4 CHAPTER

Application of Partial Differentiation
(Maxima and Minima)

4.1 : Introduction

To optimize something means to maximize or minimize some aspects of it. In the study of stability of equilibrium states of mechanical and physical systems, determination of extrema is of greatest importance, Lagrange multipliers method developed by Lagrange in 1755 is a powerful method for finding extreme values of constrained functions. In this chapter we shall be concerned with the application of differential calculus to the determination of the values of a function which are greatest or least in their immediate neighborhoods technically known as greatest and least or maximum and minimum values.

4.2 : Taylor's theorem (finite form). To obtain Lagrange's formula for the remainder after the first n terms have been taken from Taylor's Series.

If function $f(x)$ is such that

(i) $f(x)$ and all its differential coefficients upto $(n-1)$th inclusive are finite and continuous in the interval (a, b),

(ii) and $f^{n}(x)$ exists as finite differential coefficient in the interval, then

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\ldots\ldots+\frac{(b-a)^{(n-1)}}{(n-1)!}f^{(n-1)}(a)+\frac{(b-a)^n}{n!}f^n(a+\theta h)$$

where $0<\theta<1$.

Proof : Consider the auxiliary function

$$F(x)=f(b)-f(x)-(b-x)f'(x)-\ldots-\frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x)-\frac{(b-x)^n}{n!}Q \quad \ldots\ldots.(1)$$

where Q is a constant given by $F(a)=0$.

If we put $x=b$ in (1), $F(b)=f(b)-f(b)=0$.

Thus $F(x)$ is such that

(i) it is continous in the interval $[a, b]$,

(ii) differentiable in the interval (a, b),

(iii) and $F(a)=F(b)=0$.

Thus by Rolle's theorem there exists a point $c(a<c<b)$

where $F'(x)=0$, i.e. $F'(c)=0$.

Now $F'(x) = -f'(x) + f'(x) - (b-x)f''(x) + (b-x)f''(x) \ldots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) + \frac{(b-x)^n}{n!} Q$

But $F'(c) = 0$; $\therefore$ $Q = f^{(n)}(c)$.

Also since $F'(a) = 0$, we have from (1),

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) - \ldots + \frac{(b-a)^{(n-1)}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(c)$$

putting value of $Q = f^{(n)}(c)$

$$= f(a) + (b-a)f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(a + \theta h)$$

where $a + \theta h = c$ and $0 < \theta < 1$.

Remainder. The term $R_n = \frac{(b-a)^n}{n!} f^{(n)}(a + \theta h)$ which is remainder after n terms in the above Taylor's expansion, is called the Lagrange's form of the remainder after n terms of Taylor's series.

Note. In the interval $(x, x + h)$, the expansion can be put as

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^{(n)}(x - \theta h)$$

where $0 < \theta < 1$.

4.3 : Mean Value Theorem for a Function of two Variable

Functions of two or more variables often can be expanded in power series which is generalize the familiar one dimensional expansion. Let $f(x, y)$ be a function of two independent variables x and y. Let P (x, y) and Q $(x + h, y + k)$ be two neighboring points. Then $f(x + h, y + k)$ the value of f at Q can be expressed in terms of f and its derivative at P.

Statement : If f(x, y) be a differentiable function of two variable x and y defined in a certain domain R containing the point's (a, b) and (a + h, b + k). So that the line joining these two point lies wholly in R then

$$f(a + h, b + k) = f(a, b) + hf_x(a+\theta h, b+\theta k) + kf_y(a + \theta h, b+\theta k)$$

where $0 < \theta < 1$.

Proof : Let $x = a + ht$, $y = b+kt$.

So that $f(x, y) = f(a + ht, b + kt) = F(t)$ (say)

Then applying mean value theorem of a single variable to the function $F(t)$ between 0 and t. We get

$F(t) - F(0) = t\,F'(\theta t)$, $0 < \theta < 1$ ....(i)

but $F'(t) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$

$= hf_x(a+ht, b+kt) + kf_y(a+ht, b+kt)$ ....(ii)

$\therefore$ From (i) and (ii)

$F(t) - F(0) = t\{hf_x(a+h\theta t, b+k\theta t) + kf_y(a+h\theta t, b+k\theta t)\}$ ....(iii)

Putting $t = 1$.

$F(1) - F(0) = hf_x(a+\theta h, b+k\theta) + kf_y(a+\theta h, b+\theta k)$

But $F(1) = f(a+h, b+k)$ and $F(c) = f(a, b)$

Hence we get $f(a+h, b+k) = f(a, b) + hf_x(a+\theta h, b+\theta k) + kf_y(a+\theta h, b+\theta k)$

where $0 < \theta < 1$.

4.4 : Taylor's Theorem for a Function of two Variables

Statement : If $f(x, y)$ posses the continuous partial derivatives of order n in any *nbd* of a point (a, b) and if $f(a+h, b+k)$ be any point of this *nbd*, then there exists a +ve number $0 < \theta < 1$, s.t.

$f(a + h, b + k) = f(a, b)$

$$+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a, b)+........+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1}f(a,b)+R_n$$

$$f(a, b) + [hf_x(a, b) + kf_y(a, b)]\ \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)]$$

$$+\ \frac{1}{3!}[h^3 f_{xxx}(a, b) + 3h^2k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)] + ...$$

where $R_n = \dfrac{1}{n!}\left(h\dfrac{\partial}{\partial x}+k\dfrac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$ where $0 < \theta < 1$.

Where R_n is called Remainder after n terms.

Proof : Suppose that $f(x + h, y + k)$ is a function of one variable only, say x where y is assumed as constant. Expanding by Taylor's theorem for one variable, we have $f(x + \partial x, y + \partial y) = f(x, y + \partial y)$

$$+\partial x\frac{\partial}{\partial x}\left(f(x, y+\partial y)\right)\ +\frac{(\partial x)^2}{2!}\frac{\partial^2}{\partial x^2}f(x, y+\partial y)+..$$

Now expanding for y, we get

$$=\left[f(x\ y)+\partial y\frac{\partial}{\partial y}f(x, y)+\frac{(\partial y)^2}{2!}\frac{\partial^2}{\partial y^2}f(x\ y)+...\right]\ +\partial x\cdot\frac{\partial}{\partial x}$$

$$\left[f(x\ y)+\partial y\frac{\partial f(x, y)}{\partial y}+...\right]+\frac{(\partial x)^2}{2!}$$

$$\frac{\partial}{\partial x}\left[f(x, y)+\partial y\frac{\partial}{\partial y}(x, y)+....\right]+......$$

$$=\left[f(x, y)+\partial y\frac{\partial}{\partial y}f(x, y)+\frac{(\partial y)^2}{2!}\frac{\partial^2}{\partial y^2}f(x, y)+....\right]+$$

$$+\partial x\left[\frac{\partial}{\partial x}f(x,y)+\partial y\cdot\frac{\partial^2}{\partial x\partial y}f(x,y)\right]+\frac{(\partial x)^2}{2!}\left[\frac{\partial^2}{\partial x^2}f(x,\ y)+....\right]$$

Otherwise for any specfic point (a b)

$$f(a+h,\quad b+k)=f(a,\ b)+[\,hf_x(a,\ b)+kf_y(a,b)]$$

$$+\frac{1}{2!}\,[h^2 f_{xx}(a,\ b)+2hk f_{xy}(a,b)+k^2 f_{yy}(a,\ b)]$$

$$+\frac{1}{3!}\,[h^3 f_{xxx}(a,\ b)+3h^2k f_{xxy}(a,\ b)+3hk^2 f_{xyy}(a,\ b)+k^3 f_{yyy}(a\ ,b)]+....\qquad .....(i)$$

Lemma : Let $z=f(x,\ y)$ and $x=a+ht,\ y=b+kt$ so that z is a composite function of a single variable t.

$$\therefore\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$=\frac{\partial z}{\partial x}\cdot h+\frac{\partial z}{\partial y}\cdot k=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z$$

$$\frac{d^2z}{dt^2}=\frac{d}{dt}\left(\frac{dz}{dt}\right)=\frac{d}{dt}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)z$$

$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 z$, procceding in this way we can write

$$\frac{d^nz}{dt^n}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n z$$

Proof : Write $f(x,\ y)=f(a+ht,\ b+kt)=F(x)$

Now applying maclaurins theorem to the function $F(t)$ of single variable t.

$$F(t)=F(0)+tF'(0)+\frac{t^2}{2}F''(0)+......+\frac{t^{n-1}}{(n-1)!}F^{n-1}(0)+\frac{t^n}{n!}F^n(\theta t)\text{ when }0<\theta<1.$$

For $t=1$

$$F(t)=F(0)+F'(0)+\frac{1}{2!}F''(0)+......+\frac{1}{(n-1)!}F^{n-1}(0)+\frac{1}{n!}F^n(\theta),\text{ where }0<\theta<1.$$

Also $F(1)=f(a+h,\ b+k)$

Further $t=0$ we have $x=a,\ y=b$ and

$F(0)=f(a,\ b)$

$$F^n(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y)$$

$$F'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)$$

$$F''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

....

....

$$F^{n-1}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)$$

$$F^{n}(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n} f(a+\theta h,b+\theta k)$$

Cor – 1 : If 'f' be a function of n-variable x_1, x_2.....x_3 then as above $f(x_1+h_1, x_2+h_2 \ldots\ldots\ldots\ldots x_n+h_n)$

$$=f(x_1,x_2\ldots\ldots\ldots\ldots x_n)+\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\ldots\ldots\ldots\ldots+h_n\frac{\partial}{\partial x_n}\right)f$$

$$+\frac{1}{2!}\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\ldots\ldots\ldots\ldots+h_n\frac{\partial}{\partial x_n}\right)^2 f+\ldots\ldots+\frac{1}{n!}\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\ldots+h_n\frac{\partial}{\partial x_n}\right)^n f+\ldots..$$

Cor – 2 : Taylor's expansion of $f(x, y)$ about the point (a, b) in powers $(x - a)$ and $(y - b)$ can be written as

$$f(x,y)=f(a,b)+\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]f(a,b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$

....

$$+\frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)+R_n$$

Where $R_n=\frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right]^n\times f\{a+(x-a),b+(y-b)\}$

$0<\theta<1.$

4.5 : Maclaurin's Theorem

Let $f(x, y)$ be a function of two independent variable x and y. Let

(i) (x, y) be a point in the *nbd* of (0, 0)

(ii) $f(x, y)$ has derivatives in $0 \le h \le x$, $0 \le k \le y$.

(iii) $f(x, y)$ is differentiable upto nth order in $0 < h < x$, $0 < k < y$ then there exist a number $0 < \theta < 1$, s.t.

$$f(x,y)=f(0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)$$

$$+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\ldots\ldots\ldots+\frac{1}{(n-1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n-1} f(0,0)+R_n$$

$h = x,\ k = y$ in Taylors theorem we get the required result.

where $R_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y)$.

Proof. If we put a = b = 0, $h = x$, $k = y$ in Taylor's theorem we get required result.

Illustrative Examples

Example – 1 : ***Expand $e^x \tan^{-1} y$ and (a, b) = (1, 1). Then $f(x, y) = e^x \tan^{-1} y$ about (1, 1) upto the 2nd degree in (x – 1) and (y –1).***

Solution : Let $f(x, y) = e^x \tan^{-1} y$ and $(a, b) = (1, 1)$

Then $f(x, y) = f(a, b)$

$$+\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \left[\frac{1}{2!}(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$

Now $f(x, y) = e^x \tan^{-1} y$, $\therefore f(1,1) = \frac{e\pi}{4}$

$f_x(x, y) = e^x \tan^{-1} y$ $\therefore f_x(1,1) = \frac{e\pi}{4}$

$f_y(x,y) = \frac{e^x}{1+y^2}$, $\therefore f_y(1,1) = \frac{e}{2}$

$f_{xy}(x,y) = \frac{e^x}{1+y^2}$, $\therefore f_{xy}(1,1) = \frac{e}{2}$

$f_{yx}(x,y) = e^x \tan^{-1} y$, $\therefore f_{yx}(1,1) = \frac{e\pi}{4}$

$f_{yy}(x,y) = \frac{-2e^{xy}}{(1+y^2)^2}$ $\therefore f_{yy}(1,1) = \frac{-e}{2}$

Substituting this values in (1)

$$e^x \tan^{-1} y = \frac{e\pi}{4} + (x-1)\frac{e\pi}{4} + (y-1)\frac{e}{2} + \frac{1}{1}\left[(x-1)^2\frac{e\pi}{4} + 2(x-1)(y-1)\frac{e}{2} - (y-1)^2\frac{e}{2}\right]$$

$$= \frac{e}{4}\left[\pi + (x-1)\pi + 2(y-1) + \frac{1}{2}(x-1)^2\pi + 2(x-1)(y-1) - (y-1)^2\right]$$

Example – 2 : ***Expand sin xy in powers of (x – 1) and (y – π/2) upto and including 2nd degree term.***

Solution : Let $f(x, y) = \sin xy$ and $(a, b) = (1, \pi/2)$

Then $f(x, y) = f(a, b)$

$$+\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$

Now $f(x, y) = \sin xy, f\left(1, \pi/2\right) = 1$

$f_x(x, y) = y \cos xy, f_x\left(1, \pi/2\right) = \frac{\pi}{2}\cos\frac{\pi}{2} = 0$

$f_y(x, y) = x\cos xy, f_y\left(1, \pi/2\right) = 0$

$f_{xy} = \cos xy - xy \sin xy$

$f_{xy}\left(1, \pi/2\right) = 0 - \frac{\pi}{2}\cdot 1 = -\frac{\pi}{2}$

$f_{yy}(x, y) = -x^2 \sin xy, f_{yy}\left(1, \pi/2\right) = -\frac{\pi}{2}$

$f_{xx}(xy) = -y^2 \sin xy,\ f_{xx}(1, \pi/2) = \frac{-\pi^2}{4}$

$$= 1 + \frac{1}{2}\left[(x-1)^2\left(\frac{-\pi^2}{4}\right) + \left(y - \frac{\pi}{2}\right)^2\cdot\left(\frac{-\pi}{2}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left(\frac{-\pi}{2}\right)\right]$$

$$= 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{4}\left(y-\frac{\pi}{2}\right)^2 - \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right).$$

Example – 3 : ***Prove that for $0 < \theta < 1$***

$$e^{ax}\sin by = by + abxy$$
$$+\frac{1}{6}\left[a^3x^3 - 3ab^2xy^2\right]\sin b\theta y + \left[3\left(a^2bx^2y - b^3y^3\right)\cos b\theta y\right]e^{a\theta x}.$$

Solution : Let $f(x, y) = e^{ax}\sin by$ them $f(u, v) = e^{au}\sin bv$

$\therefore f(0, 0) = 0$

$$\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)f(u,v) = \left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)\left(e^{au}\sin bv\right)$$

$= xae^{au}\sin bv + ybe^{au}\cos bv =$ by, by $u = v = 0$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)^2 e^{au}\sin bv$$

$$= \frac{1}{2}\left\{x^2\frac{\partial^2}{\partial u^2}\left(e^{au}\sin bv\right) + 2xy\frac{\partial^2}{\partial u\partial v}e^{au}\sin bv + y^2\frac{\partial^2}{\partial v^2}e^{au}\sin bv\right\} = abxy \text{ for } u = 0 = v$$

$$\frac{1}{3!}\left(x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right)^3 e^{au}\sin bv$$

$$= \frac{1}{6}\left\{x^3\frac{\partial^3}{\partial u^3}e^{au}\sin bv + 3x^2y\frac{\partial^3}{\partial u^2\partial v}e^{au}\sin bv + 3xy^2\frac{\partial^3}{\partial u\partial v^2}\left(e^{au}\sin bv\right) + y^3\frac{\partial^3}{\partial y^3}\left(e^{au}\sin bv\right)\right\}$$

$$= \frac{1}{6}\left\{x^3e^{au}a^3\sin bv + 3x^2ya^2be^{au}\cos bv - 3xy^2ab^2e^{au}\sin bv - y^3b^3e^{au}\cos bv\right\}$$

$$=\frac{1}{6}e^{au}\left\{\left(a^3x^3-3ab^2xy^2\right)\sin bv+\left(3a^2bx^2y-b^3y^3\right)\cos bv\right\}$$

$$=\frac{1}{6}e^{a\theta x}\left\{\left(a^3x^3-3ab^2xy^2\right)\sin b\theta y+\left(3a^2bx^2y-b^3y^3\right)\cos b\theta y\right\}$$

Putting $u=\theta x$ and $v=\theta y$

$$\therefore\ e^{ax}\sin by=by+abxy+\frac{1}{6}e^{x\theta a}\left(a^3x^3-3ab^2xy^2\right)\sin b\theta y+\left(3a^2bx^2y-b^3y^3\right)\cos b\theta y\ ,0<\theta<1.$$

Example – 4 : ***If $f(x,y)=\sqrt{|xy|}$, prove that Taylor's expansion about the point (x, y) is not valid in any domain x – 1 includes the origin.***

Solution : $f(x,y)=\sqrt{|xy|}=\sqrt{|x|\,|y|}$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0$$

Now $f_x(x,y)=\lim_{h\to 0}\dfrac{f(x+h,y)-f(x,y)}{h}$

$$=\lim_{h\to 0}\frac{\sqrt{|x+h|\,|y|}-\sqrt{|x|\,|y|}}{h}=\lim_{h\to 0}\sqrt{y}\left(\frac{\sqrt{|x+h|}-\sqrt{|x|}}{h}\right)$$

$$=\lim_{h\to 0}\sqrt{|y|}\cdot\frac{|x+h|-|x|}{h\left(\sqrt{|x+h|}+\sqrt{|x|}\right)}$$

As $h\to 0$, we can take $x+h>0$, is $|x+h|=x+h$.
When $x>0$ and $x+h<0$ or $|x+h|=-(x+h)$, when $x<0$.

Case – 1 : For $x>0$

$$f_x(x,y)=\lim_{h\to 0}\sqrt{|y|}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\sqrt{|y|}\cdot\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{\sqrt{|y|}}{2\sqrt{|x|}}=\frac{1}{2}\sqrt{\left|\frac{y}{x}\right|}$$

Case – 2 : For $x<0$

$$f_x(x,y)=\lim_{h\to 0}\sqrt{|y|}=\frac{-(x+h)-(-x)}{h\sqrt{-(x+h)}+\sqrt{x}}=\lim_{h\to 0}\frac{(-x-h+x)\sqrt{|y|}}{h\left[\sqrt{x+h}+\sqrt{x}\right]}$$

$$=\lim_{h\to 0}\sqrt{|y|}\cdot\frac{-1}{\sqrt{x+h}+\sqrt{x}}=-\frac{1}{2}\sqrt{\left|{}^{y}\!/_{x}\right|}$$

Thus $f_x(x,y)=\begin{cases}\frac{1}{2}\sqrt{\left|\frac{4}{x}\right|} & if\ x>0\\ -\frac{1}{2}\sqrt{\left|\frac{4}{x}\right|} & if\ x<0\end{cases}$

Simillarly $f_x(x,y)=\begin{cases}\frac{1}{2}\sqrt{\left|\frac{x}{y}\right|} & if\ y>0\\ -\frac{1}{2}\sqrt{\left|\frac{x}{y}\right|} & if\ y<0\end{cases}$

Now Taylors expansion about (x, x)

$f(x + h, x + h) = f(x, x) + h[f_x(x + \theta h, x + \theta h) + f_y(x + \theta h, x + \theta h)]$

$$\Rightarrow \sqrt{|x+h||x+h|} = \begin{cases}\sqrt{|x||x|}+h\left[\frac{1}{2}+\frac{1}{2}\right] & if\ x+\theta h>0\\ \sqrt{|x||x|}+h\left[-\frac{1}{2}-\frac{1}{2}\right] & if\ x+\theta h<0\\ \sqrt{|x||x|}+h[0+0] & if\ x+\theta h=0\ \ [\because f_x(0,0)=0]\end{cases}$$

$$|x+h|=\begin{cases}|x|+h & if\ x+\theta h>0\\ |x|-h & if\ x+\theta h<0\\ |x| & if\ x+\theta h=0\end{cases} \qquad \text{......................(1)}$$

If domain $(x, x+h, x+h)$ include of origin, then x and $x + h$ must be of opposite signs, that is either $|x + h| = x + h$, $|x| = - x$ or $|x + h| = - (x + h)$, $|x| = x$. But under these condition none of equalities in (1) holds. Hence the expression is not valid.

4.6 : (a) Taylor's Theorem (finite form) for two Variables

Statement. If $f(x, y)$ and all its partial derivatives upto those of nth order inclusive are finite and continuous for all points (x, y) in the domain $a \le x \le a + h,\ b \le y \le b + k$ then

$f(a + h, b + k)$

$$= f(a,b)+\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)^2 f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)^2 f(a,b)$$

$$+\ldots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)^{n-1} f(a,b)$$

$$+.\frac{1}{n!}\left\{\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}\right)^{n} f(x,y)\right\}$$, where $x = a + \theta h$, $y = b + \theta k$.

(b) Taylor's Theorem for three variables

Statement. If $f(x, y,z)$ and all its partial derivatives upto nth order inclusive are finite and continuous at all points in the region
$a \le x \le a + h,\ b \le y \le b + k, c \le z \le c + l$ then by Taylor's *Theorem.*

$$f(a+h,\ b+k,\ c+l) = f(a,b,c) + \left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}+l\frac{\partial}{\partial c}\right)f(a,b,c)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}+l\frac{\partial}{\partial c}\right)^2 f(a,b,c)+\ldots.$$

$$+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial a}+k\frac{\partial}{\partial b}+l\frac{\partial}{\partial c}\right)^{n-1} f(a,b,c)+\ldots$$

$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}\right)^{n} f(x,y,z),\ \text{where}\ \left.\begin{array}{l} x=a+\theta h,\\ y=b+\theta k,\\ z=c+\theta l\end{array}\right]0<\theta<1.$$

(c) Maclaurin's theorem for function of two variables

Statement. If $f(x, y,z)$ and all its partial derivatives upto nth order inclusive are finite and continuous in a regin including the point(0,0) them by Maclaurin's theorem

$$f(x,y)=f(0,0) + \left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)f(u,v)+\frac{1}{2!}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^2 f(u,v)$$

$$+\ldots+\frac{1}{(n-1)!}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^{n-1} f(u,v)$$

$$+\frac{1}{n!}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^{n} f(u,v)$$

Where u and v are to be replaced by 0,0 in all the terms except the last in which these are to be replaced by θx and θy where $0 < \theta < 1$.
This is obtained but putting a =0, b =0 in Taylor's theorem for two variables.

(d) Maclaurin's theorem for function of three varibles

We have

$$f(x,y,z) = f(0,0,0) + \left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}+z\frac{\partial}{\partial w}\right)f(u,v,w)$$

$$+\ldots+\frac{1}{(n-1)!}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}+z\frac{\partial}{\partial w}\right)^{n-1} f(u,v,w)$$

$$+\frac{1}{n!}\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}+z\frac{\partial}{\partial w}\right)^{n} f(u,v,w)$$

where $u,\ v,\ w$ are to be replaced by 0, 0, 0 in all terms except the last in which these are to be replaced by $\theta x, \theta y, \theta z$, where $0 < \theta < 1$.

Example – 5 : ***Expand e^{xy} at (1, 1) by using Taylors theorem.***

Solution :

$f(x,y) = e^{xy}$ $\quad\quad f(1,1) = e$

$f_x(x,y) = ye^{xy}$ $\quad\quad f_x(1,1) = e$

$f_y(x,y) = xe^{xy}$ $\quad\quad f_y(1,1) = e$

$f_{xx}(x,y) = y^2e^{xy}$ $\quad\quad f_{xx}(1,1) = e$

$f_{xy}(x,y) = y(xe^{xy}) + e^{xy} = e^{xy}(xy+1);$ $\quad f_{xy}(1,1) = 2e$

$f_{yy}(x,y) = x^2e^{xy}$ $\quad\quad f_{yy}(1,1) = e$

By Taylor's Theorem $e^{xy} = f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)]$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1) + \ldots\right]$$

$$= e + \left[(x-1)e + (y-1)e\right] + \frac{1}{2!}\left[(x-1)^2 e + 2(x-1)(y-1)2e + (y-1)^2 e\right] + \ldots$$

$$= e\left[1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + 2(x-1)(y-1) + \frac{1}{2}(y-1)^2\right]$$

Example – 6 : ***Expand y^x upto second term at (1, 1).***

Solution :

$f(x,y) = y^x;$ $\quad\quad f(1,1) = 1$

$f_x(x,y) = y^x \log y;$ $\quad\quad f_x(1,1) = 0$

$f_y(x,y) = xy^{x-1};$ $\quad\quad f_y(1,1) = 1$

$f_{xx}(x,y) = y^x(\log y)^2;$ $\quad\quad f_{xx}(1,1) = 0$

$$f_{xy}(x,y) = y^x \cdot \frac{1}{y} + \log y \cdot xy^{x-1} = y^{x-1} + xy^{x-1}\log y;\ f_{xy}(1,1) = 1$$

$f_{yy}(x,y) = x(x-1)y^{x-2};\ f_{yy}(1,1) = 0$

Now $y^x = f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)]$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \ldots$$

$$= 1 + \left[(x-1)0 + (y-1)\cdot 1\right] + \frac{1}{2!}\left[(x-1)^2 \cdot 0 + 2(x-1)(y-1) + (y-1)^2 \cdot 0\right]$$

$$= 1 + (y-1) + \frac{1}{2}\cdot 2(x-1)(y-1) + \ldots$$

$= 1 + (y-1) + (x-1)(y-1) \ldots$

Example – 7 : ***Find the first six terms of the expansion of function $e^x \log(1+y)$ in a Taylor series in the neighourhood of the (0, 0).***

Solution : Here $f(x,y) = e^x \log(1+y)$, $\quad f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y)$, $\quad f_x(0,0) = 0$

$f_y(x,y)=\dfrac{e^x}{1+y}$, $\quad f_y(0, 0) = 1$

$f_{xx}(x, y) = e^x \log(1 + y)$, $\quad f_{xx}(0, 0) = 0$

$f_{xy}(x,y)=\dfrac{e^x}{1+y}$, $\quad f_{xy}(0, 0) = 1$

$f_{yy}(x,y)=-\dfrac{e^x}{(1+y)^2}$, $\quad f_{yy}(0, 0) = -1$

$f_{xxx}(x, y) = e^x \log(1 + y)$, $\quad f_{xxx}(0, 0) = 0$

$f_{xxy}(x,y)=\dfrac{e^x}{1+y}$, $\quad f_{xxy}(0, 0) = 1$

$f_{xyy}(x,y)=-\dfrac{e^x}{(1+y)^2}$ $\quad f_{xyy}(0, 0) = -1$

$f_{yyy}(x,y)=-\dfrac{2e^x}{(1+y)^3}$ $\quad f_{yyy}(0, 0) = 2$

....

$\therefore\ e^x \log(1 + y) = f(x, y)$

$$= f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$

$$+\frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + ...$$

$$= 0 + [x\cdot 0 + y\cdot 1] + \frac{1}{2}[x^2\cdot 0 + 2xy\cdot 1 + y^2(-1)]$$

$$+\frac{1}{6}[x^3\cdot 0 + 3x^2 y\cdot 1 + 3xy^2\cdot(-1) + y^3\cdot 2] + ...$$

$$= y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2 y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 ....$$

Example – 8 : ***Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's Theorem.***

Solution : Expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)] + \frac{1}{2!}[(x - a)^2 f_{xx}(a, b)$$

$$+2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x - a)^3 f_{xxx}(a, b)$$

$$+ 3(x - a)^2 (y - b) f_{xxy}(a, b) + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b)] + ...(1)$$

Here $f(x, y) = x^2y + 3y - 2$, $a = 1$, $b = -2$

$f(1, -2) = 1^2 \times (-2) + 3(-2) - 2 = -10$

$f_x = 2xy,$ $f_x(1,-2) = 2(1)(-2) = -4;$ $f_y = x^2+3,$ $f_y(1,-2) = 1^2+3 = 4$
$f_{xx} = 2y,$ $f_{xx}(1,-2) = 2(-2) = -4;$ $f_{xy} = 2x,$ $f_{xy}(1,-2) = 2(1) = 2$
$f_{yy} = 0,$ $f_{yy}(1,-2) = 0;$ $f_{xxx} = 0,$ $f_{xxx}(1,-2) = 0$
$f_{xxy} = 2,$ $f_{xxy}(1,-2) = 2;$ $f_{xyy} = 0,$ $f_{xyy}(1,-2) = 0$
$f_{yyy} = 0,$ $f_{yyy}(1,-2) = 0$

All higher order partial derivatives vanish.

$\therefore$ From (1), we have

$x^2y + 3y - 2 = f(x, y)$

$$= -10 + [(x-1)(-4) + (y+2)(4)] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)]$$

$$+\frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Example – 9 : ***Expand $f(x, y) = 21 + x - 20y - 4x^2 + xy + 6y^2$ in Taylor's series of maximum order about the point $(-1, 2)$.***

Solution : $f(x, y) = 21 + x - 20y - 4x^2 + xy + 6y^2$; $f(-1, 2) = -2$

$f_x = 1 - 8x + y;$ $f_x(-1, 2) = 11$
$f_y = -20 + x + 12y;$ $f_y(-1, 2) = 3$
$f_{xx} = -8;$ $f_{xx}(-1, 2) = -8$
$f_{xy} = 1;$ $f_{xy}(-1, 2) = 1$
$f_{yy} = 12;$ $f_{yy}(-1, 2) = 12$

Next derivatives all vanish.

By Taylor's Series

$$f(x, y) = f(-1, 2) + [(x+1) f_x(-1, 2) + (y-2) f_y(-1, 2)]$$

$$+\frac{1}{2!}[(x+1)^2 f_{xx}(-1, 2) + 2(x+1)(y-2) f_{xy}(-1, 2) + (y-2)^2 f_{yy}(-1, 2)]$$

$$= -2 + (x+1)\cdot 11 + (y-2)\cdot 3 + \frac{1}{2}[(x+1)^2(-8) + 2(x+1)(y-2)\cdot 1 + (y-2)^2 12]$$

$$= -2 + 11(x+1) + 3(y-2) - 4(x+1)^2 + (x+1)(y-2) + 6(y-2)^2.$$

Example – 10 : ***Find approximate value of $\sqrt{(298)^2 + (401)^2}$ by using Taylor series.***

Solution : $f(x,y) = \sqrt{x^2 + y^2}$

$$f(x+h, y+k) = \sqrt{(x+h)^2 + (y+k)^2}$$

where $x + h = 298$, $y + k = 401$

Take $x = 300$, $y = 400$

$\therefore$ $h = -2$, $k = 1$

By Taylor's series

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + ...\infty$$

From (1), $\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}2x = \frac{x}{\sqrt{x^2+y^2}}; \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$$\therefore f(x+h, y+k) = f(x, y) + \left(h\cdot\frac{x}{\sqrt{x^2+y^2}} + k\cdot\frac{y}{\sqrt{x^2+y^2}}\right) + ...$$

$$= f(x, y) + \frac{hx+ky}{\sqrt{x^2+y^2}}$$

At (300, 400),

$$f(298, 401) = f(300, 400) + \frac{(-2)300 + 1(400)}{\sqrt{90000+160000}}$$

$$\sqrt{(298)^2 + (401)^2} = \sqrt{90000+160000} - \frac{200}{500} = 500 - \frac{2}{5} = \frac{2498}{5} = 499\cdot 6$$

Hence $\sqrt{(298)^2 + (401)^2} = 499\cdot 6$.

Exercise – 4.1

1. Obtain Taylor's expansion of $tan^{-1}\left(\frac{y}{x}\right)$ about (1, 1) upto and including the second degree.
2. Expand $f(x, y) = e^{xy}$ at (1, 1) upto including the second degree term.
3. Expand $f(x, y) = sinx \,.\, siny$ in powers of $\left(x - \frac{\pi}{4}\right)$ and $\left(y - \frac{\pi}{4}\right)$ into Taylors series upto and including the terms of third order.
4. Expand $(1+x+y)^{1/2}$ at (1, 0).
5. Expand $sin\,(x+h)\,(y+k)$ by Taylor's theorem.
6. Expand $e^x cosy$ at $\left(1, \frac{\pi}{4}\right)$.
7. Obtain the expansion of $tan^{-1}\frac{y}{x}$ about (1, 1) upto the third degree term.
8. Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in powers of h, k upto and inclusive of the second degree terms.
9. Find the first six terms of the expansions of the function $e^x log(1+y)$ in a Taylor series in the neighbourhood of the point (0, 0).
10. If $f(x, y) = tan^{-1}xy$ find an approximate value of $f\,(1.1, 0.8)$ using Taylors series quadrator approximation.
11. Obtain the first degree Taylor's series approximation to the function $f(x, y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ about the point (–1, 1). Find the maximum error in the region $|x+1| < 0\cdot 1$; $|y-1| < 0\cdot 1$.