2 CHAPTER

Curvature

2.0 : Introduction

Curvature plays an important role in laying curved railway tracks etc. The rate of change of the direction of the tangent line, at a point on the curve with respect to arc lengths along a curve is called curvature of the curve. Curvature measures the degree of sharpness of the bending of a curve at that point on the curve. An important property of a curve is called curvature.

2.1 : Concept of plane curve and measure of curvature

Plane curve : Intuitively a plane curve may be regarded as a set of points (x, y) in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ determined by the range of function $f : [a, b] \to \mathbb{R}^2$, where $f(t) = (f_1(t), f_2(t))$, $a \le t \le b$ and f_1, f_2 are two real continuous functions (of a single real variable) defined on $[a, b]$ (where $a < b$) and where $\mathbb{R}$ is the set of all real numbers.

Formally the **function** $f : [a, b] \to \mathbb{R}^2$ is called a **plane curve** or a curve in $\mathbb{R}^2$. Here, the equations $x = f_1(t), y = f_2(\mathrm{t})$ are called the parametric equations of the curve. The curve $f : [a, b] \to \mathbb{R}^2$ where $f(t) = (f_1(t), f_2(t))$, $a \le t \le b$, is said to be **continuous** if f_1, f_2 are continuous on $[a, b]$. Unless otherwise stated by a **curve** we shall mean a7 curve which is **continuous.**

The function $\psi : [0, 2\pi] \to \mathbb{R}^2$ where $\psi(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$, is a **circle** which is a plane curve whose parametric equations are

$$x = a \cos t,\ y = a \sin t,\ t \in [0, 2\pi].$$

If a plane curve $\psi : [a, b] \to \mathbb{R}^2$ is given by $\psi(x) = (x, f(x))$, $a \le x \le b$ then the curve may also be represented as

$$y = f(x),\ a \le x \le b \text{ where } y = f(x)$$

is called the cartesian equation of the curve and (x, y) are rectangular cartesian co-ordinates of a point on the curve.

If $\psi : [a, b] \to \mathbb{R}^2$ be a plane curve where $\psi(t) = (f(t), g(t))$, $a \le t \le b$ then the function $\psi_1 : [t_1, t_2] \to \mathbb{R}^2$

$$\psi_1\,(t) = (f\,(t),\, g\,(t)),\ a \le t_1 \le t_2 \le b$$

is called an **arc** of the given curve $\psi : [a, b] \to \mathbb{R}^2$. The parametric equations of the arc are

$$x = f\,(t),\ y = g\,(t),\ t_1 \le t \le t_2.$$

Here we assume that the curve is **rectifiable**. Then any arc of a curve (the curve itself is also an arc of itself) has a length.

If s be the length of an arc, AP of a curve [Fig.-2.1] where P is a variable point on the curve and A is a fixed point on it then it can be shown that by formula of Tangents and Normals :

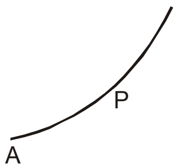


Fig. 2.1

(i) $\tan\psi = \dfrac{dy}{dx}$ (ii) $\sin\psi = \dfrac{dy}{ds}$ (iii) $\cos\psi = \dfrac{dx}{ds}$

(iv) $\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$ (v) $\dfrac{ds}{dy} = \sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$

(vi) $\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}$ (vii) $\tan\phi = r.\dfrac{d\theta}{dr}$

(viii) $\sin\phi = r.\dfrac{d\theta}{ds}$ (ix) $\cos\phi = \dfrac{dr}{ds}$ (ix) $\dfrac{ds}{d\theta} = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$

(x) $\dfrac{ds}{dr} = \sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}$ (xi) $\rho = p\sin\phi$

(xii) $\dfrac{1}{\rho^2} = \dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ (xiii) $\dfrac{1}{\rho^2} = u^2+\left(\dfrac{du}{d\theta}\right)^2$

Theorem – 1 : ***Prove that*** $\boldsymbol{\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}$ ***for the curve y = f(x)***

Proof : Let P and Q be to neighbouring points on the curve $y = f(x)$ whose co-ordinates be (x,y) and $(x + \delta y,\, y + \delta y)$

In the right angle Δ PRQ (*Fig. 2.2*)

$$(Chord\ PQ)^2 = (\delta x)^2 + (\delta y)^2$$

$$\Rightarrow \left(\frac{Chord\ PQ}{Arc\ PQ}\right)^2\left(\frac{Arc\ PQ}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

$$\Rightarrow \left(\frac{Chord\ PQ}{Arc\ PQ}\right)^2\left(\frac{\delta s}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

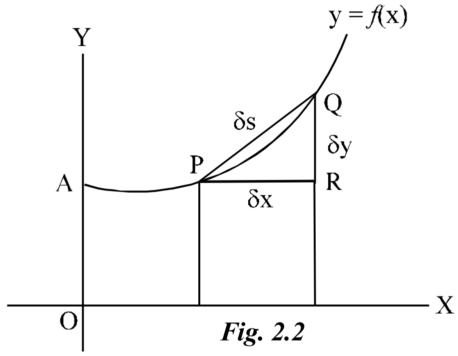


Fig. 2.2

When Q $\to$ P, $\delta x \to 0$ and $\lim_{Q \to P} \frac{Chord\ PQ}{Arc\ PQ} = 1$

So we have $\lim_{\delta x \to 0} \frac{\delta s}{\delta x} = \lim_{\delta x \to 0} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$

$$\Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ similarly } \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

2.2 : Curvature of a plane curve

Let A be a fixed point on a plane curve on which P, Q are two neighbouring points (Fig.-2.3). Let the length of arc AP be s and AQ be $s + \Delta s$.

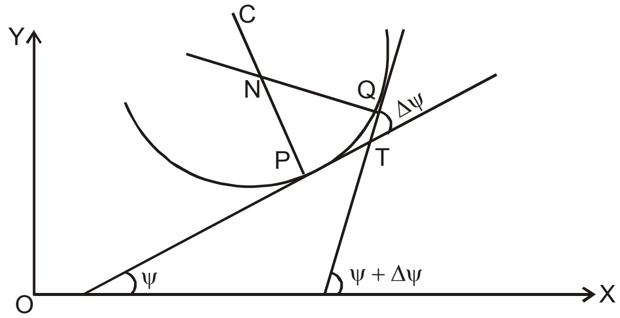


Fig. 2.3

Let the tangents at P and Q meet at T where the tangents at P and Q make angles ψ and $\psi + \Delta\psi$ with positive direction of the x-axis. Then the **whole bending** or **curvature** of the arc PQ is given by $\Delta\psi$ and the **average binding** per unit length is given by $\frac{\Delta\psi}{\Delta s}$.

We shall define the **curvature** of a curve in the immediate neighbourhood of a given point to be the **rate of deflection from the tangent** at that point. Then it will be reasonable to define **curvature** of a plane curve (shown in Fig.-2) at a given point P of the curve as the value of $\lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s}$ provided the limit exists. But $\lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s}$ (when exists) = $\frac{d\psi}{ds}$. Here, (s, ψ) are called the **intrinsic co-ordinates** of the point P on the curve [Fig.-2.4].

Then the **curvature** at P (s, ψ) of a given plane curve is defined to be $\frac{d\psi}{ds}$.

Theorem – 2 : ***Formula for radius of curvature in terms of 'S' and 'ψ' or Prove that*** $\rho = \frac{ds}{d\psi}$ ***(Intrinisic Curve)***

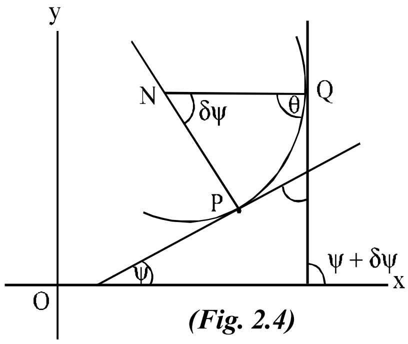
Proof : Let P be given point on the curve and Q be another point on it which is very near to 'P'. Let normals at 'P' and 'Q' meet at 'N'. Let are PQ = δS. Let tangent at 'P' makes an angle

'ψ' with the poistive direction of *x-axis* and let tangent at 'Q' makes an angle $\psi + \delta\psi$ with the positive direction of *x-axis.*

Then $\angle PNQ = \delta\psi$ (fig. 2.4)

[Since the angle between two lines is same as the angle between their normals.]

Then $C = \lim_{Q\to P}(PN) = \lim_{\delta s\to 0}(PN)$



(Fig. 2.4)

But from $\Delta\ PQN$, $\dfrac{PN}{Sin\angle PQN} = \dfrac{PQ}{sin\,\delta\psi}$

$$\therefore\ PN = \frac{PQ}{sin\,\delta\psi}.sin\,\angle PQN = \frac{PQ}{sin\,\delta\psi}.sin\,\theta$$

$$= \frac{\delta c}{\delta s}.\frac{\delta s}{\delta\psi}.\frac{\delta\psi}{sin\delta\psi}.sin\,\theta \text{ [Where Chord } \overline{PQ} = \delta c\,]$$

Let Q $\to$ P, So that $\delta c \to 0$, $\delta s \to 0$, $\delta\psi \to 0$ and $Q\to 90^0$

$$\therefore\ \lim_{Q\to P}(PN) = \lim_{\delta s\to O}\left[\frac{\delta c}{\delta s}.\frac{\delta s}{\delta\psi}.\frac{\delta\psi}{sin\,\delta\psi}.sin\,\theta\right]$$

or $\rho = 1.\dfrac{ds}{d\psi}\cdot 1.sin\,90$

or $\rho = \dfrac{ds}{d\psi}$ $\left[\begin{array}{l} \because \dfrac{\delta c}{\delta s} = \dfrac{Chord\ \overline{PQ}}{Arc\ PQ} \to 1,\ as\ Q\to P \\ \lim\limits_{\theta\to 0}\dfrac{\theta}{sin\,\theta} = 1,\ sin\,90 = 1 \end{array}\right]$

Note :
- The total bending or total curvature of the arc PQ is defined to be the anlge $\delta\psi$.
- The average curvature of the arc PQ is defind to be ratio $\delta\psi/\delta s$.
- The relation between s and ψ is called the intrinsic equation of a curve. Therefore $\rho = ds/d\psi$ is known as intrinsic formula for radius of curvature.
- The curvature of the curve at the point P is defined as the reciprocal of the radius of curvature at P. Hence the curvature at $P=1/\rho=d\psi/ds$.

Example – 1 : Find 'ρ' for the catenary whose intrinsic equation is $S = C\tan\psi$.

Solution. We have $S = C\,tan\psi$

We know $\rho = \dfrac{ds}{d\psi} = C\,sec^2\,\psi$

2.3 : Radius of curvature, centre of curvature and circle of curvature

Let the normals at P and Q (Fig.-2.4) meet at N. Here, we observe that N $\to$ C as Q $\to$ P along the curve *i.e.* as $\Delta s \to 0$. Further if $\dfrac{d\psi}{ds}. \neq 0$ then the reciprocal of the curvature at P is given by $\frac{d\psi}{ds}$ which is usually denoted by ρ and is called the **radius of curvature** of the curve at P. Thus we have $\rho = \frac{d\psi}{ds}$ If $\frac{d\psi}{ds} \neq 0$.

The circle with centre C (mentioned above) and radius ρ is called the **circle of curvature** at P and the point C is called the **centre of curvature** corresponding to P.

Remark. If the given curve is a straight line then ψ is constant and consequently $\frac{d\psi}{ds} = 0$ at every point. So curvature of a straight line is zero at every point. In this case, the radius of curvature ρ is **undefined**. We take $\rho = \infty$ (not a real number but **extended real number**) for a straight line in proving the formulae for the radius of curvature :

$$\frac{d\psi}{ds} = \cos\psi,\ \frac{d\psi}{ds} = \sin\psi,\ \sin\phi = r\,\frac{d\psi}{ds}\,\frac{d\theta}{ds},\ \cos\phi = \frac{dr}{ds}$$ where the notations are usual.

2.4 : Formulae for radius of curvature

Here, we evaluate radius of curvature when the equation of the curve is given in different forms.

I. The equation of the curve is given in Cartesian co-ordinates *i.e.* $y = f(x)$ (y is an explicit function of x)

We know that $\frac{dy}{dx} = \tan\psi$ ... (1)

where ψ is the angle made by the tangent at the point (x, y) to the curve $y = f(x)$ with x-axis.

Then differentiating (1) with respect to x, we have

$$\frac{d^2y}{dx^2} = \sec^2\psi\,\frac{d\psi}{dx} = \sec^2\psi\,\frac{d\psi}{ds}\,\frac{ds}{dx} = \sec^3\psi\,\frac{d\psi}{ds}\quad\left(\because \frac{dx}{ds} = \cos\psi\right)$$

If ρ be the radius of curvature, then

$$\rho = \frac{ds}{d\psi} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = \frac{\sec^2\psi.\sec^2\psi}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{\frac{1}{2}}(1+\tan^2\psi)}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

or $$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad ...(2)$$

where $\tan\psi = \frac{dy}{dx}$ and $\frac{d^2y}{dx^2} \neq 0$.

The relation (2) is the formula for radius of curvature when the equation of the curve is $y = f(x)$.

Note. We see from (2) that ρ is positive or negative according as $\frac{d^2y}{dx^2}$ is positive or negative, *i.e.* according as the given curve is concave upwards or downwards. If $\frac{d^2y}{dx^2} = 0$, then the formula given in (2) fails. In such case, the formula for ρ would be evaluated as

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} \quad \text{where } \frac{d^2x}{dy^2} \neq 0$$

II. The equation of the curve is given in implicit form of Cartesian co-ordinates *i.e f (x,y) = 0*

Let the equation of the curve be $f(x,y) = 0$

Differentiating both sides w.r.t x we get.

$$\frac{df}{dx} + \frac{df}{dy}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{df/dx}{df/dy} = -\frac{f_x}{f_y}$$

Again $$\frac{d^2y}{dx^2} = \frac{-f_y\frac{d}{dx}(f_x) - f_x\frac{d}{dx}(f_y)}{f_y^2} \qquad \left[f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}\right]$$

$$= \frac{-f_y\left[f_{xx} + f_{yx}\frac{dy}{dx}\right] - f_x\left[f_{xy} + f_{yy}\frac{dy}{dx}\right]}{f_y^2} = \frac{-f_y\left[f_{xx} + f_{yx}\left(-\frac{f_x}{f_y}\right)\right] - f_x\left[f_{xy} + f_{yy}\left(-\frac{f_x}{f_y}\right)\right]}{f_y^2}$$

$$= \frac{-(f_y)^2 f_{xx} - 2f_x.f_y.f_{yx} + (f_x)^2 f_{yy}}{f_y^3} \quad \text{but } \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2},\ f_{yy} = \frac{\partial^2 f}{\partial y^2},\ f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$$

$$\rho = \frac{\left[1+\left(-\frac{f_x}{f_y}\right)^2\right]^{3/2}}{-(f_y)^2 f_{xx} - 2f_xf_yf_{xy} + (f_x)^2 f_{xy}} = -\frac{\left(f_x^2 + f_y^2\right)^{3/2}}{\left(f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2\right)} \quad \text{(In magnitude)}$$

Therefore interchanging x and y in the above formula for ρ, we get

$$\rho = \frac{\{1+(dx/dy)^2\}^{3/2}}{d^2x/dy^2}$$

The formula is useful when dy/dx is infinite i.e. $dx/dy=0$ i.e. when the tangent is the perpendicular to x-axis.

If at any point of a curve $d^2y/dx^2=0$, the point is called a point of inflexion.

III. The equation of the curve is given in parametric form, *i.e.*, $x = \phi\ (t)$, $y = \psi\ (t)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$$ where prime ($'$) denotes the derivative with respect to t and $x' \neq 0$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx}$$

$$= \frac{y''x' - y'x''}{(x')^2}\frac{1}{x'} = \frac{x'y'' - y'x''}{(x')^3}$$

Substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in formula (2), we get

$$\rho = \frac{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} \quad ...(4)$$

where $\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} \neq 0$

Note :

- The value of ρ is positive or negative according as d^2y/dx^2 is positive or negative i.e. according as the curve is concave upwards or down words. However in numerical problems we shall be required to find only the length of the radius of curvature and we shall not be bothered with its sign. So we should reject the negative sign whenever we get a negative value of ρ. The curvature is zero at a point of inflexion.
- From the definition, it is clear that the value of ρ depends on the curve and not on the co-ordinate.

Illustrative Examples

Example –1 : ***Find the radius of curvature at the point (x,y) on the parabola $y^2=4ax$***

Solution : The given curve $y^2 = 4ax$

Differentiating eqn(1) w.r.t. x, we get

$$2y\frac{dy}{dx}=4a \text{ or} \frac{dy}{dx}=\frac{2a}{y} \text{ So} \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{2a}{y}\right)=-\frac{2a}{y^2}\frac{dy}{dx}=-\frac{2a}{y^2}.\frac{2a}{y}=-\frac{4a^2}{y^3}$$

$$\therefore \rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\left(1+\frac{4a^2}{y^2}\right)^{3/2}}{\frac{-4a^2}{y^3}}=\frac{\left(y^2+4a^2\right)^{3/2}}{y^3}\left(\frac{-y^3}{4a^2}\right) \quad =-\frac{1}{4a^2}\left(4ax+4a^2\right)^{3/2},$$

from eqn (1)

$$=-\frac{1}{4a^2}.a^{3/2}.8(x+a)^{3/2}=\frac{2}{\sqrt{a}}(x+a)^{3/2}$$ negative sign is negleeted as ρ is a length.

Example – 2 : ***Find the radius of curvature of parabola $y^2=4ax$ at the origin.***

Solution : Proceed example (1) we get ρ at *(x,y)* $=\frac{2}{\sqrt{a}}(x+a)^{3/2}$

$\therefore$ ρ at origin i.e. at (0,0) $=\frac{2}{\sqrt{a}}(0+a)^{3/2}=2a$

Example – 3 : ***Find the radius of curvature of the curve $y= e^x$ at the point where it crosses the y-axis.***

Solution : The given curve is $y = e^x$

$\therefore$ $dy/dx=e^x$ and $d^2y/dx^2 = e^x$

$$\therefore \rho \text{ at } (x,y)=\frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}=\frac{\left(1+e^{2x}\right)^{3/2}}{e^x}$$

The curve $y = e^x$ crosses the y-axis (i.e. the straight line $x = 0$) at the point (0,1)

$$\therefore \rho \text{ at } (0,1)=\frac{\left(1+e^0\right)^{3/2}}{e^0}=\frac{(1+1)^{3/2}}{1}=2\sqrt{2}$$

Example – 4 : ***Find the radius of curvature at (x,y) on the curve $a^2y = x^3 - a^3$***

Solution : The given curve is $a^2y = x^3 - a^3$

Differenting w.r.t. '*x*' twice we get respectively $\frac{dy}{dx}=\frac{1}{a^2}\left(3x^2\right)$ and $\frac{d^2y}{dx^2}=\frac{6x}{a^2}$

$$\therefore \rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\left[1+\left(\frac{3x^2}{a^2}\right)^2\right]^{3/2}}{\frac{6x}{a^2}}=\frac{\left(a^4+9x^4\right)^{3/2}}{a^6}.\frac{a^2}{6x}=\frac{\left(a^4+9x^4\right)^{3/2}}{6a^4x}$$

Example – 5 : ***Find the points on the parabola $y = x^2$ where the radius of curvature is 4.***

Solution : The given curve is $y = x^2$. ...(1)

Let the required point be (x_1, y_1), so eq^n (1) gives $y_1=x_1^2$

From eq^n (1), $\frac{dy}{dx}=2x=2x_1$ at (x_1,y_1) and $\frac{d^2y}{dx^2}=2$ at (x_1,y_1)

$$\therefore \rho \text{ at } (x_1, y_1) = \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left(1+4x_1^2\right)^{3/2}}{2}$$

$$\text{Given} \frac{\left(1+4x_1^2\right)^{3/2}}{2} = 4 \Rightarrow \left(1+4x_1^2\right)^{3/2} = 8$$

$$\Rightarrow 1+4x_1^2 = (8)^{2/3} = 4 \Rightarrow 4x_1^2 = 3 \Rightarrow x_1 = \pm\sqrt{3}/2$$

$\therefore y_1 = x_1^2 = \dfrac{3}{4}$. Hence the required points are $\left(\dfrac{\sqrt{3}}{2}, \dfrac{9}{4}\right)$ and $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{9}{4}\right)$

Example – 6 : ***Find the radius of curvature of the following curves.***

(a) x = a(cost + t sint), y = a(sint – t cost) ***[B.P.U.T.- 2012***

(b) $x^{2/3} + y^{2/3} = a^{2/3}$ ***, at the point (0, a)*** ***[B.P.U.T.- 2011***

Solution :(a) Given $x = a\,(\cos t + t \sin t), y = a\,(\sin t - t \cos t)$

$$\frac{dx}{dt} = a(-\sin t + t\cos t + t\sin t) = at\cos t$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\Rightarrow \frac{d^2y}{dx^2} = y_2 = \sec^2 t.\frac{dt}{dx} = \frac{\sec^2 t}{at\cos t} = \frac{\sec^3 t}{at}$$

$$\therefore \text{ We know that } \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left(1+\tan^2 t\right)^{3/2}}{\dfrac{\sec^3 t}{at}} = \left(\sec^2 t\right)^{3/2} \times \frac{at}{\sec^3 t} = at$$

$$\Rightarrow \rho = at$$

(b) $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiate w.r.t x we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \;\therefore \frac{dy}{dx} = y_1 = -\frac{y^{1/3}}{x^{1/3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y_2 = -\frac{x^{\frac{1}{3}}.\frac{1}{3}y^{\frac{-2}{3}}\frac{dy}{dx} - y^{\frac{1}{3}}.\frac{1}{3}x^{\frac{-2}{3}}}{x^{2/3}}$$

$$= -\frac{\frac{1}{3}\left[\left(-\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)\dfrac{x^{\frac{1}{3}}}{y^{\frac{2}{3}}} - \dfrac{y^{\frac{1}{3}}}{x^{\frac{2}{3}}}\right]}{x^{2/3}} \quad \left(\because \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)$$

$$= \frac{\frac{1}{3}\left[\frac{1}{y^{\frac{1}{3}}} + \frac{y^{\frac{1}{3}}}{x^{2/3}}\right]}{x^{2/3}} = \frac{\frac{1}{3}\left[\frac{x^{2/3}+y^{2/3}}{x^{2/3}y^{1/3}}\right]}{x^{2/3}} = \frac{1}{3}\left[\frac{x^{2/3}+y^{2/3}}{x^{4/3}y^{1/3}}\right] = \frac{a^{2/3}}{3x^{4/3}y^{1/3}}$$

We know $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$

$$= \frac{\left(1+\frac{y^{2/3}}{x^{2/3}}\right)}{\frac{a^{2/3}}{3x^{4/3}y^{1/3}}} = \left(\frac{x^{2/3}+y^{2/3}}{x^{2/3}}\right)^{3/2} \times \frac{3x^{4/3}y^{1/3}}{x^{2/3}} = \left(\frac{a^{2/3}}{x^{2/3}}\right)^{3/2} \cdot \frac{3x^{4/3}y^{1/3}}{a^{2/3}} = \frac{a}{x} \cdot \frac{3x^{4/3}y^{1/3}}{a^{2/3}}$$

or $\rho = 3\ (axy)^{1/3}$

$\because \rho\big]_{(0,3)} = 0$

Example – 7 : ***Show that the ellipse x = a cos t, y = b sin t, a > 0, b > 0 has its largest curvature on its major axis and its smallest curvature on its minor axis.***

Solution : Equation of the ellipse is

$x = \text{a cos t}; y = \text{b sin t}$ ....(1)

$x' = -\text{a sin t}; y' = \text{b cos t}$

$x'' = -\text{a cos t}; y'' = -\text{b sin t}$

$$\text{Curvature } \rho = \frac{x'\,y'' - x''y'}{\left(x'^2+y'^2\right)^{3/2}} - \frac{(-a\sin t)(-b\sin t) - (-a\cos t)(b\cos t)}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}$$

$$\rho = \frac{ab\left(\sin^2 t+\cos^2 t\right)}{\left[a^2\sin^2 t + b^2\cos^2 t\right]^{3/2}}$$

$$\rho = \frac{ab}{\left[a^2(1-\cos 2t)+b^2(1+\cos 2t)\right]^{3/2}}$$

$$\rho = \frac{ab}{\left[\left(a^2+b^2\right)-\left(a^2-b^2\right)\cos 2t\right]^{3/2}}$$

(i) ρ will be maximum, when cos $2t$ is maximum and max cos $2t = 1$
i.e., cos $2t$ = cos 0, cos 2π $\quad\therefore$ t = 0, π
$\therefore$ From (1), $x = \pm a$, $y = 0$
$\therefore$ ρ is maximum at $(a, 0)$ and $(-a, 0)$
i.e., ρ is maximum on its (a, 0) and (–a, 0)

(ii) ρ will be minimum, when cos 2t is minimum and minimum value of cos 2t = –1
$\therefore$ 2t = π, – π $\quad\therefore\ t = \frac{\pi}{2}, -\frac{\pi}{2}$ $\quad\therefore$ x = 0, y = ± b
$\therefore$ ρ is minimum on minor axis.

Example – 8 : ***Show that the radius of curvature of any point of the astroid x = a cos³ θ, y = a sin³ θ is equal to three times the length of the perpendicular from the origin to the tangent.***

[B.P.U.T. - 2008, 2014, 2016

Solution : Given that $x = cos^3\theta$, $y = a\ sin^3\ \theta$

$$\therefore \frac{dx}{d\theta} = -3a\,cos^2\,\theta\,sin\,\theta, \text{ and } \frac{dy}{d\theta} = 3a\,sin^2\,\theta.cos\,\theta$$

$$\Rightarrow \frac{dy}{dx} = y_1 = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{3a\,sin^2\,\theta\,cos\,\theta}{3a\,cos^2\,\theta\,sin\,\theta} = -tan\,\theta$$

$$\Rightarrow \frac{d^2y}{dx^2} = y_2 = -sec^2\,\theta\frac{d\theta}{dx} = -sec^2\,\theta \times \frac{1}{-3a\,cos^2\,\theta.\,sin\,\theta} = \frac{1}{3a\,cos^4\,\theta\,sin\,\theta}$$

Now equation of the tangent at any point θ is

$y - a\ sin^3\theta = -tan\theta\ (x - a\ cos^3\theta)$

$$\Rightarrow y - a\,sin^3\,\theta = -x\frac{sin\,\theta}{cos\,\theta} + a\,sin\,\theta.\,cos^2\,\theta$$

$$\Rightarrow y\,cos\theta - a\,sin^3\,\theta.cos\theta = -x\,sin\,\theta + a\,sin\,\theta.cos^3\,\theta$$

$$\Rightarrow x\,sin\,\theta + y\,cos\,\theta - a\,sin\,\theta\,cos\,\theta\left(sin^2\,\theta + cos^2\,\theta\right) = 0$$

$$\Rightarrow x\,sin\,\theta + y\,cos\theta = a\,sin\,\theta\,cos\theta$$

∴ Length of the perpendicular from the origin (0,0) to the tangent $x\ sin\theta + y\ cos\theta - a\ sin\theta$. cosθ is

$$\rho = \frac{0 + 0 + a\,sin\,\theta.\,cos\,\theta}{\sqrt{sin^2\,\theta + cos^2\,\theta}} = a\,sin\,\theta.\,cos\,\theta$$

Now we have to find ‘ρ’

We know that $\rho = \dfrac{\left(1 + y_1^2\right)^{3/2}}{y_2}$

$$= \frac{\left(1 + tan^2\,\theta\right)^{3/2}}{\dfrac{1}{3a\,cos^4\,\theta.\,sin\,\theta}} = sec^3\,\theta.3a\,cos^4\,\theta.\,sin\,\theta = \frac{3a\,cos^4\,\theta.\,sin\,\theta}{cos^3\,\theta}$$

or $\rho = 3a\,sin\,\theta.\,cos\,\theta$

= 3 × length of the prependicular from the origin to the tangent.

Example – 9 : ***Show that the radius of curvature at a point of the curve x=ae^θ (sinθ – cosθ) and y=ae^θ (sinθ + cosθ) is twice the distance of the tangent at the point from the origin.***

[B.P.U.T. - 2012

Solution : Given that $y = ae^{\theta}(sin\theta + cos\theta)$ and $x = ae^{\theta}(sin\theta - cos\theta)$

$$\Rightarrow \frac{dy}{d\theta} = ae^{\theta}(cos\,\theta - sin\,\theta) + (sin\,\theta + cos\,\theta)ae^{\theta}$$

$$= ae^{\theta}\,cos\,\theta - ae^{\theta}\,sin\,\theta + ae^{\theta}\,sin\,\theta + ae^{\theta}\,cos\,\theta = 2ae^{\theta}\,cos\,\theta$$

and $\dfrac{dx}{d\theta} = ae^{\theta}(\cos\theta + \sin\theta) + ae^{\theta}(\sin\theta - \cos\theta) \quad = ae^{\theta}\left[(\cos\theta + \sin\theta) + (\sin\theta - \cos\theta)\right]$

$= 2ae^{\theta}\sin\theta$

Now $\dfrac{dy}{dx} = y_1 = \dfrac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$

Then $\dfrac{d^2y}{dx^2} = y_2 = -\operatorname{cosec}^2\theta.\dfrac{d\theta}{dx} \quad = -\dfrac{\operatorname{cosec}^2\theta}{2ae^{\theta}\sin\theta} = -\dfrac{\operatorname{cosec}^3\theta}{2ae^{\theta}}$

$$\therefore \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = -\frac{\left(1+\cot^2\theta\right)^{3/2}}{\dfrac{\operatorname{cosec}^3\theta}{2ae^{\theta}}} = -\operatorname{cosec}^3\theta \times \frac{2ae^{\theta}}{\operatorname{cosec}^3\theta} = -2ae^{\theta}$$

Now the equation of the tangent at any point ‘θ’ on the curve is

$y - ae^{\theta}(\sin\theta + \cos\theta) = \dfrac{\cos\theta}{\sin\theta}\left\{x - ae^{\theta}(\sin\theta - \cos\theta)\right\}$

$\Rightarrow y\sin\theta - ae^{\theta}\sin^2\theta - ae^{\theta}\sin\theta.\cos\theta = x\cos\theta - ae^{\theta}\sin\theta\cos\theta + ae^{\theta}\cos^2\theta$

$\Rightarrow x\cos\theta - y\sin\theta = -ae^{\theta}$

Then length of the perpendicular form the origin to the tangent is

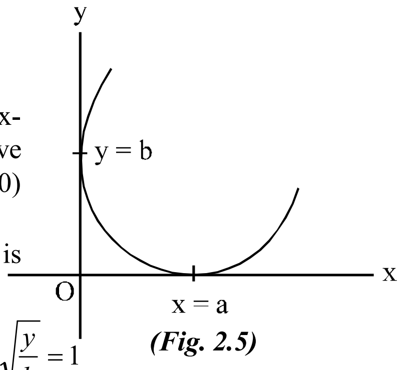
$\rho = \dfrac{0 + 0 - ae^{\theta}}{\sqrt{\sin^2\theta + \cos^2\theta}} = -ae^{\theta}$

i.e. The radius of curvature = twice the distance of the tangent from the origin.

Example – 10 : ***Find the radius of curvature for*** $\sqrt{\dfrac{x}{a}} - \sqrt{\dfrac{y}{b}} = 1$ ***at the points where it touches the co-ordinate axes.***

Solution : The given equation of the curve is

$\sqrt{\dfrac{x}{a}} - \sqrt{\dfrac{y}{b}} = 1$ ......(1)



(Fig. 2.5)

Since the curve touches to the co-ordinate axes i.e. to x-axis and y-axis, hence they must be tangent to the curve (*fig. 2.5*), X-axis will be a tangent at the curve at (a,0) and y-axis will be tangent at (0,b)

Radius of curvature of the curve at (a,0) where x-axis is a tangent to the curve.

Shifting the origin to (a,0) equation will be $\sqrt{\dfrac{x+a}{a}} - \sqrt{\dfrac{y}{b}} = 1$

Squaring both sides,

$\dfrac{x}{a} + 1 + \dfrac{y}{b} - 2\sqrt{\dfrac{y}{b}\left(\dfrac{x}{a} + 1\right)} = 1$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}=2\sqrt{\frac{y}{b}\left(\frac{x}{a}+1\right)}$$

Again Squaring $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+2\frac{xy}{ab}=4\frac{y}{b}\left(\frac{x}{a}+1\right)$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-2\frac{xy}{ab}-4\frac{y}{b}=0 \quad ....(2)$$

Dividing the lowest degree term from equation (2), we see the *x-axis* is the tangent to the curve at (a,0)

Now dividing equation (2) by y, we get

$$\frac{1}{a^2}.\frac{x^2}{y}-\frac{y}{b^2}-\frac{2x}{ab}-\frac{4}{b}=0$$

Taking $\lim x \to 0$ and $y \to 0$ that $\lim\limits_{x\to 0}\frac{x^2}{y}=2\rho$

(Newton's formula), we get

$$\frac{1}{a^2}.2\rho-0-0-\frac{4}{b}=0$$

$$\Rightarrow \rho=\frac{4}{b}\times\frac{a^2}{2} \Rightarrow \rho=\frac{2a^2}{b}$$

In the same way by shifting the origin to (0,b) where *y-axis* is a tangent to the curve we find.

$$\rho=\frac{2b^2}{a}$$

Example – 11 : ***Show that the ratio of the radii of curvature at points on the two curves $xy=a^2$ and $x^3=3a^2y$. Which have the same abscissa varies as the square root of the ratio of the co-ordinates.***

Solution : The two given curves are

$xy = a^2$...........(i)

$x^3 = 3a^2y$ ....(ii)

Let (x,y_1) be any point on (i) and (x,y_2) any point on (ii)

Diff. (i) w.r.t x we get

$$x\frac{dy}{dx}+y=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x} \quad ....(iii) \quad \therefore \left.\frac{dy}{dx}\right]_{(x,y_1)}=-\frac{y_1}{x}$$

Again differentiating (iii) we get

$$\frac{d^2y}{dx^2}=\frac{-x\frac{dy}{dx}+y}{x^2}, \therefore \left.\frac{d^2y}{dx^2}\right]_{(x,y_1)}=\frac{-x\left(-\frac{y_1}{x}\right)+y_1}{x^2}=\frac{2y_1}{x^2}$$

$$\text{Now } \rho_1 = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{y_1^2}{x^2}\right)^{3/2}}{\frac{2y_1}{x^2}}$$

$$= \frac{\left(x^2+y_1^2\right)^{3/2}}{x^3} \times \frac{x^2}{2y_1} = \frac{\left(x^2+\frac{a^4}{x^2}\right)^{3/2}}{2xy_1} \quad \left[\begin{array}{l} \therefore xy_1 = a^2 \\ \Rightarrow y_1 = \frac{a^2}{x} \end{array}\right]$$

$$= \frac{\left(x^4+a^4\right)^{3/2}}{2x^4y_1} \qquad \text{......(iv)}$$

Again from curve (ii), we get

$$3x^2 = 3a^2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x^2}{a^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{2x}{a^2}$$

$$\therefore \left.\frac{dy}{dx}\right]_{(x,y_2)} = \frac{x^2}{a^2},\ \therefore \left.\frac{d^2y}{dx^2}\right]_{(x,y_2)} = \frac{2x}{a^2}$$

$\therefore$ Radius of curvature of the 2nd curve

$$\therefore \rho_2 = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{x^4}{a^4}\right)^{3/2}}{\frac{2x}{a^2}} = \frac{\left(x^4+a^4\right)^{3/2}}{2a^4x} \qquad \text{...(v)}$$

$$\text{Now } \frac{\rho_1}{\rho_2} = \frac{\left(x^4+a^4\right)^{3/2}}{2x^4y_1} \times \frac{2xa^4}{\left(x^4+a^4\right)^{3/2}} = \frac{a^4}{x^3y_1} = \frac{a^4}{x^{3/2}\cdot x^{3/2}\cdot y_1} \qquad \text{...(vi)}$$

Now when $xy_1 = a^2 \Rightarrow x = \frac{a^2}{y_1} \Rightarrow x^{3/2} = \frac{a^3}{y_1^{3/2}} \Rightarrow x^{3/2} = a\sqrt{3}\sqrt{y_2}$

$$\text{Now (vi) will be } \frac{\rho_1}{\rho_2} = \frac{a^4}{\frac{a^3}{y_1^{3/2}}\cdot a\sqrt{3}\sqrt{y_2}\cdot y_1} = \frac{y_1^{3/2}}{\sqrt{3}\, y_1\cdot\sqrt{y_2}}$$

$$\text{or } \frac{\rho_1}{\rho_2} = \frac{1}{\sqrt{3}}\sqrt{\frac{y_1}{y_2}} \text{ or } \frac{\rho_1}{\rho_2} \propto \sqrt{\frac{y_1}{y_2}}$$

Example – 12 : ***Show that*** $\frac{3\sqrt{3}}{2}$ ***is the least value of*** $|\rho|$ ***for*** $y = \log x$

Solution : Given that $y = \log x$

$$\Rightarrow \frac{dy}{dx} = y_1 = \frac{1}{x} \text{ and } \frac{d^2y}{dx^2} = y_2 = -\frac{1}{x^2}$$

$$\therefore \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} = \frac{\left(1+\frac{1}{x^2}\right)^{3/2}}{-\frac{1}{x^2}} = \frac{\left(x^2+1\right)^{3/2}}{x^3}\left(-x^2\right)$$

$$\text{or } \rho = \frac{\left(x^2+1\right)^{3/2}}{-x}, \text{ or } |\rho| = \frac{\left(x^2+1\right)^{3/2}}{x}$$

$$\therefore \frac{d}{dx}|\rho| = \frac{x.\frac{3}{2}\left(x^2+1\right)^{\frac{1}{2}}.2x - 1\left(x^2+1\right)^{3/2}}{x^2}$$

$$= \frac{\left(x^2+1\right)^{\frac{1}{2}}\left[3x^2-\left(x^2+1\right)\right]}{x^2} = \frac{\left(x^2+1\right)^{\frac{1}{2}}\left(2x^2-1\right)}{x^2} = \left(x^2+1\right)^{\frac{1}{2}}.\left(2-\frac{1}{x^2}\right)$$

For maximum and minima of $|\rho|$

Let $\frac{d}{dx}|\rho| = 0$

$$\Rightarrow \frac{\left(x^2+1\right)^{\frac{1}{2}}\left(2x^2-1\right)}{x^2} = 0$$

$\Rightarrow 2x^2 - 1 = 0$ $\quad$ ($\because x^2+1$ gives imaginary values)

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$$\text{Now} \frac{d^2}{dx^2}|\rho| = \left(x^2+1\right)^{\frac{1}{2}}\left(0+\frac{2}{x^3}\right)+\left(2-\frac{1}{x^2}\right).\frac{1}{2}.\frac{1}{\sqrt{x^2+1}}.2x$$

$$= \frac{2\sqrt{x^2+1}}{x^3} + \frac{2x^2-1}{x\sqrt{x^2+1}}$$

By putting $x = \frac{1}{\sqrt{2}}$, we get $\frac{d^2}{dx^2}|\rho|$ is +ve $\left(i.e = \frac{30}{\sqrt{5}}\right)$

and if $x = -\frac{1}{\sqrt{2}}$ we get $\frac{d^2}{dx^2}|\rho|$ is the $\left(i.e - \frac{30}{\sqrt{5}}\right)$

So minimum value of $|\rho|$ can be found out by putting $x = \frac{1}{\sqrt{2}}$

$\therefore$ Least value of $|\rho|$

$$= \frac{\left(1+\frac{1}{2}\right)^{3/2}}{\frac{1}{\sqrt{2}}} \left(\because |\rho| = \frac{\left(1+x^2\right)^{3/2}}{x}\right) = \frac{\left(\frac{3}{2}\right)^{\left(1+\frac{1}{2}\right)}}{\frac{1}{\sqrt{2}}} = \frac{3}{2}.\frac{\sqrt{3}}{\sqrt{2}} \times \sqrt{2}$$

or $|\rho| = \frac{3\sqrt{3}}{2}$ minimum

Example – 13 : ***Apply Newton's formula to find the radius of curvature at the origin for the cycloid $x = a\,(\theta + \sin\theta),\ y = a\,(1 - \cos\theta)$***

Solution : Here $x = a\,(\theta + \sin\theta)$ $y = a\,(1 - \cos\theta)$

$$\frac{dx}{d\theta} = a\,(1 + \cos\theta) \quad \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

$$\therefore \frac{dy}{dx} = 0 \text{, when } \theta = 0$$

Here initial line is tangent to origin

$$\text{Then } \rho = \lim_{\theta\to 0}\frac{x^2}{2y} = \lim_{\theta\to 0}\frac{a^2(\theta+\sin\theta)^2}{2a(1-\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{a}{2}\left[\frac{2(\theta+\sin\theta)(1+\cos\theta)}{\sin\theta}\right]$$

$$= \lim_{\theta\to 0}\frac{a}{2}\left[\frac{2(1+\cos\theta)^2 - \sin\theta(\theta+\sin\theta)}{\cos\theta}\right]$$

$= 4a$ (by Hospital's Rule)

Example – 14 : ***Find radius of curvature ρ at the pole for the curve $r = a\sin n\theta$, by Newton's method.***

Solution : The curve is $r = a\sin n\theta$. Here 'r' and 'θ' are 0. Hence initial line is tangent to the curve at origin.

$$\text{We know, } \rho = \lim_{\theta\to 0}\frac{r}{2\theta} = \lim_{\theta\to 0}\frac{a\sin n\theta}{2\theta}$$

$$= \lim_{\theta\to 0}\frac{\sin n\theta}{n\theta}\cdot\frac{na}{2} = \frac{na}{2}$$

Example – 15 : ***Show that the curvature of the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the folium $x^3 + y^3 = 3axy$ is $-8\sqrt{2}/3a$.***

Solution : The given equation is $x^3 + y^3 = 3axy$

Diff. both sides w.r.t x. we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx} \Rightarrow x^2 + y^2\frac{dy}{dx} = ay + ax.\frac{dy}{dx} \quad \text{........(1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

$$\Rightarrow \left.\frac{dy}{dx}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{a.\frac{3a}{2}-\frac{9a^2}{4}}{\frac{9a^2}{4}-a.\frac{3a}{2}} = \frac{\frac{6a^2-9a^2}{4}}{\frac{9a^2-6a^2}{4}} = \frac{-3a^2}{3a^2} = -1$$

Again Differentiating (1) w.r.t x we get

$$2x+2y\left(\frac{dy}{dx}\right)^2+y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx}+a\frac{dy}{dx}+ax\frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2a\frac{dy}{dx}-2x-2y\left(\frac{dy}{dx}\right)^2}{y^2-ax}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right]_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{2a(-1)-2\left(\frac{3a}{2}\right)-2\left(\frac{3a}{2}\right)(-1)^2}{\frac{9a^2}{4}-a\frac{3a}{2}} = -\frac{32}{3a}$$

Hence the curvature at $\left(\frac{3a}{2},\frac{3a}{2}\right)$

i.e. $\frac{1}{\rho} = \frac{y_2}{\left(1+y_1^2\right)^{3/2}} = \frac{-\frac{32}{3a}}{\left\{1+(-1)^2\right\}^{3/2}} = -\frac{32}{3a}.2^{-3/2} = -\frac{8\sqrt{2}}{3a}$

Example – 16 : ***Find the point of the curve y =e^x, at which the curvature is maximum and show that the tangent at the point forms with the axes of co-ordinates a triangle whose sides are in the ratio 1: $\sqrt{2}$:$\sqrt{3}$.***

Solution : Given that $y = e^x$,

$\therefore \frac{dy}{dx^2} = e^x$ and $\frac{d^2y}{dx} = e^x$

$\therefore \rho = \frac{\left(1+e^{2x}\right)^{3/2}}{e^x}$...........(A) $\qquad \left(\because \rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}\right)$

Now $\frac{d\rho}{dx} = \frac{e^x.\frac{3}{2}\left(1+e^{2x}\right)^{\frac{1}{2}}.2.e^{2x}-\left(1+e^{2x}\right)^{3/2}.e^x}{e^{2x}} = \frac{e^x\left(1+e^{2x}\right)^{\frac{1}{2}}\left[3e^{2x}-\left(1+e^{2x}\right)\right]}{e^{2x}}$

For 'ρ' to be maximum or minimum,

Let $\frac{d\rho}{dx} = 0 \Rightarrow \frac{e^x\left(1+e^{2x}\right)^{\frac{1}{2}}\left[2e^{2x}-1\right]}{e^{2x}} = 0$

$\Rightarrow 2e^{2x}-1 = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow 2x = log\frac{1}{2}\ \left(\because a^x = N\right) \Rightarrow x = \frac{1}{2}log\frac{1}{2}\ (log_a N = x)$

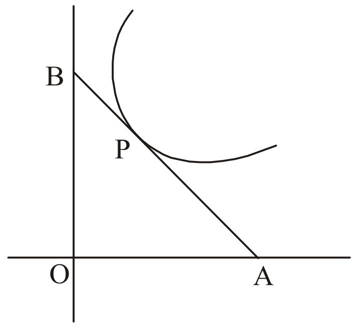
From (A) we conclude that, if $x \to \infty$ then $\rho \to \infty$ and when $x \to -\infty$, then 'ρ ' must have the minimum value i.e. zero. (*fig. 2.6*)

Thus for $x = \frac{1}{2}\log\frac{1}{2}$, the value of '$\rho$ ' must be maximum

$$\therefore y = e^x = e^{\frac{1}{2}\log\frac{1}{2}} = e^{\log\sqrt{\frac{1}{\sqrt{2}}}} = \frac{1}{\sqrt{2}}$$

So the required Point at which 'ρ' is maximum is $\left(\frac{1}{2}\log\frac{1}{2}.\frac{1}{\sqrt{2}}\right)$

$\therefore$ Equation of the tangent at P $\left(\frac{1}{2}\log\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ is



(Fig. 2.6)

$$y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x - \frac{1}{2}\log\frac{1}{2}\right) \quad \left[\begin{array}{l}\because \text{Slope of tangent i.e.} \\ \frac{dy}{dx} = e^{\frac{1}{2}\log\frac{1}{2}} = \frac{1}{\sqrt{2}}\end{array}\right]$$

$$\Rightarrow \sqrt{2}y - 1 = x - \frac{1}{2}\log\frac{1}{2}$$

$$\Rightarrow x - y\sqrt{2} = \frac{1}{2}\log\frac{1}{2} - 1$$

Reducing to the intercept form, we have

$$\frac{x}{\frac{1}{2}\log\frac{1}{2} - 1} + \frac{y}{-\frac{1}{\sqrt{2}}\left(\frac{1}{2}\log\frac{1}{2} - 1\right)} = 1$$

$\therefore$ x-intercept i.e. $OA = \frac{1}{2}\log\frac{1}{2} - 1$ and y-intercept i.e. $OB = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\log\frac{1}{2} - 1\right)$ [length cannot be –ve]

Sinec OAB is a right angled triangle, hence

$$AB = \sqrt{\left(\frac{1}{2}\log\frac{1}{2} - 1\right)^2 + \frac{1}{2}\left(\frac{1}{2}\log\frac{1}{2} - 1\right)^2}$$

$$= \left(\frac{1}{2}\log\frac{1}{2} - 1\right)\sqrt{\left(1 + \frac{1}{2}\right)} = \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{1}{2}\log\frac{1}{2} - 1\right)$$

$\therefore$ lengths of the triangle OAB are $\dfrac{\frac{1}{2}\log\frac{1}{2} - 1}{\sqrt{2}}, \left(\frac{1}{2}\log\frac{1}{2} - 1\right), \left(\frac{1}{2}\log\frac{1}{2} - 1\right)\frac{\sqrt{3}}{\sqrt{2}}$

So the required ratio of the sides of the triangles are $1:\sqrt{2}:\sqrt{3}$

Example – 17 : ***The tangents at two points P,Q on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are at right angles. Show that if ρ_1 and ρ_2 be the radii of curvatures at these points, then*** $\rho_1^2 + \rho_2^2 = 16a^2$

Solution : Equation of cycloid is

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

$$\therefore \frac{dx}{d\theta} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2.\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

$\therefore$ Slope of the tangent at $P = \cot\frac{\theta}{2}$

Since tangent at Q is perpendicular to that at P. ($\therefore$ They are at right angles)

$\therefore$ Slope of the tangent at Q$= -\tan\frac{\theta}{2} = \cot\left(90^0 + \frac{\theta}{2}\right)$

If ϕ be the angle at Q, then $\phi = 2\left(90^0 + \frac{\theta}{2}\right) = 180 + \theta = \pi + \theta$

Now $\frac{d^2y}{dx^2} = -\operatorname{cosec}^2\frac{\theta}{2}.\frac{1}{2}.\frac{d\theta}{dx} = -\frac{1}{2\sin^2\frac{\theta}{2}}.\frac{1}{a(1-\cos\theta)} = -\frac{1}{4a\sin^4\frac{\theta}{2}}$

$\therefore$ Radius of curvature at P i.e.

$$\rho_1 = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \cot^2\frac{\theta}{2}\right)^{3/2}}{-\frac{1}{4a\sin^4\frac{\theta}{2}}} = -\operatorname{cosec}^3\frac{\theta}{2}.4a\sin^4\frac{\theta}{2}$$

$= -4a\sin\frac{\theta}{2}$ (Numercial value)

Putting $\pi + \theta$ for θ in the value of ρ_1 we get

$$\rho_2 = -4a\sin\left(90^0 + \frac{\theta}{2}\right) = -4a\cos\frac{\theta}{2}$$

$$\therefore \rho_1^2 + \rho_2^2 = 16a^2\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) = 16a^2$$

Example – 18 : If ρ_1 and ρ_2 are the various of the extermities of a focal chord of a parabola whose semi latus rectum is 'l' then prove that $(\rho_1)^{-2/3} + (\rho_2)^{\frac{-2}{3}} = (l)^{-\frac{2}{3}}$.

Solution : Let the equation of the parabola be $y^2 = 4ax$. Its focus is (a,0) and semi-latus rectum l=2a. Any point 'ρ' on the parabola is $(at^2, 2at)$

Let Q be the other extermity of the focal chord through P, then co-ordinates of Q are $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

From (1), $\frac{dy}{dx} = \frac{2a}{y} \therefore \left.\frac{dy}{dx}\right]_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$

Again $\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{4a^2t^2}.\frac{1}{t} = -\frac{1}{2at^3}$

$$\therefore \text{ At P, } \rho_1 = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{1}{t^2}\right)^{3/2}}{-\frac{1}{2at^3}} = \frac{\left(t^2+1\right)^{3/2}}{t^3}\left(-2at^3\right)$$

$= -l\left(1+t^2\right)^{3/2}$ $\qquad (\because 2a = l)$

$= l\left(1+t^2\right)^{3/2}$ ....(1) (Numerical Value)

Replaceing t by $-\frac{1}{t}$ for the point 'Q'

$\rho_2 = l\left(1+\frac{1}{t^2}\right)^{3/2} = \frac{l}{t^3}\left(1+t^2\right)^{3/2}$............(2)

Now from (1) $\rho_1^{-2/3} = l^{\frac{-2}{3}}\left(1+t^2\right)^{-1}$

and from (2) $\rho_2^{-\frac{2}{3}} = \frac{l^{-2/3}}{t^{-2}}\left(1+t^2\right)^{-1}$

$\therefore (\rho_1)^{-2/3} + (\rho_2)^{\frac{-2}{3}} = l^{\frac{-2}{3}}\left(1+t^2\right)^{-1}\left[1+\frac{1}{t^{-2}}\right] = l^{\frac{-2}{3}}\frac{\left(1+t^2\right)}{\left(1+t^2\right)}$

or $(\rho_1)^{-\frac{2}{3}} + (\rho_2)^{-\frac{2}{3}} = l^{-\frac{2}{3}}$

Example –19 : ***Prove that for the ellipse*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$***, the radius of curvature*** $'\rho' = \frac{a^2b^2}{P^3}$***, whese 'P' is the perpendicular from the centre upon the tangent at any point (x,y).***

[B.P.U.T. - 2009, 2011

Solution : Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

or $b^2x^2 + a^2y^2 = a^2b^2$........(1)

Differentiating w.r.t x, we get

$b^2.2x + a^2.2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}.\frac{x}{y}$ ....(2)

We know that any point on the ellipse is *(acos ϕ, bsinϕ)*

$$\therefore \left.\frac{dy}{dx}\right]_{(a\cos\phi, b\sin\phi)} = -\frac{b^2}{a^2}\cdot\frac{a\cos\phi}{b\sin\phi} = -\frac{b}{a}\frac{\cos\phi}{\sin\phi}$$

Again from (2)

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y.1 - x\frac{dy}{dx}}{y^2}\right) \qquad \left(\because \frac{dy}{dx} = -\frac{b^2}{x^2}\cdot\frac{x}{y}\right)$$

$$= -\frac{b^2}{a^2y^2}\left(y + x.\frac{b^2x}{a^2y}\right) = -\frac{b^2}{a^2y^2}\left(\frac{a^2y^2 + b^2x^2}{a^2y}\right)$$

$$= -\frac{b^2}{a^4y^3}\left(a^2b^2\right) = -\frac{b^4}{a^2y^3} = -\frac{b^4}{a^2b^3\sin^3\phi} = -\frac{b}{a^2}\operatorname{cosec}^3\phi$$

$$\therefore \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+\frac{b^2}{a^2}\frac{\cos^2\phi}{\sin^2\phi}\right\}^{3/2}}{-\frac{b}{a}\operatorname{cosec}^3\phi} = \frac{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}{a^3\sin^3\phi} \times \frac{a^2\sin^3\phi}{-b}$$

$$= \frac{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}{ab} \text{...........(3)}$$

From co-ordinate geometry,
Equation of the tangents at *(a cos ϕ, b sin ϕ)* is

$$\frac{xa\cos\phi}{a^2} + \frac{yb\sin\phi}{b^2} = 1$$

$$\Rightarrow \frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$$

$\therefore$ Length of perpendicular from origin to the tangent

i.e. $P = \dfrac{1}{\sqrt{\dfrac{\cos^2\phi}{a^2} + \dfrac{\sin^2\phi}{b^2}}} = \dfrac{ab}{\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}}$

or, $\left(b^2\cos^2\phi + a^2\sin^2\phi\right)^{\frac{1}{2}} = \dfrac{ab}{P}$

Putting this value in (3) we get

$$\rho = \frac{\left(\frac{ab}{P}\right)^3}{ab} = \frac{a^2b^2}{P^3}.$$

2.5 : Radius of Curvature at the Origin

There are methods of finding the radius of curvature at the origin. Now we shall discuss them.

Newton's method of finding the radius of curvature at the origin :

When the curve passes through the origin and the axis of x is the tangent at the origin, we get

$$f(0) = 0 \text{ and } \left(\frac{dy}{dx}\right)_{x=0} = 0, \text{ i.e. } f'(0) = 0$$

Now by Maclaurin's theorem y can be expanded as

$$y = y(0) + xf'(0) + \left(\frac{x^2}{2!}\right)f''(0) + \left(\frac{x^3}{3!}\right)f'''(0) + \ldots.$$

$$y = 0 + 0 + \left(\frac{x^2}{2!}\right)f''(0) + \left(\frac{x^3}{3!}\right)f'''(0) + \ldots \text{ using equation.}$$

Dividing both sides x^2, we get $\frac{y}{x^2} = \frac{1}{2!}f''(0) + \frac{x}{3!}f'''(0) + \ldots$

Since the curve passes through the origin, therefore $x \to 0$ and $y \to 0$

Hence taking limit of both sides of equation when $x \to 0$, $y \to 0$ we get,

$$\lim_{x\to 0}\frac{y}{x^2} = \frac{1}{2!}f''(0) \text{ [Other terms vanish]}$$

$$\therefore \rho \text{ (at the origin)} = \frac{\left[1 + \{f'(0)\}^2\right]^{3/2}}{f''(0)} = \frac{1}{f''(0)} \; [\because y_1(0) = 0] = \lim_{x\to 0, y\to 0}\frac{x^2}{2y}$$

Similarly, we can prove that if a curve passes through the origin and the axis of y is the tangent there then ρ (at the origin) $= \lim_{x\to 0, y\to 0}\frac{y^2}{2x}$

The above two formulae are known as Newton's formulae.

Note :

1. If $ax + by = 0$ (a, b are real constants and $a^2 + b^2 \neq 0$) be the tangent at the origin, then the radius of curvature at the origin is given by

$$\rho = \frac{1}{2}\sqrt{a^2 + b^2} \lim_{\substack{x\to 0\\ y\to 0\\ \frac{y}{x}\to -\frac{a}{b}}} \left(\frac{x^2 + y^2}{ax + by}\right) \quad \ldots(3)$$

2. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of tangent (or tangents) at the origin is obtained by equating to zero, the terms of the lowest degree in the equation.

For example, if the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$ the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the curve, *i.e.*, $x^2 - y^2 = 0$, *i.e.*, $x + y = 0$ and $x - y = 0$.

3. If a curve passes through the origin and is given by a rational integral, algebraic equation, then the tangents at the origin can easily be obtained by equating to zero the lowest degree terms in the equation of the curve.

2.6 : Expansion Method for Finding the Radius of Curvature at the Origin

When a curve passes through the origin, but neither of the co-ordinate axes is a tangent at the origin, we cannot apply Newton's formula to find the radius of curvature at the origin. In such cases we can apply the following method known as the expansion method.

Since the curve passes through the origin, therefore $f(0) = 0$, i.e., the value of y at $x = 0$ is 0. i.e. $f(0) = 0$

Let $(dy/dx)_{(0,0)} = f'(0) = \alpha$ and $(d^2y/dx^2)_{(0,0)} = f''(0) = \beta$

$$\therefore \rho \text{ (at origin)} = \frac{(1+\alpha^2)^{3/2}}{\beta}$$

Now by Maclaurin's theorem, we have $y = y(0) + xf'(0) + \left(\frac{x^2}{2!}\right)f''(0) + \ldots$

$= \alpha x + \frac{1}{2}\beta x^2 + \ldots$ because the curve passes through the origin.

To get the values of α and β we should obtain from the equation of the curve an expansion for y in ascending powers of x by algebraic or trigonometric methods. The coefficient of x in this expansion will be equal to α and the coefficient of x^2 will be equal to $\frac{1}{2}\beta$ as it is clear form the Maclaurin's expansion for y.

Putting these values of α and β in eq^n, we shall get the ρ at the origin.

2.7 : Newtonian Method

If a curve passes through the origin and the axis of x is tangent there, then $\lim \frac{x^2}{2y}$ as $x \to 0$ is the radius of curvature at the origin. Now $\frac{x^2}{2y}$ assume the indeterminant form $\left(\frac{0}{0}\right)$ is $x \to 0$

$$\lim_{x\to 0}\frac{x^2}{2y} = \lim_{x\to 0}\frac{2x}{2y} = \lim_{x\to 0}\frac{1}{y} = \frac{1}{y_2(0)}$$

We know $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+0)^{3/2}}{y_2(0)} = \dfrac{1}{y_2(0)}$

Thus at the origin where x axis is a tangent i.e. $\rho = \lim\limits_{x\to 0}\dfrac{x^2}{2y}$

and where y-axis is a tangent at origin (0,0) to the curve, we have $\rho = \lim_{y \to 0} \frac{y^2}{2x}$

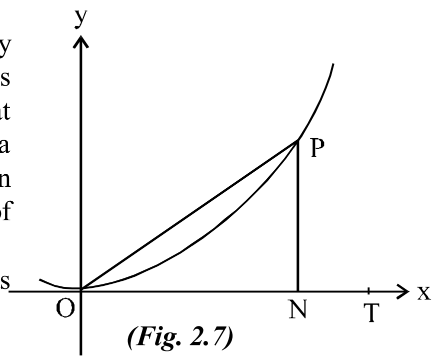
These two formulaes are given by Newton.

Generalised Newtonian Formula :

If a curve passes through the origin and x-axis is the tangent then we have

$$\lim \frac{x^2 + y^2}{2y} = \lim \left(\frac{x^2}{2y} + \frac{y}{2} \right) = \lim \frac{x^2}{2y} = \rho \text{ at the origin } (\textit{fig. 2.7})$$

Here $x^2 + y^2 = OP^2$ is the square of the distance of any point $P\,(x,y)$ on the curve from the origin 'O' and 'y' is the distance of the point 'P' from the tangent *x-axis* at zero. If OT be the tangent at any given point 'O' of a curve and PM the length of the perpendicular drawn from any point P to the tangent at 'O' then the radius of curvature at 'O' is $\lim \frac{OP^2}{2PN}$. When the point 'P' tends to zero as it's limits.



(Fig. 2.7)

Illustrative Examples

Example – 1 : ***Find the radius of curvature of $x^2 + 2x + y^2 = 0$ at the origin***

Solution : The given curve is $x^2 + 2x + y^2 = 0$ ...(1)

Differentiating eq^n (1) w.r.t.x we get

$$2x + 2 + 2y\frac{dy}{dx} = 0$$

i.e. $\frac{dy}{dx} = -\left[\frac{(x+1)}{y}\right]$

Differentiating eq^n (2) w.r.t. x, $2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\left(\frac{d^2y}{dx^2}\right) = 0$

$$\Rightarrow 1 + \left(\frac{x+1}{y}\right)^2 + y\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{y}\left[1 + \left(\frac{x+1}{y}\right)^2\right]$$

$$\therefore \ \rho \text{ at } (x,y) = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left[1 + \left(\frac{x+1}{y}\right)^2\right]^{3/2}}{-\frac{1}{y}\left[1 + \left(\frac{x+1}{y}\right)^2\right]} = y\sqrt{1 + \left(\frac{x+1}{y}\right)^2}$$

neglecting negative sign

$$= \sqrt{(x+1)^2 + y^2}$$

$$\therefore \ \rho \text{ at origin} = \sqrt{(0+1)^2 + 0^2} = 1$$

Example – 2 : ***By using Maclaurin's expansion, find the radius of curvature at the origin of the curve $y= x^3+5x^2+6x$.***

Solution : The given curve is $y= x^3+5x^2+6x$ ...(1)

which obviously passes through the origin.

Let $\left(\frac{dy}{dx}\right)_{(0,0)} = \alpha$ and $\left(\frac{d^2y}{dx^2}\right)_{(0,0)} = \beta$

Then Maclaurin's expansion, we get for this curve $y = \alpha x + \frac{1}{2}\beta x^2 + \ldots$ ...(2)

Comparing (1) and (2) we get $\alpha = 6$, $\frac{\beta}{2} = 5$ i.e $\beta = 10$

Hence ρ at the origin $= \left(\frac{1+\alpha^2}{\beta}\right)^{3/2} = \frac{(1+36)^{3/2}}{10} = \frac{1}{10}\left(37\sqrt{37}\right)$

Example – 3 : ***Find the radius of curvature at the origin of the curve***

$$y^2 - 3xy - 4x^2 + x^3 + x^4y + y^5 = 0$$

Solution : Putting $y = \alpha x + \frac{\beta}{2!}x^2 + \ldots$ and equating coefficients of x^2 and x^3 equal to zero we get

$\alpha^2 - 3\alpha - 4 = 0$ ...... (i) and $\alpha\beta - \frac{3\beta}{2!} + 1 = 0$ .....(2)

Equation (1) gives $\alpha = -1$ and 4

From (2) we get, when $\alpha = -1$ and $\beta = 2/5$ and when $\alpha = 4$, $\beta = -2/5$

$\therefore$ Radius of curvature at the origion$= \frac{\left(1+\alpha^2\right)^{3/2}}{\beta}$

$\therefore$ When ($\alpha = 1$, $\beta = 2/5$) radius of curvature at the origin $= \frac{(1+1)^{3/2}}{2/5} = 5\sqrt{2}$

and when ($\alpha = 4$, $\beta = -2/5$), radius of curvature at the origin $= \frac{\left(1+4^2\right)^{3/2}}{-2/5} = -\frac{85\sqrt{\{(17)\}}}{2}$

Therefore radius of curvature at the origin are $5\sqrt{2}$ and $-\frac{85\sqrt{17}}{2}$.

Example – 4 : ***Find the radius of curvature at the origin for the curve $a(y^2 - x^2) = x^3$***

Solution : The given curve passes through the origin equating to zero the lowest degree terms in the equation of the curve, the tangents at origin are $y^2 - x^2 = 0$, i.e., $y = \pm x$. Thus neither of the co-ordinate axes is tangent at the origin. So we can not apply Netwon's formula for finding ρ at origin.

From the equation of the curve we have $y^2 = x^2 + \frac{x^3}{a} = x^2\left(1 + \frac{x}{a}\right)$

$$\therefore y = \pm x\left(1+\frac{x}{a}\right)^{1/2} = \pm x\left(1+\frac{1}{2}.\frac{x}{a}+...\right) \text{ (by binomial theorem)}$$

Let $\left(\frac{dy}{dx}\right)_{(0,0)} = y_1(0) = \alpha$ and $\left(\frac{d^2y}{dx^2}\right) = y_2(0) = \beta$

Then by Maclaurin's expansion, we get for this curve as

$$y = y(0) + x.y_1(0) + \frac{x^2}{2!}.y_2(0) + ...$$

i.e. $y = \alpha x + \frac{1}{2}\beta x^2 + ...$

Comparing eq^{n}s (1) and (2) we get $\alpha = 1$, $\beta = \frac{1}{a}$ or $\alpha = -1$, $\beta = -\frac{1}{a}$

Now ρ (at origin) $= \dfrac{\left(1+\alpha^2\right)^{3/2}}{\beta}$

When $\alpha = 1$, $\beta = \frac{1}{a}$, we have ρ (at origin) $= \dfrac{(1+1)^{3/2}}{1/a} = \left(2\sqrt{2}\right)a$

and when $\alpha = -1$, $\beta = -\frac{1}{a}$, we have ρ (at origin) $= \dfrac{(1+1)^{3/2}}{\frac{-1}{a}} = -\left(2\sqrt{2}\right)a$

Example – 5 : ***Apply Newton's formula to find the radius of curvature at the origin for the curve $5x^3 + 7y^3 + 4x^2y + xy^2 + 2x^2 + 3xy + y^2 + 4x = 0$.***

Solution : The given curve is $5x^3 + 7y^3 + 4x^2y + xy^2 + 2x^2 + 3xy + y^2 + 4x = 0$

The given curve passes through the origin (0,0). Equating to zero the lowest degree terms in the equation of the curve, we get the tangent at origin as $4x = 0$, i.e., $x = 0$, i.e., *y–axis*

By Newton's formula ρ (at the origin) $= \lim\limits_{x\to 0, y\to 0}\left(\frac{y^2}{2x}\right)$

Now dividing each term in the equation of the curve by $2x$, we get

$$\frac{5}{2}x^2 + 7y\left(\frac{y^2}{2x}\right) + 2xy + \frac{1}{2}y^2 + x + \frac{3}{2}y + \frac{y^2}{2x} + 2 = 0$$

Taking limits of boths sides of eqn (1) when $x\to 0$, $y\to 0$ and remembereing the $\lim\limits_{x\to 0}\left(\frac{y^2}{2x}\right) = \rho$ (at origin) we get

$0 + 0.\rho + 0 + 0 + 0 + 0 + \rho + 2 = 0$ i.e. (at origin) $\rho = -2 = 2$ (numerically).

Example – 6 : ***Apply Newton's formula to find the radius of curvature at the origin of the following Curves.***

(a) $x^4 - y^4 + x^3 - y^3 + x^2 - y^2 + y = 0$ (b) $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$

Solution : (a) Equating to zero the lowest degree term i.e. $y = 0$. It is clear that *x-axis* is the tangent to the curve at the origin.

Dividing through out the equation by y we get $x^2.\frac{x^2}{y} - y^3 + x.\frac{x^2}{y} - y^2 + \frac{x^2}{y} - y + 1 = 0$

Taking limits and $x \to 0$, $y \to 0$, we get

$$\lim_{x\to 0}\left(x^2.\frac{x^2}{y} - y^3 + x.\frac{x^2}{y} - y^2 + \frac{x^2}{y} - y + 1\right) = 0 \text{ ....(1)}$$

But according to Newtons formula $\rho = \lim_{x\to 0}\frac{x^2}{2y} \Rightarrow 2\rho = \lim_{x\to 0}\frac{x^2}{y}$

So, (1) gives

$$0.2\rho - 0 + 0.2\rho - 0 + 2\rho - 0 + 1 = 0 \Rightarrow \rho = -\frac{1}{2}$$

(b) $\boldsymbol{2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0}$

Equating to zero the lowest degree term i.e. $2x = 0$

$\Rightarrow x = 0$, it is clear that *y-axis* is the tangent at the origin. Now dividing by x, we get

$$2x^3 + 3y^2\frac{y^2}{x} + 4xy + y - \frac{y^2}{x} + 2 = 0$$

Taking $\lim x \to 0, y \to 0$. So that $\lim_{y\to 0}\frac{y^2}{x} = 2\rho$ Newton's form ......, we get $-2\rho + 2 = 0$

$\Rightarrow \rho = 1$

2.8 : Polar Equations

To prove that

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \text{, for the curve } r = f(\theta)$$

Or, **Find radius of curvature at any point *P* (r, θ) of the curve *r* = *f*(θ)**

Proof. Let s denote arcual length of any point P (r.θ) form some fixed point A on the curve.

We take another point Q (r + δr, θ + δθ) on the curve near P.

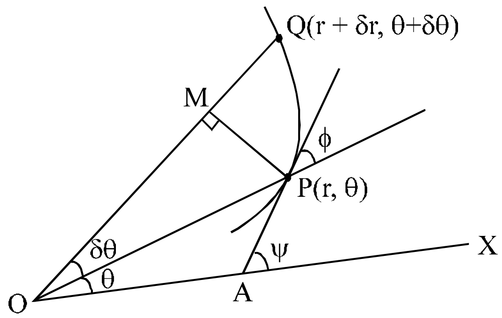
Let Arc AQ = s + δs so that Arc PQ = δs

Draw PM ⊥ OQ

PM = $r\sin\delta\theta$, MQ = OQ – OM

$$= (r + \delta r - r\cos\delta\theta)$$

$$= r(1 - \cos\delta\theta) + \delta r = 2r\sin^2\left(\frac{\delta\theta}{2}\right) + \delta r$$



(Fig. 2.8)

Since $PQ^2 = PM^2 + MQ^2$

$$= r^2\sin^2\delta\theta + \left\{2r\sin^2\left(\frac{\delta\theta}{2}\right) + \delta r\right\}^2$$

$$\Rightarrow \frac{(PQ)^2}{(\delta\theta)^2} = r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left\{2r\sin\left(\frac{\delta\theta}{2}\right)\frac{\sin\left(\frac{\delta\theta}{2}\right)}{2.\left(\frac{\delta\theta}{2}\right)} + \frac{\delta r}{\delta\theta}\right\}^2$$

$$\Rightarrow \frac{(PQ)^2}{(ArcPQ)^2}.\frac{(Arc\ PQ)^2}{(\delta\theta)^2} = r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left\{r\sin\left(\frac{\delta\theta}{2}\right).\frac{\sin\left(\frac{\delta\theta}{2}\right)}{\left(\frac{\delta\theta}{2}\right)} + \frac{\delta r}{\delta\theta}\right\}^2$$

When Q $\rightarrow$ P, $\delta\theta \rightarrow 0$ and chord PQ $\rightarrow$ Arc PQ
So we obtain

$$\lim_{\delta\theta\to 0}\left(\frac{\delta s}{\delta\theta}\right)^2 = \lim_{\delta\theta\to 0} r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \lim_{\delta\theta\to 0}\left\{r\sin\left(\frac{\delta\theta}{2}\right)\frac{\sin\left(\frac{\delta\theta}{2}\right)}{\left(\frac{\delta\theta}{2}\right)} + \frac{\delta r}{\delta\theta}\right\}^2$$

$$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Again in the right angled ΔPMQ

$$\sin\angle PQM = \frac{PM}{PQ} = \frac{r\sin\delta\theta}{PQ} = \frac{r\sin\delta\theta}{\delta s}.\frac{\delta s}{PQ} = \frac{r\sin\delta\theta}{\delta\theta}.\frac{\delta\theta}{\delta s}.\frac{\delta s}{PQ}$$

When Q$\rightarrow$ P, chord PQ $\rightarrow$ Arc PQ and $\angle$PQM $\rightarrow \phi$

So we have $\sin\phi = r\dfrac{d\theta}{ds}$

Again $\cos\angle PQM = \dfrac{MQ}{PQ} = \dfrac{(r+\delta r) - r\cos\delta\theta}{PQ} = \dfrac{2r\sin^2\left(\frac{\delta\theta}{2}\right) + \delta r}{PQ}$

$$= 2r\sin\left(\frac{\delta\theta}{2}\right).\frac{\sin\left(\frac{\delta\theta}{2}\right)}{2.\left(\frac{\delta\theta}{2}\right)}.\frac{\delta\theta}{\delta s}.\frac{\delta s}{PQ} + \frac{dr}{\delta\theta}.\frac{\delta\theta}{\delta s}.\frac{\delta s}{PQ}$$

As Q $\rightarrow$ P, $\delta\theta \rightarrow 0$, chord PQ $\rightarrow$ Arc PQ and $\angle$PQM $\rightarrow \phi$

So we have $\cos\phi = 0 + \dfrac{dr}{d\theta}.\dfrac{d\theta}{ds} \Rightarrow \cos\phi = \dfrac{dr}{ds}$

Hence $\tan\phi = \dfrac{r\dfrac{d\theta}{ds}}{\left(\dfrac{dr}{ds}\right)} = r\left(\dfrac{d\theta}{dr}\right)$

$\therefore \tan\phi = r\dfrac{d\theta}{dr}$, here $\psi = \theta + \phi$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}.\frac{d\theta}{ds}$$

$\Rightarrow \dfrac{d\psi}{ds} = \dfrac{d\theta}{ds}\left(1 + \dfrac{d\phi}{d\theta}\right)$ ....(1), again $\tan\phi = \dfrac{r}{\left(\dfrac{dr}{d\theta}\right)}$.

Differentating both sides w.r.t. θ

We get $\sec^2\phi\dfrac{d\phi}{d\theta} = \dfrac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{1}{\left(1 + \tan^2\phi\right)}\left\{\frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}\right\} \Rightarrow \frac{d\phi}{d\theta} = \frac{1}{\left(1 + r^2\left(\frac{d\theta}{dr}\right)^2\right)}\left[\frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}\right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Since from equation (1)we have, $\dfrac{d\psi}{ds} = \dfrac{d\theta}{ds}\left(1 + \dfrac{d\phi}{d\theta}\right)$

$$= \frac{1}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}\left\{1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right\}$$

$$= \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}$$

where $r_1 = \dfrac{dr}{d\theta}$ and $r_2 = \dfrac{d^2r}{d\theta^2}$