore k^2 = 0 \Rightarrow k = 0.$

Hence the equation of asymptote is $y = k. \Rightarrow \boxed{y = 0}$.

Oblique asymtotes

Here on putting $y = m$ and $x = 1$ in homogeneous terms of 4th degree, 3rd degree 2nd degree and 1st degree separately we get,

$$\phi_4(m) = m - 3m^2 + 3m^3 - m^4 \Rightarrow \phi_4'(m) = 1 - 6m + 9m^2 - 4m^3$$

$$\Rightarrow \phi_4''(m) = -6 + 18m - 12m^2$$

$$\phi_2(m) = -m + m^2 \Rightarrow \phi_2'(m) = (-1 + 2m) \Rightarrow \phi_4''(m) = 18 - 24m$$

$$\phi_0(m) = -2$$

Now on equating $\phi_4(m) = 0 \Rightarrow m - 3m^2 + 3m^3 - m^4 = 0$

$\Rightarrow m^4 - 3m^3 + 3m^2 - m = 0$

$\Rightarrow m^4 - m^3 - 2m^3 + 2m^2 + m^2 - m = 0$

$\Rightarrow (m-1)(m^3 - 2m^2 + m) = 0$

$\Rightarrow (m-1)(m^3 - m^2 - m^2 + m) = 0$

$\Rightarrow (m-1)(m^2(m-1) - m(m-1)) = 0$

$\Rightarrow (m-1)(m-1)(m^2 - m) = 0$

$\Rightarrow (m-1)^2 \cdot m(m-1) = 0$

$\Rightarrow m(m-1)^3 = 0$

$\Rightarrow m = 0, 1.$ (Thrice)

We know $c = \frac{-\phi_3(m)}{\phi_4'(m)}$

For $m = 0$, $c = 0$ $\left(\because \phi_3(m) = 0 \;\&\; \phi_4'(0) \neq 0\right)$

Hence the equation of asymptote is $y = mx + c \Rightarrow \boxed{y = 0}$

For $m = 1$, $\phi_4'(1) = 0$ and $\phi_4''(1) = 0$

So c is given by the formulae

$$\boxed{\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0}$$

$$\Rightarrow \frac{c^3}{3!}(18 - 24m) + \frac{c^2}{2!}\cdot 0 + c\cdot(-1 + 2m) + 0 = 0$$

$$\Rightarrow \frac{c^3}{6}\cdot 6(3 - 4m) - c + 2mc = 0$$

$$\Rightarrow 3c^3 - 4mc^3 - c + 2mc = 0$$

For $m = 1$, $3c^3 - 4c^3 - c + 2c = 0$

$$\Rightarrow -c^3 + c = 0$$

$$\Rightarrow c\,(1 - c^2) = 0$$

$$\Rightarrow c = 0, \pm 1.$$

Hence the equation of asymptotes are $y = mx + c$

For $m = 1$, $c = 0$, $\quad y = x$

$c = \pm 1 \quad y = x \pm 1$

Hence the equation of asymptotes parallel to y – axis are $x = k$.

$$\Rightarrow \boxed{x = 0, \& \; x = 1}$$

Example – 5 : ***Findout the asymptote*** $\left(\frac{\mathbf{a^2}}{\mathbf{x^2}}\right) - \left(\frac{\mathbf{b^2}}{\mathbf{y^2}}\right) = \mathbf{1}$**.**

Solution: Here $\left(\frac{a^2}{x^2}\right) - \left(\frac{b^2}{y^2}\right) = 1$

$$\Rightarrow a^2y^2 - b^2x^2 = x^2y^2 \Rightarrow x^2y^2 + b^2x^2 - a^2y^2 = 0 \;.............(1)$$

Inspections for asymptote parallel to x-axis

Here on dividing both sides of equation (1) by x^2, we get

$$y^2 + b^2 - \frac{a^2y^2}{x^2} = 0$$

$\therefore$ When $x \to \infty$, $y \to k$, $k^2 + b^2 = 0 \Rightarrow \mathrm{k} \pm \mathrm{ib}$

Which is impossible as k is a real no.

Which shows that there exists no asymptote parallel to x-axis.

Inspection for asymptote parallel to y-axis

Here on dividing both sides of equation (1) by y^2, we get

$$x^2 + \frac{b^2x^2}{y^2} - a^2 = 0$$

$\therefore$ When $x \to \infty$, $x \to k$, $k^2 - a^2 = 0 \Rightarrow \boxed{k = \pm a}$

Hence the equation of aymptotes parallel to y-axis are $\boxed{x = \pm a}$

Example – 6 : ***Form the equation of the quartic curve which has x = 0, y = 0, y = x and y = – x four asymptotes, which passes through the point (a, b) and cuts its asymptotes again in eight points lying upon the circle $x^2 + y^2 = a^2$.***

Solution: The combined equation of the asymptotes is $xy(y^2 - x^2) = 0$

The required curve passes through the points of intersection of the asymptotes and the given circle $x^2 + y^2 - a^2 = 0$.

Let the equation of the curve be $xy(y^2 - x^2) + k(x^2 + y^2 - a^2) = 0$

Now this curve also passes through the point (a, b). Then

$ab(b^2 - a^2) + k(a^2 + b^2 - a^2) = 0$ i.e. $k = \left(\frac{a}{b}\right)\left(a^2 - b^2\right)$

Substituting this value of k in eq^n(2), we get

$bxy(y^2 - x^2) + a\,(a^2 - b^2)\,(x^2 + y^2 - a^2) = 0$ as the required equation of the curve.

Example – 7 : ***Show that the eight points of intersection of the curve $xy(x^2 - y^2) + x^2 + y^2 = a^2$ and its asymptotes lie on a circle whose centre is at the origin.***

Solution : The equation of the curve is $xy\,(x^2 - y^2) + x^2 + y^2 - a^2 = 0$

This is of order 4, therefore it will have four asymptotes.

Asmyptotes parallel to the y-axis is $x = 0$ & asymptotes parallel to the x-axis is $y = 0$

Here $\phi_4(m) = m(1 - m^2) = m - m^3$, $\phi'_4(m) = 1 - 3m^2$, $\phi_3(m) = 0$, as there are no third degree terms

Here, $\phi_4(m) = 0$ gives $m(1 - m^2) = 0$ or $m = 0, \pm 1$.

We know, $c = \dfrac{-\phi_3(m)}{\phi'_4(m)} = -\dfrac{0}{1 - 3m^2} = \dfrac{0}{3m^2 - 1}$

When $m = \pm 1$, $c = 0$ and the corresponding asymptotes are $y = x$, $y = -x$

Hence, the asymptotes are $x = 0$, $y = 0$, $y = -x$, $y = x$

$\therefore$ Combined equation of asymptotes is $xy\,(x - y)\,(x + y) = 0$

or $xy\,(x^2 - y^2) = 0$

Substracting eq^n(2) from eq^n(1) we get $x^2 + y^2 = a^2$, which represents a circle whose centre is at the origin.

$\therefore$ The points of intersection of the curve (1) and its asymptotes (2) are $n(n - 2)$ i.e., $4(4 - 2)$ i.e. 8.

Therefore eight points of intersection of the curve (1) and its asymptotes lies on the circle $x^2 + y^2 = a^2$, whose centre is at the origin.

Example – 8 : ***Show that the eight points of the curve $x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$ and its asymptotes lie on a rectangular hyperbola.***

Solution : The equation of the given curve is $x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$

or $(x^2 - y^2)(x^2 - 4y^2) + x^2 - y^2 + x + y + 1 = 0$

or $(x - y)(x + y)(x - 2y)(x + 2y) + x^2 - y^2 + x + y + 1 = 0$

By inspection, the combined equation of the asymptotes of (1) is $(x - y)(x + y)(x - 2y)(x + 2y) = 0$

or $x^4 - 6x^2y^2 + 4y^4 = 0$

Now each asymptote of (1) will cut it in 4 – 2 i.e. 2 points. Therefore the four asymptotes will cut it in 4 × 2 i.e., 8 points.

Subtracting eq^n(2) from eq^n(1) we get

$x^2 - y^2 + x + y + 1 = 0$

The curve (3) passes through the eight points of intersection of (1) and (2). Also the conic (3) is a rectangular hyperbola because in its equation the sum of the coefficients of x^2 and y^2 is zero.

Hence the eight points of intersection of (1) and (2) lies on a rectangular hyperbola.

Exercise – 1.1

Find all the asymptotes of the followings :

1. $y(x - y)^2 = x + y$ 2. $x^3 + y^3 = 9xy$
3. $xy^2 = y^2 + x^3$ 4. $y^3 - x^2y + 2y^2 + 4y + 1 = 0$
5. $y^3 = x^2(x - a)$ 6. $y^3 - 2y^2x - yx^2 + 2x^3 + y^2 - 6xy + 5x^2 - 2y + 2x + 1 = 0$
7. $x(x - y)^4 = 4y(x - 1)^3$
8. Find the asymptotes following curves :
   (a) $x(y - x)^2 - x(y - x) = 2$
   (b) $y^3 - x^2y + 2xy^2 - y + 1 = 0$
   (c) $x^2y - xy^2 + 3x^2 - 2y^2 = 0$
   (d) $x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0$
   (e) $xy^2 - x^2y - 3x^2 - 2xy + y^2 + x - 2y + 1 = 0$
   (f) $(x^2 - y^2)(x + 2y) = y^2 - y + 1$
   (g) $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$
9. Show that the real asymptotes of the curve $x^2(x^2 + y^2) = a^2(y^2 - x^2)$ are $x = \pm a$.
10. Show that $x + y \pm a = 0$ are the asymptotes of the curve $(x + y)^2(x^2 + xy + y^2) + a^2(x - y) - a^2x^2 = 0$
11. Show that $x - y = 0$ is the real only asymptote of the curve $x(y^2 - 3ay + 2a^2) = y^3 - 3ax^2 + a^3$
12. Show that the curve $a^4y^2 + x^5(x - 2a) = 0$ has no asymptotes.
13. Show that the equation to the tangent to the curve $x^3 + y^3 - 3ax^2 = 0$ which is parallel to it's asymptote is $x + y = 4a$.
14. Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$ and which passes through the points (0, 0), (2, 0) and (0, 2).
15. Find the equation of the cubic curve whose asymptotes are $x + a = 0$, $y - a = 0$ and $x + y + a = 0$ and which touches the axis of x at the origin and passes through the point $(-2a, 2a)$.

16. What do you understand by an asymptote ?
17. How many asymptotes can a curve of degree *'n'* have ?
18. Write the number of points which asymptotes of an algebraic curve of degree '*n*' cut the curve.
19. Discuss the existence of the asymptotes of the curve $y^2 = x(x+1)^2$.
20. Find the asymptotes parallel to the coordinate axes of the curve $x^2y - 3x^2 - 5xy + 6y + 2 = 0$.
21. Show that $x = 1$ is the only asymptote of the curve $xy^2 = (x+y)^2$.

Answers

1. $y = 0, y = x + \sqrt{2}, y = x - \sqrt{2}$
2. $y = -x - 3$
3. $x = 1, y = x - \frac{1}{2}, y = x + \frac{1}{2}$
4. $y = 0, y = x\ 1, y = -x - 1$
5. $y = x - \frac{a}{3}$
6. $y = x, y = 2x + 1, y = -x - 2$
7. $x = 0, y = 0, y = 2x + \frac{3}{2}, 2y + 4x = 15$
8. (a) $x = 0, y = x, y = x + 1$ (b) $y = 0, y + x = \pm 1$
   (c) $x = -2, y = -3, y = x + 1$ (d) $y = x, y + x - 1 = 0, y + x + 1 = 0$ ***[B.P.U.T. - 2011***
   (e) $y + 3 = 0, x + 1 = 0, y = x + 4$ (f) $x - y = 1/6, x + y = -1/2, x + 2y = 1/3$
   (g) $x = 0, y = x, y = x + 1$
13. $x^3 - 6x^2y + 11xy^2 - 6y^3 - 4x + 24y = 0$
14. $xy(x+y) + a(y^2 - x^2 + xy) - 6a^2y = 0$
15. n asymptotes
16. $n(n-2)$ points
17. No asymptotes
18. $x = 2, x = 3, y = 3$.
19. No asymptotes
20. $x = 2, x = 3, y = 3$

1.13 : Working rule for obtaining asymptotes to polar curves

(1) Express the equation of curve in the form $\frac{1}{r} = f(\theta)$

(2) Solve $f(\theta) = 0$
Let $\theta = \alpha$ be the root.

(3) The equation of the required asymptote is $r\,sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$

1.14 :Circular Asymptotes

Let the equation of a curve be $r = f(\theta)$

If $\lim_{\theta \to \infty} f(\theta) = l$, then the circle $r = l$ is circular asymptote of the curve $r = f(\theta)$.