3

CHAPTER

Partial Differentiation

3.0 Introduction

The functions studied so far are of single independent variable. There are functions which depends on two or more variables. The area of the rectangle depends on it's sides x and y units of length, the volume of parallelopiped depends on its length, breadth and height i.e. the area of a rectangle is a function of its two independent variable x & y. Then we have to write $A = f(x,y) = xy$. In general, if f is a function of several independent variables $x,y,z...$ we express this fact by the symbol $f(x,y,z...)$

In case of $y = f(x)$, only a single dependent variables which can be expressed in terms of a single independent variables. Here x is called independent variable, y is called dependent variables. In case of $z = f(x,y)$,z is a function of two variables, where z be the dependent variable and x,y are called two independent variable.

In case of $u = f(x,y,z)$, u is a function of three variables, where 'u' be the dependent variable and x, y, z are called independent variables.

Limit :

If the values of the function $z = f(x,y)$ can be made as close as we like to a fixed number l by taking the point (x,y) close to the point (x_0, y_0), but not equal to (x_0, y_0) then we say that l is the limit of f as the point (x,y) approaches the point (x_0,y_0)

We write this in symbols as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$$

We read this as : the limit of f as (x,y) approaches (x_0, y_0) is l.

Remark : The condition

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

is very much stronger than the requirement

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$$

The latter allow (x,y) to vary on a particular line through (a,b), $y = b$ or $x = a$; while in the former (x,y) is allowed to approach (a,b) in every possible manner. However, the partial limiting process has a useful role to pay in the concept that we given below.

Now we have two equivalent definitions of limit as given below.

Definition -1 : The limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is the number l if any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all points either.

(i) $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ implies that $|f(x,y) - l| < \varepsilon$ or

(ii) $0 < |x - x_0| < \delta$ and $0 < |y - y_0| < \delta$ implies that $|f(x,y) - l| < \varepsilon$

Example –1 : ***If*** $f(x,y) = \dfrac{y^2 - x^2}{y^2 + x^2}$***, for (x,y) ≠ (0,0) show that*** $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ ***does not exist.***

Solution : Here the point (x,y) can approach the origin infinitely in many way.
For example, if we approach the origin along x-axis, then y = 0. In this case,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} = \frac{-x^2}{x^2} = -1$$

On the other hand, if we approach the origin along y-axis, then $x = 0$ and in this case we get,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} = \frac{y^2}{y^2} = 1$$

Thus we get different values of limits depending on how we approach the origin. Hence we have shown that for any open disc centred at the origin, there are points at which takes on the value +1 and –1. Therefore f cannot have the limit as $(x,y) \to (0,0)$
This example leads us to a general rule for non-existence of the limit, which we can state as follows.
If we get two or more different value for

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

as we approach (x_0, y_0) along different paths (Path is another name for a curve joining (x,y) to a given point (x_0, y_0) then

$$\lim_{(x,y)\to(0,0)} f(x,y), \text{ does not exist.}$$

We state the following theorem without proof.

Theorem – 1 :

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = l_1$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = l_2$

then

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} \left[f(x,y) + g(x,y)\right] = l_1 + l_2$

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} \left[f(x,y) - g(x,y)\right] = l_1 - l_2$

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} \left[f(x,y) \cdot g(x,y)\right] = l_1 l_2$

(iv) $\lim\limits_{(x,y)\to(x_0,y_0)} \left[K\, f(x,y)\right] = K\, l_1, k$ being any constant, and

(v) $\lim\limits_{(x,y)\to(x_0,y_0)} \left[\dfrac{f(x,y)}{g(x,y)}\right] = \dfrac{l_1}{l_2}$, if $l_2 \neq 0$

Definition - 2 : A function $f(x,y)$ is said to be continuous at a point (x_0, y_0) if

(i) f is defined at (x_0, y_0)

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists and

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Example –2 : ***Determine the points at which f (x,y) is continuous if***

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & if\ (x,y)\neq(0,0) \\ 0 & if\ (x,y)=(0,0) \end{cases}}$$

Solution : We approach (0,0) along the line $y = m\,x$, m being an arbitrary constant. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xmx}{x^2+m^2x^2} = \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}$$

Here we get different limits for different values of m.

Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So f is not continuous at (0,0)

When $(a,b) \neq (0,0)$, $a^2 + b^2 \neq 0$ and $\lim_{(x,y)\to(a,b)} f(x,y) = \dfrac{ab}{a^2+b^2} = f(a,b)$

Therefore f is continuous at all points $(a,b) \neq (0,0)$. So we conclude that f is continuous at all points except the origin.

Here we make a note : If one or more of three conditions in the definition of continuity of $f(x,y)$ fails to hold, then f is discontinuous at the point under consideration.

3.1 : State & Explain Partial Derivatives upto Three Variables

Partial Derivatives :

So far we have studied about derviatives of functions of a single variable i.e. $y = f(x)$

$\dfrac{dy}{dx} = \lim_{\delta x\to 0} \dfrac{f(x+\delta x)-f(x)}{\delta x} = f\,(x)$, In that cases if a dependent variable is a function of single independent variable. In order to find the derivative of a function of two variables the following procedure is adopted.

If a dependent variable is a function of two or more independent variable, in that cases partial derivaties exists i.e. $z = f(x,y)$. The function is differentiated with respect to one of the independent variables while other is treated as constant.

Consider a function of two independent variables x and y. Let $z = f(x,y)$. If the variable x under goes a change δx while the variable y remains constant, then z undergoes a change δz.

$\delta z = f(x + \delta x, y) - f(x,y)$

We say that z possesses partial derivative w.r.t. x and denoted by

$$\frac{\partial z}{\partial x} = \lim_{\delta x\to 0} \frac{f(x+\delta x, y)-f(x,y)}{\delta x} = \frac{\partial f}{\partial x} = f_x$$

Simillarly z possesses partial derivative w.r.t y and denoted by

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} = \frac{\partial f}{\partial y} = f_y$$

The symbols $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are used to denoted the operators for partial differentiation.

$\therefore$ The partial derivative of $z = f(x,y)$ with respect to x is the ordinary derivative of $f(x,y)$ when y is regarded as constant. i.e.

$\frac{\partial z}{\partial x}$ = partial derivative of z w.r.t. x treating y being constant.

The partial derivative of $f(x,y)$ w.r.t y is the ordinary derivative of $f(x,y)$ when x is regarded as constant. i.e.

$\frac{\partial z}{\partial y}$ = partial derivative of z w.r.t. y treating x being constant.

So all the rules of differentiation such as differention of sum product, quotient of functions and chain rule are also applicable here.

3.2 : Higher Order Partial Derivatives

Let $z = f(x,y)$ be a function of two variables. Then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of two variables x & y, $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = r, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = t$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{xy} = s, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{yx} = s$$

But they are equal (as in most case) When partial derivatives are continuous.

Note - 1 : In general $f_{xy} = f_{yx}$

Note - 2 : In all ordinary cases $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

On similar lines, we can define partial derivatives of orders higher than three.

Illustrative Examples

Example – 1 : ***If $z = x^2y + xy^2$, Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$***

Solution : $z = x^2y + xy^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2y + xy^2\right) = \frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial x}\left(xy^2\right) = 2xy + y^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2y + xy^2\right) = \frac{\partial}{\partial y}\left(x^2y\right) + \frac{\partial}{\partial y}\left(xy^2\right) = x^2 + 2xy$$

Example –2 : ***If*** $z = sin\left(\frac{x}{y}\right)$***, Find*** $\frac{\partial z}{\partial x}$ ***and*** $\frac{\partial z}{\partial y}$

Solution : Given $z = sin\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left\{sin\left(\frac{x}{y}\right)\right\} = cos\left(\frac{x}{y}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}cos\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left\{sin\left(\frac{x}{y}\right)\right\} = cos\left(\frac{x}{y}\right).\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = cos\left(\frac{x}{y}\right).\left(\frac{-x}{y^2}\right) = \frac{-x}{y^2}cos\left(\frac{x}{y}\right)$$

Example – 3 : ***If*** $u = (x^2 + y^2 + z^2)^{-1/2}$ ***show that*** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Solution : Given $u = (x^2 + y^2 + z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = \frac{-1}{2}\left(x^2+y^2+z^2\right)^{-3/2}.\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$= \frac{-1}{2}\left(x^2+y^2+z^2\right)^{-3/2}.2x = -x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{-x\left(x^2+y^2+z^2\right)^{-3/2}\right\}$$

$$= -\left[x.\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-3/2} + \left(x^2+y^2+z^2\right)^{-3/2}.\frac{\partial}{\partial x}(x)\right]$$

$$= -\left[x.\left(\frac{-3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}.2x + \left(x^2+y^2+z^2\right)^{-3/2}\right]$$

$$= -\left[\frac{-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \left(x^2+y^2+z^2\right)^{-3/2}\right] = -\left[\frac{-3x^2+\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{5/2}}\right]$$

$$= -\left[\frac{-2x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}\right] = \frac{2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

Similarly $\frac{\partial^2 u}{\partial y^2} = \frac{2y^2-z^2-x^2}{\left(x^2+y^2+z^2\right)^{5/2}}$, $\frac{\partial^2 u}{\partial z^2} = \frac{2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{2y^2-z^2-x^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{2x^2-y^2-z^2+2y^2-z^2-x^2+2z^2-x^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}} = \frac{0}{\left(x^2+y^2+z^2\right)^{5/2}} = 0$$

3.3 : Partial Differential Equation of First Order

Defination : A partial differential equation is a relation between the independent variables, dependent variable and it's partial derivatives.

If $z = f(x,y)$, z is called a function of two independent variables x & y. The partial derivatives.

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s, \frac{\partial^2 z}{\partial y^2} = t$$

An equation involving one or more partial derivatives is called a partial differentiation. The order of a partial differential equation (P.D.E) is the order of the highest partial derivative in the equation and it's degree is the degree of this derivative.

Example : $x\frac{\partial u}{\partial y} + 4y\frac{\partial u}{\partial x} = 2u + 3xy$ (Linear), $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = \frac{1}{c^2}\frac{\partial u}{\partial t}\right)$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 8u = 0$

A (P.D.E) are linear since partial derivatives as well as the dependent variable occur in the first degree.

A (P.D.E) is said to be homogeneous if each term of the equation contains either the dependent variable or one of its derivatives otherwise it is said to be non-homogenous.

FORMATION OF A P.D.E.

In case of ordinary differential equations which arises from the elimination of arbitary constants, in case of partial differential equations can be formed by the eliminations of arbitary constant or by the elimination of arbitary functions from a relation involving three or more variable.

Solutions of a partial -Differential Equation :

It is clear from above examples that a partial differential equation can results both from elimination of arbitary constants and from the elimination of arbitary functions. The selection $f(x,y,z,a,b) = 0$ of first order partial which contains two arbitary constants is called a **complete integral**.

A solution obtained from the complete integral by assigning particular values to the arbitary constants is called a **particular integral**.

Definition : A solution of a partial differential equation is said to be a complete solution or a complete integral if it contain as many arbitary constants as there are independent variables.

Definition : A general solution or a general integral of a partial equation is a relation involving arbitary functions which provides a solution to that equation.

Lagrange's Linear Equation is an equation of the type :

Pp + Qq =R

Where P, Q, R are functions of x, y, z and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$

Working Rule :

Step –1 : Write down the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step –2 : Solve the above auxiliary equations, let the two solutions be u = C_1, and v = C_2

Step –3 : Then $f(u,v)=0$, or $u=\phi(v)$ is the required solution of $Pp + Qq = R$

Method of Multipliers :

Let the auxiliary equations be $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

l, m, n may be constants or function of *x, y, z* then we have $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$

l, m, n are chosen in such a way that $lP + mQ + nR = 0$

Thus $ldx + mdy + ndz = 0$

Solve this differential equation, if the solution is $u = C_1$.

Similarly, choose another set of multipliess (l_1, m_1, n_1) and if the second solution is $v = C_2$.

$\therefore$ Required solution is $f(u,v)=0$

Illustrative Examples

Example –1 : ***Solve*** $z = ax + by + a^2 + b^2$

Solution : $z = ax + by + a^2 + b^2$ ....(1)

Differentiating partially to equation (1), with respect to *x*,

$$\frac{\partial z}{\partial x} = a \qquad ....(2)$$

Again differentiating partially to equation (1) w.r.t *y* we have $\frac{\partial z}{\partial y} = b$

Putting these values in (1) we get

$$\Rightarrow z = \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

$$\Rightarrow z = px + qy + p^2 + q^2 \left(\because \frac{\partial z}{\partial x} = P, \frac{\partial z}{\partial y} = q\right)$$

Example – 2 : ***Solve*** $z = f(x^2 - y^2)$

Solution : $z = f(x^2 - y^2)$ ....(1)

Differentiating the eq^n (1) partially w.r.t. *x*

$$\frac{\partial z}{\partial x} = f'\left(x^2 - y^2\right).2x \qquad ....(2)$$

Again differentiating partially w.r.t. *y* the eq^n (1)

$$\frac{\partial z}{\partial y} = f'\left(x^2 - y^2\right).(-2y) \qquad .....(3)$$

Divide the eq^n (2) by the eq^n (3) we have

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{f'\left(x^2-y^2\right)(2x)}{f'\left(x^2-y^2\right)(-2y)} = \frac{-x}{y}$$

$$\Rightarrow \frac{p}{q} = -\frac{x}{y} \Rightarrow py + qx = 0$$

It's the required partial differential equation of 1st order.

Example – 3 : ***Find the differential equation of all spheres of fixed radius 3 having their centres in the xy plane.***

Solution : From the co-ordinate geometry of three dimensions, the equation of any sphere of radius 3 having centre (a,b,c) in xy plane written as

$(x-a)^2 + (y-b)^2 + (z-c)^2 = 9$

i.e $(x-a)^2 + (y-b)^2 + z^2 = 9$ $(as\ c = 0)$ ....(1)

$\therefore$ Differentiating partially the eq^n (1) w.r.t. x we have

$$\Rightarrow 2(x-a) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow x - a + z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow a = x + z\frac{\partial z}{\partial x} = x + zp$$

Again differentiating partially to the eq^n (1) w.r.t. y we have

$$\Rightarrow 2(y-b) + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow y - b + z\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow b = y + z\frac{\partial z}{\partial y} = y + zq$$

Putting the value of a & b in the eq^n (1) we have

$$\Rightarrow (x - x - zp)^2 + (y - y - zq)^2 + z^2 = 9$$

$$\Rightarrow z^2p^2 + z^2q^2 + z^2 = 9$$

$$\Rightarrow z^2\left(p^2 + q^2 + 1\right) = 9$$

$\therefore$ This is the required partial differential equation of all spheres of fixed radius '3' having their centre in the xy plane.

Example –4 : ***Solve $f(xy + z^2, x + y + z)$***

Solution : $f(xy + z^2, x + y + z) = 0$ ....(1)

Let $u = xy + z^2$, $v = x + y + z$

So, the equation reduced to $f(u,v) = 0$

Differentiating w.r.t x

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial f}{\partial u}(y + 2zp) + \frac{\partial f}{\partial v}(1 + p) \qquad ....(1)$$

Again differentiating w.r.t. y

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}.\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}.\frac{\partial z}{\partial y}\right) \qquad ...(2)$$

$$= \frac{\partial f}{\partial u}(x + 2zq) + \frac{\partial f}{\partial v}(1 + q) \qquad ....(3)$$

Elliminating $\frac{\partial f}{\partial u}$ & $\frac{\partial f}{\partial v}$ we have $\begin{vmatrix} y+2zp, & 1+p \\ x+2zq, & 1+q \end{vmatrix} = 0$

$\Rightarrow (y + 2zp)\,(1+q) - (1+p)\,(x + 2zq) = 0$

$\Rightarrow (y + 2zp) + qy + 2zpq - x - 2zq - px - 2zpq$

$\Rightarrow (1 + q)y - (1+p)x + 2z\,(p - q) = 0$

This is the required partial differential equation of first order.

3.4 : State Homogeneous Function and Euler's Theorem for Two Variables

Homogeneous Function :

A function $f(x,y)$ is said to be homogeneous in x & y of degree 'n' if $f(tx, ty) = t^n f(x,y)$ where t is any constant. If it can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$

A function $f(x,y)$ is said to be homogeneous in x & y of degree n if sum of all powers of x & y is equal to n.

Example –1 : ***Test the following functions are homogeneous. (i.e. power of each term is same)***

(i) $f(x,y) = x^2y + xy^2$

(ii) $f(x,y) = 2xy^2 - 3x^2y + (x-y)^3$

(iii) $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$

(iv) $f(x,y) = x^2 + 2xy + y^2 + 3x + 3y$

Solution : (i) Given $f(x,y) = x^2y + xy^2$, Here $f(tx, ty) = t^2x^2.\,ty + tx.\,t^2y^2$

$= t^3x^2y + t^3xy^2 = t^3\,(x^2y + xy^2) = t^3 f(x,y)$, Hence f is homogeneous of degree 3.

(ii) $f(x,y) = 2xy^2 - 3x^2y + (x-y)^3$, $f(tx, ty) = 2.\,tx.\,t^2y^2 - 3.t^2x^2.ty + (tx - ty)^3$

$= 2t^3xy^2 - 3t^3x^2y + t^3\,(x-y)^3 = t^3\,[2xy^2 - 3x^2y + (x-y)^3] = t^3 f(x,y)$

Hence f is homogeneous of degree 3.

(iii) $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$, $f(tx,ty) = \sin^{-1}\left(\frac{tx}{ty}\right) = \sin^{-1}\left(\frac{x}{y}\right) = t^0 \sin^{-1}\left(\frac{x}{y}\right)$

Hence f is homogeneous of degree '0'

(iv) $f(x,y) = x^2 + 2xy + y^2 + 3x + 4y$ is not homogeneous.

Notes :

(1) If each terms in the expression of a function is of the same degree then function is homogeneous.

(2) If z is a homogeneous function of x & y of degree n then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also homogeneous of degree n –1

(3) If $z = f(x,y)$ is homogenous of degree n then we can put it in the from $z = x^n \phi\left(\frac{y}{x}\right)$

(4) Similarly a function $f(x,y,z)$ is said to be homogeneous of degree n in the variable x,y,z if $f(x,y,z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right)$ or $y^n \phi\left(\frac{x}{y}, \frac{z}{y}\right)$ or $z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right)$ otherwise $f(tx, ty, tz) = t^n f(x,y,z)$

3.5 : Euler's Theorem On Homogeneous Functions

Theorem –1 : If z is a homogeneous function of x and y of degree n then $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$

Proof : Since z is a homogeneous function of x and y of degree n we can put it as $z=x^n\phi\left(\frac{y}{x}\right)$.....(1)

Differentiating (1) partialy w.r.t, x and y we have

$$\frac{\partial z}{\partial x}=nx^{n-1}\phi\left(\frac{y}{x}\right)+x^n\phi'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right),\ \frac{\partial z}{\partial y}=x^n\phi'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right)$$

$$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nx^n\phi\left(\frac{y}{x}\right)-yx^{n-1}\phi'\left(\frac{y}{x}\right)+yx^{n-1}\phi'\left(\frac{y}{x}\right)=nx^n\phi\left(\frac{y}{x}\right)=nz$$

Eulers theorem can be extended to a homogeneous function of any number of variables. Thus if z is a homogeneous function of degree n in x,y,z then $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+z\frac{\partial z}{\partial z}=nz$

Theorem –2 : Given z is a homogeneous function of degree n is x and y then by Euler's theorem

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

Proof : Since z is a homogeneous function of x & y of degree n.

By Euler's theorem, we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$..........(i)

Differentiating (i) partially w.r.t x we have $\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$ ....(ii)

Differentitating (i) Particularly w.r.t y we have $x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$, $\left(\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}\right)$

$$\therefore x\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}\text{..........(iii)}$$

Multiplying (ii) by x, (iii) by y and adding

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

or $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+nz=n.nz$

or $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n^2z-nz=n(n-1)z$

Note :

(1) If z is a homogeneous function of x, y of degree n and $z=f(u)$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\cdot\frac{f(u)}{f'(u)}$

(2) If $f(u)=v(x,y,z)$ where v is a homogeneous function in x, y, z of degree n then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{n\,f(u)}{f'(u)}$

(3) If z is a homogeneous function of degree n then $x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$,

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$

Illustrative Examples

Example –1: ***Verify Euler's theorem :***

(a) If $u = x^2 + y^2 + z^2$***, then,*** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u$

(b) If $u = x^3 + y^3 + z^3 + 3xyz$***, then,*** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3u$

(c) If $u=\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$ ***then,*** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u$

(d) If $u=\frac{x^2}{zy^3}+\frac{y^2z^2}{x^6}$, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-2u$

Solution : (a) $u = x^2 + y^2 + z^2$. Here, u is a homogeneous function in three variables x, y and z of degree 2. Therefore, by Euler's Theorem, we get.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u$$

(b) Given that $u = x^3 + y^3 + z^3$. Here, u is a homogeneous function in x, y and z of degree 3. Therefore, by Euler's Theorem we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3u$$

(c) Given that $u=\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$

Here, u is a homogeneous function in variables x, y and z of degree zero. Therefore by Euler's Theorem, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$$

(d) Given that $u=\frac{x^2}{zy^3}+\frac{y^2z^2}{x^6}$. Here u is a function in x, y and z of degree –2. Therefore, by Euler's Theorem, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-2u$$

Example –2 : ***If*** $u = sin^{-1}\left\{\dfrac{(x+y)}{(\sqrt{x}+\sqrt{y})}\right\}$ ***then verify*** $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$ ***is equal to*** $\dfrac{1}{2}$ ***tan u***

Solution : Here $sin\,u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = f$ (say)

$\therefore$ f is a homogeneous function in x and y of degree $\left(1-\dfrac{1}{2}\right)$ i.e. $\dfrac{1}{2}$

$\therefore$ By Euler's Theorem of f we get $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=\dfrac{1}{2}f$

or, $x\dfrac{\partial}{\partial x}(sin\,u)+y\dfrac{\partial}{\partial y}(sin\,u)=\dfrac{1}{2}sin\,u$

or, $x\,cos\,u\dfrac{\partial u}{\partial x}+y\,cos\,u\dfrac{\partial u}{\partial y}=\dfrac{1}{2}sin\,u$

or , $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\dfrac{sin\,u}{cos\,u}$

or, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}tan\,u$

Example – 3 : ***Prove that if*** $z = sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$***, then*** $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=tan z$

Solution : Given $z = sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, Let $u=\dfrac{x^2+y^2}{x+y}$

So that $z = sin^{-1}\,u$, Here $u = sin\,z$

Given $u=\dfrac{x^2+y^2}{x+y}$, Since u is a homogeneous function of x & y of degree 1

So by Euler's theorem$= x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1\cdot u=u$

Given $u = sin\,z$,

$\dfrac{\partial u}{\partial x}=cos\,z.\dfrac{\partial z}{\partial x},\dfrac{\partial u}{\partial y}=cos\,z.\dfrac{\partial z}{\partial y}$

$\therefore x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=u$

$\Rightarrow x\,cos\,z\dfrac{\partial z}{\partial x}+y\,cos\,z\dfrac{\partial z}{\partial y}=sin\,z \Rightarrow cos\,z\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)=sin\,z$

$\Rightarrow\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)=\dfrac{sin\,z}{cos\,z}=tan\,z$

Example – 4 : ***If*** $\boldsymbol{cos u = \frac{x-y}{\sqrt{x}+\sqrt{y}}}$***, prove that*** $\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2} cot u}$.

Solution : $cos\,u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x[1+(y/x)]}{\sqrt{x}\left[1+\sqrt{y/x}\right]} = x^{1/2}\left\{\frac{1+(y/x)}{1+\sqrt{y/x}}\right\}$

i.e., $cos\,u = x^{1/2} g(y/x)$

$\Rightarrow$ cos u is homogeneous of degree $\frac{1}{2}\left(n=\frac{1}{2}\right)$

Applying Euler's theorem for the function cos u we have,

$$x\frac{\partial}{\partial x}(cos\,u) + y\frac{\partial}{\partial y}(cos\,u) = \frac{1}{2}\cdot cos\,u$$

i.e., $x(-sin\,u)\frac{\partial u}{\partial x} + y(-sin\,u)\frac{\partial u}{\partial y} = \frac{1}{2}cos\,u$

or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{cos\,u}{-sin\,u}$

Thus $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}cot\,u$.

Exercise

1. Find the first order partial derivatives of the followings

(a) $z = x^3y^2$ (b) $z = tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ (c) $u = log\,(x^2+y^2)$ (d) $u = sin^{-1}\left(\frac{x}{y}\right)$

(e) $u = x^y$ (f) $u = xe^y + ye^x$ (g) $u = sin\left(\frac{x}{y}\right)$ (h) $u = \frac{x}{y}tan^{-1}\left(\frac{y}{x}\right)$

2. Find the first order partial derivative for each of the functions.

(a) $u = xyz$ (b) $u = x^y + y^z + z^x$ (c) $u = e^{xyz}$ (d) $u = ln\left(\frac{x+y}{z}\right)$

3. If $u = e^x(x\cos y - y\sin y)$, verify $u_{xx} + u_{yy} = 0$.
4. If $u = log\,(x^3 + y^3 + z^3 - 3xyz)$ show that

(a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ (b) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -9(x+y+z)^{-2}$

5. If $u = log\,(tanx)$ prove that $sin\,2x\frac{\partial u}{\partial x} + sin\,2y\frac{\partial u}{\partial y} = 2$.

6. If $u = e^{xyz}$, prove that $\frac{\partial^2 u}{\partial x\,\partial y\,\partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

7. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$ prove that $V_{xx} + V_{yy} + V_{zz} = m\,(m+1)\,r^{m-2}$
8. If $u = \log(x^3 + y^3 - x^2y - xy^2)$ show that $u_{xx} + 2u_{xy} + u_{yy} = -4(x+y)^2$.
9. If $u = \dfrac{x^3 + y^3}{\sqrt{x+y}}$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{5}{2}u$.
10. If $u = sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ show that $\dfrac{\partial u}{\partial x} = \dfrac{-4}{x}\dfrac{\partial u}{\partial y}$.
11. Verify Euler's theorem in following cases

    (a) $z = xy + \dfrac{(x+y)^4}{xy}$ (b) $z = axy + byz + czx$

    (c) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$
12. If $u = \ln\left(\dfrac{x^2+y^2}{xy}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.
13. If $z = \dfrac{x^2+y^2}{xy}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.
14. If $u = x^2 + y^2 + z^2$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2u$.
15. If $u = tan^{-1}(x^2 + y^2 + z^2)$, show that $xu_x + yu_y + zu_z = sin\,2u$.
16. If $u = sin^{-1}\left(\dfrac{x}{y}\right) + tan^{-1}\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Answers

1. **(a)** $3x^2y^2, 2x^3y$ **(b)** $\dfrac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)}, \dfrac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}$

(c) $\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}$ **(d)** $\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}$

(e) $yx^{y-1}, x^y \ln x$ **(e)** $e^y + ye^x, xe^y + e^x$

(g) $cos\left(\dfrac{x}{y}\right)\cdot\dfrac{1}{y}, cos\left(\dfrac{x}{y}\right)\cdot\left(\dfrac{-x}{y^2}\right)$

2. **(a)** yz, zx, xy **(b)** $yx^{y-1} + z^x \ln z, x^y \ln x + zy^{z-1} + y^z \ln y$

(c) $yz\,e^{xyz}, zx\,e^{xyz}, xy\,e^{xyz}$ **(d)** $\dfrac{1}{x+y}, \dfrac{1}{x+y}, \dfrac{-1}{z}$.

Objective type Questions with Answers

Multiple type or dash fill up type questions. carries 2 Marks

1. Answer the following questions :

(a) Find $\frac{\partial}{\partial y}$ [sin $(x + 2yz)$]

Ans. $2z \cos(x + 2yz)$

(b) Write True or False :

(i) If $f(x, y) = \tan(\tan^{-1}x + \tan^{-1}y)$, then $\frac{f_x}{f_y} = \frac{1+y^2}{1+x^2}$

Ans. T

(ii) If $f(x, y) = x^y$, then $f_{yy} = x^{y-1}(\ln x)^2$

Ans. F

(c) If $z = \sin(x^2\ y^2)$, then $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 =$ ________

[$4x^2y^2 \cos^2(x^2y^2)$, $4x^2y^2(x^2 + y^2)$, $4x^2y^2(x^2+y^2) \cos^2(x^2y^2)$, none]

Ans. $4x^2y^2(x^2 + y^2) \cos^2(x^2\ y^2)$

(d) If $u = x \cos y$, what is $\frac{\partial^2 u}{\partial x^2}$?

Ans. 0

(e) Find $\frac{\partial}{\partial x} ln\left(3x + y^3\right)$.

Ans. $\frac{3}{3x + y^3}$

(f) If $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$, find $xu_x + yu_y + zu_z$.

Ans. $\frac{1}{2}u$

2. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x, y)$ is given as

(i) $ax^2 + by^2 + cxy + dx + ey + f$ (ii) $3x^2 - 4x^2y^2 + 6y^2$

(iii) $\frac{x+2y}{y+2x}$ (iv) $\frac{x}{x^2+y^2}$ (v) e^{x+2y}

(vi) $\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ (vii) $x^2 \cos y + y^2 \sin x$

(viii) $x^y + y^x$ (ix) $x \log y \log(x + y^2)$ (x) $\tan xy$

Ans. (i) $2ax + cy + d$, $2by + cx + e$ (ii) $2x(3 - 4y^2)$, $4y(3 - 2x^2)$

(iii) $\dfrac{-3y}{(y+2x)^2}, \dfrac{3x}{(y+2x)^2}$ (iv) $\dfrac{y^2-x^2}{(y^2+x^2)^2}, -\dfrac{2xy}{(y^2+x^2)^2}$

(v) e^{x+2y}, $2\ e^{x+2y}$ (vi) $\dfrac{1}{\sqrt{y^2-x^2}} - \dfrac{y}{x^2+y^2}, \dfrac{-x}{y\sqrt{y^2-x^2}} + \dfrac{x}{x^2+y^2}$

(vii) $2x\cos y + y^2\cos x$, $2y\sin x - x^2\sin y$ (viii) $yx^{y-1} + y^x \ln y$, $x^y \ln x + xy^{x-1}$

(ix) $\log y\ \log(x+y^2) + \dfrac{x\log y}{x+y^2}, \dfrac{x}{y}\ \log(x+y^2) + \dfrac{2xy\log y}{x+y^2}$

(x) $y\sec^2 xy$, $x\sec^2 xy$

3. If $z = xy\ f\left(\dfrac{y}{x}\right)$, find the value of $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} =$ ———.

Ans. 2z

4. If $x = r\cos\theta$, $y = r\sin\theta$, find the value of $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 =$ ———.

Ans. 1

5. If $u = f\left(\dfrac{y}{x}\right)$, then find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} =$ ———.

Ans. 0

6. Using the definition of partial derivative, find $f_x(1, 2)$ and $f_y(1, 2)$ where $f(x, y) = x^2 + xy + y^2$.

Ans. 4, 5

7. If $u = \log\sqrt{x^2+y^2}$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2}$ at $x = 1, y = 1$.

Ans. 0

8. If $u = e^{ax}\sin by$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} =$ ———.

Ans. $(a^2 - b^2)u$

9. If $u = e^{xyz}$, then find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$.

Ans. $3xyz\ e^{xyz}$

10. If $f(x, y, z) = \dfrac{z}{x} + \dfrac{x}{y}$, find the value of $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} =$ ——— .

Ans. 0

11. If $u = x^2 + y^2 + z^2$, then find the value of $x\,u_x + y\,u_y + z\,u_z =$ ——— .

Ans. 2u