Example – 9 : $f(x)=\begin{cases}\frac{1}{2}+x & \text{if } -\frac{1}{2}<x<0\\ \frac{1}{2}-x & \text{if} 0<x<\frac{1}{2}\end{cases}$ $p = 2L = 1$

Sol^n : Let the fourier series of the periodic function f(x) of period of length in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ be given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n Cos(2n\pi x)+b_n Sin(2n\pi x)\right) \quad ......(1)$$

Then $a_0=2\int_{-1/2}^{1/2}f(x)dx=2\left[\int_{-1/2}^{0}f(x)dx+\int_0^{1/2}f(x)dx\right]$

$$=2\left[\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)dx+\int_0^{1/2}\left(\frac{1}{2}-x\right)dx\right]$$

$$=2\left[\left|\frac{x}{2}+\frac{x^2}{2}\right|_{-1/2}^{0}+\left|\frac{x}{2}-\frac{x^2}{2}\right|_0^{1/2}\right]$$

$$=2\left[0-\left(-\frac{1}{4}+\frac{1}{8}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)\right]=2\left[2\left(\frac{1}{4}-\frac{1}{8}\right)\right]$$

$$=1-\frac{1}{2}=\frac{1}{2}$$

$$a_n=2\int_{-1/2}^{1/2}f(x)cos(2n\pi x)dx=2\left[\int_{-1/2}^{0}f(x)cos(2n\pi x)dx+\int_0^{1/2}f(x)cos(2n\pi x)dx\right]$$

$$=2\left[\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)cos(2n\pi x)dx+\int_0^{1/2}\left(\frac{1}{2}-x\right)cos(2n\pi x)dx\right]$$

$$=2\left[\left|\left(\frac{1}{2}+x\right)\frac{sin(2n\pi x)}{2n\pi}+\frac{cos(2n\pi x)}{4\pi^2n^2}\right|_{-1/2}^{0}+\left|\left(\frac{1}{2}-x\right)\frac{sin(2n\pi x)}{2n\pi}-\frac{cos(2n\pi x)}{4\pi^2n^2}\right|_0^{1/2}\right]$$

$$=2\left[0+\frac{1}{4n^2\pi^2}(1-cos\,n\pi)-\frac{1}{4n^2\pi^2}(cos\,n\pi-1)\right]$$

$$=\frac{1}{n^2\pi^2}(1-cos\,n\pi)=\frac{1}{n^2\pi^2}\left[1-(-1)^n\right]$$

$\therefore a_n = 0$ if n be even

$=\frac{2}{n^2\pi^2}$ if n be odd.

$$b_n=2\int_{1/2}^{1/2}f(x)sin(2n\pi x)dx=2\left[\int_{-1/2}^{0}f(x)sin(2n\pi x)dx+\int_0^{1/2}f(x)sin(2n\pi x)dx\right]$$

$$= 2\left[\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)sin(2n\pi x)dx + \int_{0}^{1/2}\left(\frac{1}{2}-x\right)sin(2n\pi x)dx\right]$$

$$= 2\left[\left|\left(\frac{1}{2}+x\right)\frac{(-)cos(2n\pi x)}{2n\pi} + \frac{sin(2n\pi x)}{4n^2\pi^2}\right|_{-1/2}^{0} + \left|\left(\frac{1}{2}-x\right)\frac{(-)cos(2n\pi x)}{2n\pi} - \frac{sin(2n\pi x)}{4n^2\pi^2}\right|_{0}^{1/2}\right]$$

$$= 2\left[(-)\frac{1}{2}\cdot\frac{1}{2n\pi} + 0 + \frac{1}{2}\cdot\frac{1}{2n\pi}\right] = 2\left[-\frac{1}{4n\pi} + \frac{1}{4n\pi}\right] = 0$$

Hence on putting the values of a_0, a_n, b_n in (1), we obtain the required fourier series of the given periodic function $f(x)$ as

$$f(x) = \frac{1}{4} + \frac{2}{\pi^2}\left[\frac{1}{1^2}cos(2\pi x) + \frac{1}{3^2}cos(6\pi x) + \frac{1}{5^2}cos(10\pi x) + ...\right]$$

Example – 10: $f(x) = \pi \sin \pi x$ if $0 < x < 1$, $\pi = 2L = 1$

Sol^n : Let the fourier series of the periodic function $f(x)$ of period of length 1 in the interval (0,1) be given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos(2n\pi x) + b_n Sin(2n\pi x)\right) \quad ......(1)$$

Then, $a_0 = 2\int_0^1 f(x)dx = 2\int_0^1 \pi\, sin\, \pi x\, dx = \frac{-2\pi}{\pi}\left|cos\, \pi x\right|_0^1$

$= -2[\cos \pi - 1] = 4$

$$a_n = 2\int_0^1 f(x)\,cos(2n\pi x)dx = 2\pi\int_0^1 cos(2n\pi x)\,sin\,\pi x\,dx$$

$$= \pi\int_0^1\left\{sin(2n+1)\pi x - sin(2n-1)\pi x\right\}dx$$

$$= \pi\left|\frac{cos(2n-1)\pi x}{(2n-1)\pi} - \frac{cos(2n+1)\pi x}{(2n+1)\pi}\right|_0^1$$

$$= \left[\left\{\frac{(1)^{2n-1}-1}{(2n-1)}\right\} - \left\{\frac{(-1)^{2n-1}-1}{(2n-1)}\right\}\right] = \left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) + (-1)^{2n-1}\left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}$$

$$= \left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) + (-1)^{2n}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = 2\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = \frac{-4}{4n^2-1}$$

$$a_n = \frac{4}{4n^2-1}$$

$$\therefore a_1 = \frac{-4}{13}, a_2 = \frac{-4}{35}, a_3 = \frac{-4}{57}..........$$

Again $b_n = 2\int_0^1 f(x)$ sin $(2n\pi x)$ dx

$= \pi\int_0^1 2$ sin $(2n\pi x)$ sin $\pi x\, dx$

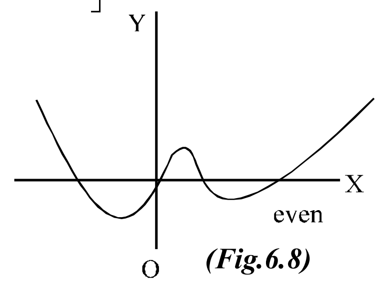
$$= \pi\int_0^1 \{cos(2n-1)\pi x - cos(2n+1)\pi x\}dx$$

$$= \pi\left|\frac{sin(2n-1)\pi x}{(2n-1)\pi} - \frac{sin(2n+1)\pi x}{(2n+1)\pi}\right|_0^1 = 0$$

Hence on putting the values of a_0, a_n, b_n in (1), we obtain the required fourier series of the given function $f(x)$ as

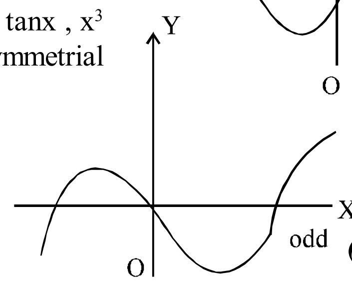
$$f(x) = 4\left[\frac{1}{2} - \frac{1}{1\cdot 3}cos(2\pi x) - \frac{1}{3\cdot 5}cos(4\pi x) - \frac{1}{5\cdot 7}cos(6\pi x) - ......\right]$$

6.8 : *Even And Odd Functions*



(Fig.6.8)

A function *f(x)* is said to be even if $f(-x) = f(x)$. e.g. cos x sec x, x^2 are all even functions. *(fig.6.8)* Graphically an even function is symmetrical about y-axis. A function $f(x)$ is said to be odd if $f(-x) = -f(x)$. e.g. sin x, tanx , x^3 are odd functions. Graphically an odd function is symmetrial about the origin *(fig.6.9)*



(Fig.6.9)

$$\int_{-l}^{l} f(x)dx = \begin{cases} 2\int_0^l f(x)dx, & when\ f(x)\ even \\ 0, & when\ f(x)\ odd \end{cases}$$

Expansions of even or odd functions :

We know that a periodic function *f(x)* defined in $(-l, l)$ can be represented by Fourier series

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx, a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{nx}{l}dx$

Case – 1 : When *f(x)* is even function

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{2}{l}\int_0^l f(x)dx$$

since *f(x)* $cos\left(\frac{n\pi x}{l}\right)$ is also even function $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\ cos\left(\frac{n\pi x}{l}\right) = \frac{2}{l}\int_0^l f(x)\ cos\left(\frac{n\pi x}{l}\right)$.

Again since *f(x)* $sin\left(\frac{n\pi x}{l}\right)$ is odd function $b_n = \frac{1}{l}\int_{-l}^{l} f(x) sin\left(\frac{n\pi x}{l}\right) = 0$

Hence, if a periodic function *f(x)* is even, it's Fourier expansion contains only cosine terms

$$a_0 = \frac{2}{l}\int_0^l f(x)dx\ ,\ a_n = \frac{2}{l}\int_0^l f(x) cos\left(\frac{n\pi x}{l}\right)dx$$

Case – II When *f(x)* is an odd function

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = 0$$

since $cos\left(\frac{n\pi x}{l}\right)$ is an even function there fore *f(x)* $cos\left(\frac{n\pi x}{l}\right)$ is an odd function

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)cos\left(\frac{n\pi x}{l}\right)dx = 0$$

Again $sin\left(\frac{n\pi x}{l}\right)$ is an odd function that *f(x)* $sin\left(\frac{n\pi x}{l}\right)$ is an even function

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_{0}^{l} f(x)sin\left(\frac{n\pi x}{l}\right)dx$$

Thus, if a periodic function *f(x)* is odd, its Fourier expansion contains only sine terms and

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)sin\left(\frac{n\pi x}{l}\right)dx$$

Illustrative Examples

Example – 1 : Show that for $-\pi < x < \pi$

$$sin\ ax = \frac{2\sin a\pi}{\pi}\left(\frac{\sin nx}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \ldots\right)$$

Sol^n: Clearly *f(x)* = sin ax is an odd function of x, $a_0 = a_n = 0$

Let $\sin ax = \sum b_n \sin nx$, where $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi}\sin ax.\sin nx dx\right] = \frac{1}{\pi}\left[\int_0^{\pi}(\cos(n-a)x - \cos(n+a)x)\, dx\right]$$

$$= \frac{l}{\pi}\left\{\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\sin n\pi\cdot\cos a\pi - \cos n\pi\ \sin a\pi}{(n-a)} - \frac{\sin n\pi.\cos a\pi + \cos n\pi.\sin a\pi}{n+a}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\sin a\pi}{(n-a)} + \frac{\sin a\pi}{n+a}\right\} = \frac{\sin a\pi}{\pi}\left[\frac{n+a+n-a}{n^2-a^2}\right] = (-1)^n\frac{2n.\sin a\pi}{\pi(n^2-a^2)}$$

$$\sin ax = \sum_{n=1}^{\infty}(-1)^n\frac{2n\sin a\pi}{\pi(n^2-a^2)}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} + \ldots\right]$$

Example – 2 : **Express *f(x)* = x as a Fourier series in the interval $-\pi < x < \pi$.**

Sol^{n} ***:*** Since $f(-x) = -x = -f(x)$

$\therefore$ *f(x)* is an odd function and hence $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Where $b_n = \dfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \dfrac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$

$= \dfrac{2}{\pi}\left[x\left(\dfrac{-\cos nx}{n}\right) - \left(\dfrac{-\sin nx}{n^2}\right)\right]_0^{\pi} = -\dfrac{2\cos\pi}{n}$

$\therefore\ b_1 = \dfrac{2}{1}, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4$ etc.

Hence the series is

$x = 2\left(\sin x - \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x - \dfrac{1}{4}\sin 4x + ...\right)$

Example – 3 : Obtain Fourier series for the equation $f(x) = x \sin x$ in the interval $-\pi < x < \pi$. Hence deduce that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - .....$$

Sol^{n} ***:*** The given function *f(x)* = x sin x is an even function of x in the interval $-\pi < x < \pi$; consequently $b_n = 0$.

Hence the Fourier expansion of given function (x sin x) would contain only cosine terms i.e.

$$f(x) = x\sin x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad ....(1)$$

where $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \dfrac{2}{\pi}\int_0^{\pi} f(x)dx$

and $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \dfrac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

Since $f(x) = x \sin x$; therefore

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx \quad = \frac{1}{\pi}\left[\{-x\cos x\}_0^{\pi} + \{\sin x\}_0^{\pi}\right] = 1 \quad ....(2)$$

and $a_n = \dfrac{2}{\pi}\int_0^{\pi} x\ \sin x\cos nx\,dx$

$a_n = \dfrac{1}{\pi}\left[\int_0^{\pi} x\{\sin(n+1)x - \sin(n-1)x\}dx\right]$

$$= \frac{1}{\pi}\left[\left[(-x)\frac{\cos(n+1)x}{(n+1)}\right]_0^{\pi} + \left[\left\{\frac{\sin(n+1)x}{(n+1)^2}\right\}_0^{\pi}\right] - \frac{1}{\pi}\left[-\left\{-x\frac{\cos(n-1)x}{(n-1)}\right\} - \left\{\frac{\sin(n-1)x}{(n-1)^2}\right\}_0^{\pi}\right]_0^{\pi}\right]$$

$[\because \sin(n+1)\pi = 0\ \&\ \sin(n-1)\pi = 0]$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{n+1} + \frac{\pi\cos(n-1)\pi}{n-1}\right] = (-1)\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1}\right] = -\frac{2\cos n\pi}{n^2-1} \quad ....(3)$$

(For $n \neq 1$ and n being an integer)

If n = 1, then

$$a_n = a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[\left\{-x\frac{\cos 2x}{2}\right\}_0^{\pi} + \left\{\frac{\sin 2x}{4}\right\}_0^{\pi}\right] = -\frac{1}{2} \quad ....(4)$$

Substituting these values of a_0 and a_n in equation (1); we get

$$f(x) = x\sin x = a_0 + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\left\{-\frac{2\cos n\pi}{(n^2-1}\right\}\cos nx$$

$$= 1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + ...$$

$$= 1 - 2\left[\frac{\cos x}{4} + \frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5}......\right] ....(5)$$

This is required series

Substituting $x = \frac{\pi}{2}$ in (5); we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 2\left[\frac{-1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + ...\right]$$

or $$\frac{\pi}{2} = 1 - 2\left[\frac{-1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + ...\right]$$

Dividing 2 on both sides, we get

or $$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ....$$

***Example – 4 :* Find a Fourier series to represent x^2 in the interval $(-l, l)$**

***Sol^n* :** Since $f(x) = x^2$ is an even function is $(-l, l)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos \pi x}{l}$$

Then $$a_0 = \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{2l^2}{3}$$

$$a_n = \int_0^l x^2\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[x^2\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 2x\left(\frac{-\cos n\pi x/l}{n^2\pi^2/l^2}\right) + 2\left(\frac{-\sin n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \frac{4l^2(-1)^n}{n^2\pi^2} \quad \left[\because \cos n\pi = (-1)^n\right]$$

$$\therefore\ a_1 = \frac{4l^2}{\pi^2},\ a_2 = 4l^2/2^2\pi^2,\ a_3 = -4l^2/3^2\pi^2$$

substituting these values

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \ldots.\right)$$

which is the required Fourier series

Example – 5 : Prove that in the interval $-\pi < x < \pi$, $x \cos x = \frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx$

Proof : Clearly $f(x) = x \cos x$ is an odd, function i.e. $a_0 = a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x.\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x + \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{(n+1)} - \frac{\cos(n-1)\pi}{(n-1)}\right\}\right], (n \neq 1),\ = \left\{-\frac{1}{n+1} - \frac{1}{n-1}\right\} = \frac{-2n}{n^2-1}, \text{if n is odd}$$

$$\text{but } b_n = \left[\frac{1}{(n+1)} + \frac{1}{(n-1)}\right] = \frac{2n}{n^2-1} \text{ if n is even}$$

If n = 1,

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\cdot\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{-1}{2}$$

$$\therefore\quad x\cos x = -\frac{1}{2}\sin x + \frac{4\sin 2x}{2^2-1} - \frac{6\sin 2x}{3^2-1} + \ldots.$$

$$= -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx$$

Example – 6 : For a function f(x) defined by $f(x) = |x|$, $-\pi < x < \pi$, obtain Fourier series,

$$\textbf{\textit{Deduce that }} \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots. = \frac{\pi^2}{8}$$

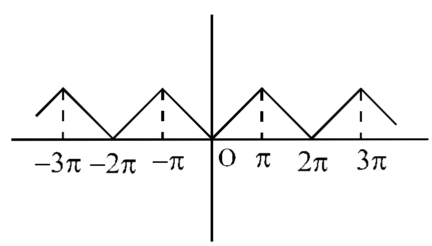


Fig. 6.10

Sol^n: Since $|x|$ is an even function, of x, $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi}|x|dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n} - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$= 0$, if n is even. $= \dfrac{-4}{\pi n^2}$, if n is odd

$$\therefore |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$$

If we substitute x = 0, in the above result, we get *(fig.6.10)*

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}$$

Example – 7: Find the Fourier series of the periodic function f (x) , $f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$

and f (x + 2π) = f(x). Sketch the graph of f (x) and the two partial sums. Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

Sol: $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)dx + \int_0^\pi k\,dx\right] = \frac{1}{\pi}\left[-k[x]_{-\pi}^{0} + k[x]_0^\pi\right]$$

$$= \frac{1}{\pi}\{-k\pi + k\pi\} = \frac{1}{\pi} \times 0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx\,dx + \int_0^\pi k\cos nx\,dx\right]$$

$$a_n = \frac{1}{\pi}\left\{(-k)\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + k\left[\frac{\sin nx}{n}\right]_0^\pi\right\} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_0^\pi k\sin nx\,dx\right] = \frac{2k}{\pi n}(1 - \cos n\pi) = \begin{cases} 0 & \text{for 'n' even} \\ \dfrac{4k}{\pi n} & \text{for 'n' odd} \end{cases}$$

The function is continuous at all points of $[-\pi, \pi]$ except $\pm\pi$ **.**

$$\therefore f(x) = \frac{4k}{\pi}\left\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right\}$$

Which holds at all pionts with exception of discontinuities $\pm\pi$ **.**

At x = ± π the sum of the series *(fig. 6.11)*

$$= \frac{1}{2}\{f(\pi-)+f(-\pi+)\} = \frac{1}{2}(1-1) = 0$$

At x = $\pi/2$, a point of continuity

$$1 = \frac{4k}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}-....\right\}$$

$$1-\frac{1}{3}+\frac{1}{5}.........=\frac{\pi}{4k}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4k}$$

$$\text{If } k = 1,\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}.$$

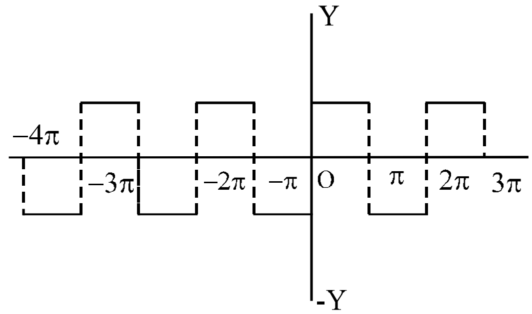


Fig. 6.11

Example – 8 : A function is defined as follows

$$f(x)=\begin{cases}-x, & -\pi<x<0\\ x, & 0<x<\pi\end{cases}$$

Show that $f(x) = \frac{\pi}{2}-\frac{4}{\pi}\left[\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+...\right]$

Deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

Sol^n: $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}-x\,dx+\int_0^{\pi}x\,dx\right]$$

$$\frac{1}{\pi}\left[\int_0^{\pi}x\,dx+\int_0^{\pi}x\,dx\right] = \frac{2}{\pi}\int_0^{\pi}x\,dx=\pi$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}-x\cos nx+\int_0^{\pi}x\cos nx\,dx\right] = \frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=\frac{2}{\pi n^2}(\cos n\pi-1)$$

$$=\begin{cases}0, & \text{for 'n' even}\\ \dfrac{-4}{\pi n^2}, & \text{for 'n' odd}\end{cases}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}-x\sin nx\,dx+\int_0^{\pi}x\sin nx\,dx\right] = \frac{2}{\pi}\left[\int_0^{\pi}x\sin nx\,dx\right]=0$$

We, thus, obtain the series

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]$$

Thus the series converges at all points and its sum is equal to the given function.

Sum of the series at 2π, is the same as at 0.

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

6.9 : *Extension of a function*

Somtimes a function is defined on $[0, a]$ or $[-a, 0]$ where $a > 0$, and we are required to re define it on $[-a, a]$ such that it is either an even function or an odd function and coincides with $f(x)$ on $[0, a]$. This re defined function is called the extension of the given function. If the redefined function is even, then it is called the even extension of the given function and it is an odd extension if it is odd. The following procedure is adopted to find an even or an odd extension.

Even Extension :

Let $f(x)$ be defined on $[0, a]$.Then the even extension of $f(x)$ on $[-a, a]$ is given by

$$\phi(x) = \begin{cases} f(x), 0 \le x \le a \\ f(x), -a \le x < a \end{cases}$$

Odd Extension :

Let $f(x)$ be defined on $[0, a]$.Then the odd extension of $f(x)$ on $[-a, a]$ is given by

$$\phi(x) = \begin{cases} f(x), 0 \le x \le a \\ -f(-x), -a \le x < 0 \end{cases}$$

Following examples illustrate this procedure

Illustration – 1 : Let the function $f(x) = x^2 + x$ be defined on the interval [0, 1]. Find the odd and even extensions of $f(x)$ in the interval [–1,1]

Sol^n : **Even Extension**

Let $\phi(x)$ be the even extension of $f(x)$ on $[-1,1]$

$$\text{Then } \phi(x) = \begin{cases} f(x) & 0 \le x \le 1 \\ f(-x) & -1 \le x < 0 \end{cases} = \begin{cases} x^2 + x, & 0 \le x \le 1 \\ x^2 - x, & -1 \le x < 0 \end{cases}$$

Odd Extension :

Let $g(x)$ be the odd extension of $f(x)$ on $[-1,1]$

$$\text{Then } \phi(x) = \begin{cases} f(x) & 0 \le x \le 1 \\ -f(-x) & -1 \le x < 0 \end{cases} = \begin{cases} x^2 + x, & 0 \le x \le 1 \\ -x^2 - x, & -1 \le x < 0 \end{cases}$$

Illustration – 2 : Let $f(x) = \sin x - \cos x$ be defined on $[0, \pi]$. Find the even and odd extension of $f(x)$ on $[-\pi, \pi]$

Sol^n : Let $\phi(x)$ be the even extension of $f(x)$ on $[-\pi, \pi]$

$$\phi(x)=\begin{cases} f(x) & 0\le x\le\pi \\ -f(-x) & -\pi\le x<0 \end{cases} = \begin{cases} \sin x-\cos x, & 0\le x\le\pi \\ \sin(-x)-\cos(-x), & -\pi\le x<0 \end{cases}$$

$$\therefore\ \phi(x)=\begin{cases} \sin x-\cos x, & 0\le x\le\pi \\ -\sin x-\cos x & -\pi\le x<0 \end{cases}$$

Odd Extension :

Let g(x) be the odd extension of $f(x)$ on $[-\pi,\pi]$

Then $g(x)=\begin{cases} f(x) & 0\le x\le\pi \\ -f(-x) & -\pi\le x<0 \end{cases} = \begin{cases} \sin x-\cos x, & 0\le x\le\pi \\ -[\sin(-x)-\cos(-x)] & -\pi\le x<0 \end{cases}$

$$\therefore\ g(x)=\begin{cases} \sin x-\cos x, & 0\le x\le\pi \\ \sin x+\cos x & -\pi\le x<0 \end{cases}$$

Half-Range Fourier Series :

Somtimes a function $f(x)$ defined on the interval $[0, \pi]$ satisfies Dirchlet's conditions and we require to expand it as a series of (i) Sines on.y. (ii) cosines only. If we want to expand $f(x)$ over the interval $[0, \pi]$ as a series of sines only, then we extend $f(x)$ over the interval $[-\pi,\pi]$ such that it becomes an odd fucntion on $[-\pi,\pi]$. In case $f(x)$ is to be expanded over $[0,\pi]$ as a series of cosines only, then $f(x)$ is extended over $[-\pi,\pi]$ as an even function.

The Sine Series :

To obtain a series consisting of only sine terms in the interval $[0, \pi]$, we define a second function $\phi(x)$ on $[-\pi,\pi]$ such that

$$\phi(x)=\begin{cases} f(x) & \text{for } 0<x<\pi \\ -f(-x) & \text{for } -\pi<x<0 \end{cases}$$

This function $\phi(x)$ is an odd function of x on the interval $[-\pi,\pi]$. Therefore its fourier series consists of sine terms only and is given by $\sum_{n=1}^{\infty} b_n \sin(nx)$.

Where $b_n=\frac{2}{\pi}\int_0^{\pi}\phi(x)\ \sin(nx)\,dx$

$$=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx\left[\because \phi(x)=f(x) \text{ for } 0<x<\pi\right]$$

The sum of the fourier series is equal to

$$=\frac{1}{2}\left[\lim_{x\to a^-}\phi(x)+\lim_{x\to a^+}\phi(x)\right]=\frac{1}{2}\left[\lim_{x\to a^-} f(x)+\lim_{x\to a^+} f(x)\right]$$

at every point 'a' between 0 and π.

At $x = 0$ or π, the sum of the series is zero.

The cosines series :

To obtain a series consisting of only cosine terms in the interval $[0,\pi]$, we define a new function $\phi(x)$ on the interval $[-\pi,\pi]$such that

$$\phi(x)=\begin{cases} f(x) & \text{for } 0<x<\pi \\ f(-x) & \text{for } -\pi<x<0 \end{cases}$$

This new function $\phi(x)$ is an even function of x on the interval $[-\pi,\pi]$. Therefore its fourier series consists of cosine terms only and is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi}\phi(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx \left[\because \phi(x) = f(x) \, for \, 0 < x < \pi\right]$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\phi(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

The sum of the series is equal to

$$\frac{1}{2}\left[\lim_{x\to a^-}\phi(x)+\lim_{x\to a^+}\phi(x)\right]=\frac{1}{2}\left[\lim_{x\to a^-}f(x)+\lim_{x\to a^+}f(x)\right]$$

at every point 'a' between 0 and π.

At $x = 0$, the sum of the fourier series is $\lim_{x\to 0^+} f(x)$ and at $x = \pi$, the sum is $\lim_{x\to\pi^-} f(x)$.

Half Range Series

Some times it is required to represent a function $f(x)$ by a Fourier series in the interval $(0, \pi)$ and not in the full interval $(-\pi, \pi)$. Since $f(x)$ is not defined in the interval $(-\pi, 0)$. We can choose $f(-x) = f(x)$ in the interval $(-\pi, 0)$. In that case $f(x)$ behaves as an even function for which $b_n = 0$.

Hence the half-range cosine series is $f(x) = \frac{a_0}{2}+\sum a_n \cos nx$

We have $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$

If we choose $f(-x) = -f(x)$ and the interval $(-\pi, 0)$, then $f(x)$ behaves as an odd function for which $a_0 = a_n = 0$.

Hence the half range sine series is given by $f(x) = \sum b_n \sin nx$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

Similarly, the half range cosine series $(0, l)$ is given by

$$F(x)=\begin{cases} f(x) & if \; 0\le x\le l \\ -f(x) & if \; -l\le x\le 0\end{cases}$$

$$f(x) = \frac{a_0}{2}+\sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx, a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and half-range sine series in the interval $(0, l)$ is given by

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

Where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Illustrative Examples

Example – 1. *Show that a constant 'C' can be expanded in a infinite sine series*

$$\frac{4c}{\pi}\left\{\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{3}+\ldots\right\} \text{in the range } 0 < x < \pi .$$

Sol^n : Let C = $\sum b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} C\sin nx\,dx = \frac{2C}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{2c}{n\pi}\left[1-\cos n\pi\right] = \frac{2c}{n\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, & \text{for 'n' even} \\ \frac{4c}{n\pi}, & \text{for 'n' odd} \end{cases}$$

The function is continuous at all points $[-\pi, \pi]$ except $\pm\pi$.

$$\therefore f(x) = \frac{4c}{\pi}\left\{\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right\}$$

Example – 2 : **Obtain cosine and sine series for** $f(x) = x$ **in the interval** $0 \le x \le \pi$ **. Hence show that**

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$$

Sol^n : **The sine series.** The function may be expanded as an odd function $f(x) = x$, in $-\pi < x < \pi$, periodic with period 2π.

$\therefore a_n = 0$ for n = 0, 1, 2, 3 .....

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$$= \begin{cases} \frac{2}{n}, & \text{for } n \text{ odd} \\ -\frac{2}{n} & \text{for } n \text{ even} \end{cases}$$

Hence for all points between 0 and π, $x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$

$$x = 2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots\right]$$

Put $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 2\left[\frac{\sin(\pi/2)}{1}-\frac{\sin\pi}{2}+\frac{\sin(3\pi/2)}{3}+\cdots\right]$$

$$\frac{\pi}{2} = 2\left[1-\frac{1}{3}+\frac{1}{5}-\cdots\right] \quad \therefore 1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4}.$$

According to half range series the sum of the series must be at $x = 0, \pi$ and this fact can be verified directly as well. The representation holds at $x = 0$, but not at $x = \pi$.

(ii) The cosine series :

The function may be extended as an even periodic function , i.e. $f(x) = x$, in $[-\pi, \pi]$ with period 2π.

$\therefore\ b_n = 0$, for n = 1, 2, 3 ....

and $a_0 = \dfrac{2}{\pi}\int_0^{\pi} x\,dx = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi}\frac{\left[(-1)^n - 1\right]}{n^2}$$

$$= \begin{cases} 0, \text{for } n, \text{even} \\ -4/\pi n^2, \text{for } n, \text{odd} \end{cases}$$

Hence for all points between 0 & π.

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]$$

The function being continuous, the relation holds for all x.

According to half range series must be $f(0+) = 0$, at $x = 0$ and $f(\pi) = \pi$, at $x = \pi$ which can be found directly from the above relation

At $x = 0$, or π, we get

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

Example – 3 : Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$ and hence find the sum of the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Sol^n: Since $f(x) = x^2$, is an even function $b_n = 0$,

Let $x^2 = \dfrac{a_0}{2} + \sum a_n \cos nx$, Then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^2$$

$$\therefore\ x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right]$$

Put $x = 0$ in the above result

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

Example – 4 : Find half range cosine series for the function $f(x) = (x - 1)^2$ in the interval $0 < x < 1$.

Sol^n : The graph of $f(x) = (x - 1)^2$ in $0 < x < 1$, is symmetrical about y - axis and therefore represents an even function in $(-1, 1)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

When $a_0 = 2\int_0^1 f(x)dx = 2\int_0^1 (x-1)^2 dx$

$$= 2\left[\int_0^1 (x^2 + 1 - 2x)dx\right] = 2\left[\left[\frac{x^3}{3}\right]_0^1 + [x]_0^1 - [x^2]_0^1\right] = 2\left[\frac{1}{3} + 1 - 1\right] = \frac{2}{3}$$

$$a_n = 2\int_0^1 f(x)\cos n\pi x\, dx = 2\left[\int_0^1 (x-1)^2 \cos n\pi x\, dx\right]$$

$$= 2\left\{\left[(x-1)^2 \frac{\sin n\pi x}{n\pi} - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1\right. = \frac{4\cos n\pi}{n^2\pi^2} = \frac{4(-1)^n}{n^2\pi^2}$$

Hence the desired required series will be $f(x)$ over the half range (0, 1) is

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

Example – 5 : **Obtain the half range sine series for e^x in $0 < x < 1$.**

Sol^n : Let $e^x = \sum b_n \sin n\pi x$, since $l = 1$,

Now $b_n = 2\int_0^1 e^x \sin n\pi x$

$$= 2\left\{\left[e^x\left(\frac{-\cos n\pi x}{n\pi}\right) - e^x\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \int_0^1 e^x\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)dx\right\}$$

$$= 2\left[e^x\left(\frac{-\cos n\pi}{n\pi}\right) + e^0\frac{\cos 0}{n\pi}\right] - \frac{2}{n^2\pi^2}\int_0^1 e^x \sin n\pi x\, dx = \frac{2}{n\pi}[-e(-1)^n + 1] - \frac{1}{n^2\pi^2}(b_n)$$

$$b_n = \frac{2n\pi}{n^2\pi^2 + 1}\left[1 - (-1)^n e\right]$$

Hence $e^x = 2\pi\left[\frac{1+e}{\pi^2+1}\sin \pi x + \frac{2(1-e)}{4\pi^2+1}\sin 2\pi x + \frac{3(1+e)}{9\pi^2+1}\sin 3\pi x + \ldots\right]$

Example – 6 : **If f (x) =** $\begin{cases} \mathbf{x}, & \mathbf{0 < x < \pi / 2} \\ \mathbf{\pi - x}, & \mathbf{\pi / 2 < x < \pi} \end{cases}$

Show that (i) $f(x) = \dfrac{4}{\pi}\left[\sin x - \dfrac{1}{3^2}\sin 3x + \dfrac{1}{5^2}\sin 5x.......\right]$

(ii) $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{1}{1^2}\cos 2x + \dfrac{1}{3^2}\cos 6x + \dfrac{1}{5^2}\cos 10x + ...\right]$

Sol^n : For half range sine series

let $f(x) = \sum b_n \sin nx$

Where $b_n = \dfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\cdot\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \frac{2}{\pi}\left[0 + \frac{\frac{\pi}{2}}{n}\cos\frac{n\pi}{2} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

Clearly $b_n = 0$, when n is even

$$b_1 = \frac{4}{\pi}, b_2 = \frac{4}{\pi\cdot 3^2}\sin\frac{3\pi}{2} = \frac{-4}{9\pi}$$

Thus $f(x) = \dfrac{4}{\pi}\left[\sin x - \dfrac{1}{9}\sin 3x + \dfrac{1}{25}\sin 5x.....\right]$

(ii) For half range cosine series,

let $f(x) = \dfrac{a_0}{2} + \sum a_n \cos nx$

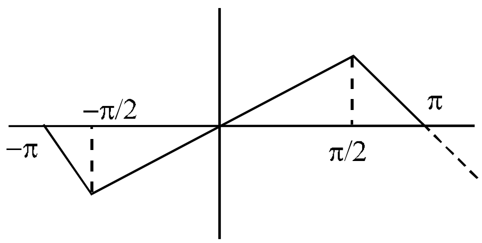
Where $a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)dx$

$$\frac{2}{\pi}\left[\int_0^{\pi/2} x dx + \int_{\pi/2}^{\pi}(\pi - x)dx\right] = \frac{\pi}{2}$$

Similarly it can be show that

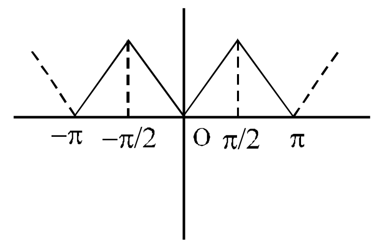
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx dx\right] = \frac{2}{\pi n^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$



(a)
(f(x) is an odd function)

Fig. 6.12



(b)
(f(x) is an even function)

Fig. 6.13

(i) If n = 4k , then $a_n = \frac{2}{\pi n^2}[2\cos 2k\pi - \cos 4k\pi - 1]$

$\Rightarrow$ $a_4 = a_8 = a_{12} = \ldots\ldots.. = 0$

(ii) If n = 4k + 1, or 4k + 3, then again it can be seen that $a_n = 0$

$\Rightarrow$ $a_1, a_3, a_7, a_9 \ldots\ldots\ldots = 0$

(iii) If n = 4k + 2

then $a_n = \frac{2}{\pi n^2}[2\cos(2k+1)\pi - \cos(4k+2)\pi - 1]$

$$= \frac{2}{\pi n^2}[2\cos\pi - \cos 2\pi - 1] = \frac{2}{\pi(4k+2)^2}(-2-1-1) = \frac{-2}{\pi(2k+1)^2}$$

$$a_2 = \frac{-2}{k}, a_6 = \frac{-2}{9\pi}, a_{10} = \frac{-2}{25k}$$

Where k = 0, 1, 2.........

Thus the half range cosine series will be

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left[\cos 2x + \frac{\cos 6x}{9} + \frac{\cos 10x}{25} + \ldots..\right]$$

Below in the first diagrams all have extended the given function *f (x)* in the interval (0, π) as an odd function in the interval (– π, π). In the second figure the same function *f (x)* in (0, π) has been entereded as an even function in (–π, π).

Example – 7 : Obtain a half range consine series for

$$f(x) = \begin{cases} kx & 0 \le x \le l/2 \\ k(l-x) & l/2 \le x \le l \end{cases}$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots..$

Sol^n : Since the interval is (0, *l*), we can expand *f (x)* either as a cosine or a sine series

For a cosine seires let

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \;= \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right] = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} kx\cdot\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[kx.\frac{l}{n\pi}\sin\frac{n\pi x}{l} + k.\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_0^{l/2} + \frac{2}{l}\left[\left((\alpha - x)\frac{l}{n\pi}\frac{\sin n\pi x}{l} - (-1)\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right]_{\frac{l}{2}}^{l}$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right] = \frac{2kl}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)$$

If $n = 1$, $a_1 = \frac{2kl}{\pi^2}\left(2\cos\frac{\pi}{2} - 1 - \cos\pi\right) = \frac{2kl}{\pi^2}(0 - 1 + 1) = 0$

Simillalrly it can be seen that $a_3 = a_4 = a_5 = a_7 = a_8 = ...0$

Again $a_2 = \frac{2kl}{\pi^2}[2\cos\pi - 1 - \cos 2\pi] = \frac{2kl}{\pi^2}(-4) = \frac{-8kl}{2^2\pi^2}$

Simillarly, $a_6 = \frac{-8kl}{6^2\pi^2}$, $a_{10} = \frac{-8kl}{10^2\pi^2}$ and so on

$$\therefore \quad f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + ...\right)$$

Put $x = l$

$$kl = \frac{kl}{4} - \frac{8kl}{\pi^2}\times\frac{1}{2^2}\left(\cos 2\pi + \frac{1}{3^2}\cos 6\pi + \frac{1}{5^2}\cos 10\pi + ...\right)$$

$$1 = \frac{1}{4} + \frac{8}{\pi^2} - \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

$$\frac{\pi^2}{8} = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

***Example – 8 :* Expand** $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \end{cases}$ **in a sine series.**

Sol

Sol[n] : The sine series for $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \text{ where } b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx\,.$$

Here $l = 2$.

$$\therefore b_n = \int_0^2 f(x)\,sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 x\,sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\,sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\left(-\frac{2x}{n\pi}\right)cos\left(\frac{n\pi x}{2}\right) + \left(\frac{4}{n^2\pi^2}\right)sin\left(\frac{n\pi x}{2}\right)\right]_0^1 + \left[-\frac{(2-x)}{n\pi}2\,cos\left(\frac{n\pi x}{2}\right) - \frac{4}{n^2\pi^2}sin\left(\frac{n\pi}{2}\right)\right]_1^2$$

$$= \left[\frac{-2}{n\pi}cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}sin\left(\frac{n\pi}{2}\right)\right] + \left[\frac{-4\,sin\,n\pi}{n^2\pi^2}\right] - \left[\frac{-2}{n\pi}cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{8}{n^2\pi^2}sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & if\ \ n\ is\ even \\ \dfrac{8(-1)^{(n-1)/2}}{n^2\pi^2} & if\ \ n\ is\ odd \end{cases}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\cdot\left[\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right]sin\left(\frac{(2n-1)nx}{2}\right)$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}sin\left(\frac{nx}{2}\right) - \frac{1}{3^2}sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}sin\left(\frac{5\pi x}{2}\right) - \cdots\right].$$

Example – 9 : **Find a Fourier sine series for $f(x) = ax + b$ in $0 < x < l$.**

Sol^n : $b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\,sin\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{l}\int_0^l (ax+b)\,sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\frac{-(ax+b)l\,cos\left(\frac{n\pi x}{l}\right)}{n\pi} + \frac{al^2\,sin\left(\frac{n\pi x}{l}\right)}{n^2\pi^2}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-l(al+b)\,cos\,n\pi}{n\pi} + \frac{bl}{n\pi}\right] = \frac{2}{l}\left[\frac{b-(al+b)(-1)^n}{n}\right].$$

$\therefore$ This sine series for $f(x) = ax + b$ in $0 < x < l$ is given by

$$ax + b = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{b-(al+b)(-1)^n}{n}\right]sin\left(\frac{n\pi x}{l}\right).$$

Example – 10 : **Expand $f(x) = \cos x$; $0 < x < \pi$ in half range sine series.**

Sol^n : The Fourier sine series of $f(x)$ in $0 < x < \pi$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx.$$

Hence $b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x dx = \frac{2}{\pi}\left[\frac{\sin^2 x}{2}\right]_0^{\pi} = 0$

If $n > 1$,

$$b_n = \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x + \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos(\pi+n\pi)}{n+1} - \frac{\cos(\pi-n\pi)}{n-1}\right) + \left(\frac{1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{2n}{n^2-1}\right] = \frac{1}{\pi}\left[\frac{2n\cos n\pi + 2n}{n^2-1}\right] = \frac{2n}{\pi}\left[\frac{(-1)^n + 1}{n^2-1}\right]$$

$$\text{Thus } b_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{4n}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4(2n)\sin(2n\pi)}{\pi\left[(2n)^2 - 1\right]}$$

$$= \frac{4}{\pi}\left[\frac{2\sin 2x}{2^2-1} + \frac{4\sin 4x}{4^2-1} + \frac{6\sin 6x}{6^2-1} + \cdots\right] = \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{2\sin 4x}{15} + \frac{3\sin 6x}{35-1} + \cdots\right]$$

Exercise – 6.1

1. Derive the expression for Fourier series co-efficients a_0, a_n, b_n.
2. Expand $f(x) = x^2$ for $0 \le x \le 2\pi$ in a Fourier series.
3. (a) Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$, and hence deduce

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.. = \frac{\pi^2}{12}$$

   (b) Obtain the Fourier series is represent e^{-ax} from $-\pi < x < \pi$

4. A saw tooth wave is given by *f (x)* = x, $-\pi < x < \pi$.

   Show that $f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{1}\sin n\pi$

5. Prove that in the range $-\pi < x < \pi$.

   $$\cos hax = \frac{2a^2}{\pi}\sin ha\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + a^2}\cos nx\right]$$

6. Find a series of sine and cosine of multiple of x which will represent *f (x)* in the interval $-\pi < x < \pi$. Where $f(x) = \begin{cases} 0, -\pi < x < 0 \\ \frac{n\pi}{4}, 0 < x < \pi \end{cases}$

   and hence deduce that

   $$\frac{\pi^2}{8} = \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

7. Find a Fourier series in the expansion of *f (x)* = x cosx, $-\pi < x < \pi$.
8. Find the Fourier series of the following functions

   (a) $f(x) = \begin{cases} 1, \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1, \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$ (b) $f(x) = \begin{cases} x, \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, \text{if } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$

   (c) $f(x) = \begin{cases} \frac{\pi}{2} + x, \text{ if } -\pi < x < 0 \\ \frac{\pi}{2} - x, \text{ if } 0 < x < \pi \end{cases}$ (d) $f(x) = \begin{cases} 0, \text{ if } -2 < x < 0 \\ 1, \text{ if } 0 < x < 2 \end{cases}$

9. A triangular wave is represented by

   $$f(x) = \begin{cases} x, 0 < x < \pi \\ -x, -\pi < x < 0 \end{cases}$$

   Represent *f (x)* as Fourier series
10. Obtain Fourier series for the function *f (x)* given by

    $$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, \ 0 \le x \le \pi \end{cases}$$

    and hence deduce that

    $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.. = \frac{\pi^2}{8}$$

11. Find Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ where

$$f(x) = \begin{cases} \pi + x, \text{for } -\pi < x < 0 \\ \pi - x, \text{for } 0 < x < \pi \end{cases}$$

12. Show that if $f(x)$ is an even function i.e. $f(-x) = f(x)$; then its real Fourier series expansion contains no sine terms.
13. Show that if $f(x)$ is an odd function i.e. $f(-x) = -f(x)$; then it's real Fourier series expansion contains no cosine term and no constant term.
14. If $f(x) = \begin{cases} 0, \text{for } -l < x < 0 \\ 1, \text{for } 0 < x < l \end{cases}$

    Find Fourier series for the range $-l < x < l$

15. If $f(x) = \begin{cases} \pi x, 0 \le x \le 1 \\ \pi(2 - x), \ 1 \le x \le 2 \end{cases}$

    Show that in the interval (0, 2)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

16. Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$.
17. Obtain the Fourier series $f(x) = \pi x$, in $0 \le x \le 2$.
18. Prove that in the interval $- x < x < \pi$.

$$x \cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2 - 1} \sin nx$$

19. Represent the following functions $f(x)$ by a half range Fourier sine series :
    (i) $f(x) = 1 (0 < x < l)$
    (ii) $f(x) = e^x \ (0 < x < 1)$
20. Represent the following functions $f(x)$ by a half range Fourier cosine series
    (i) $f(x) = x - \frac{x}{l} (0 < x < l)$
    (ii) $f(x) = \sin \frac{\pi x}{l} (0 < x < l)$
    (iii) $f(x) = x(\pi - x) \ (0 < x < \pi)$
21. Expand $f(x) = \begin{cases} \frac{1}{4} - x, \text{ if } 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, \text{if } \frac{1}{2} < x < 1 \end{cases}$

    as the Fourier series of sine terms.
22. Obtain the half range sine series for the function $f(t) = t - t^2$, $0 < t < 1$.

Answers

1. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, a_n = \int_{-\pi}^{\pi} f(x)\cos nx dx$ $\qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

2. $\frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{\pi^2}\cos nx - \frac{4\pi}{4}\sin nx\right)$

3. **(a)** $-\frac{\pi^3}{3} + 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right)$

$$+2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right)$$

(b) $\frac{2a \sinh a\pi}{\pi}\left[\frac{1}{2a^2} + \sum \frac{(-1)^n}{a^2+n^2}\cos nx + \sum \frac{n(-1)^n}{a^2+n^2}\sin nx\right]$

7. $x \cos x = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{2n(-1)^n}{(n-1)(n+1)}\sin nx$

8. **(a)** $\frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - .....\right)$

(b) $\frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25}\sin 5x....$

(c) $\frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right)$

(d) $\frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + ...\right)$

9. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,2,3.....\text{odd}}\frac{\cos nx}{n^2}$

10. $f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right)$

11. $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right)$

14. $\frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + ...\right)$

16. $f(t) = \frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - ....\right)$

17. $-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}$

19. **(i)** $\frac{4}{\pi}\left(\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}-...\right)$

(ii) $\sum_{n=1}^{\infty}\frac{2n\pi}{1+n^2\pi^2}(1-e\cos nx)\sin n\pi x$

20. **(i)** $\frac{1}{2}+\frac{4}{\pi^2}\left(\cos\frac{\pi x}{l}+\frac{1}{9}\cos\frac{3\pi x}{l}+\frac{1}{25}\cos\frac{5\pi x}{l}+....\right)$

(ii) $\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{1}{1.3}\cos\frac{2\pi x}{l}+\frac{1}{3.5}\cos\frac{4\pi x}{l}+\frac{1}{5.7}\cos\frac{6\pi x}{l}+...\right)$

(iii) $\frac{\pi^2}{6}-\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+...\right)$

21. $f(x)=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}+\frac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x+....$

22. $\frac{8}{\pi^3}\left(\frac{\sin\pi t}{1^3}+\frac{\sin 3\pi t}{3^3}+\frac{\sin 5\pi t}{5^3}+.....\right)$

6.10 : *Complex form of Fourier Series*

Let $f(x)$ be a periodic function of period 2π defined in $(\alpha\ \ \alpha+2\pi)$. The Fourier series of $f(x)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad ...(1)$$

Euler's formulae.

We know that $e^{inx}=\cos nx+i\sin nx$; $e^{-inx}=\cos nx-i\sin nx$

Hence $\cos nx=\frac{e^{inx}+e^{-inx}}{2}$ and $\sin nx=\frac{e^{inx}-e^{-inx}}{2i}=\frac{-i}{2}\left(e^{inx}-e^{-inx}\right)$

Substituting these value in (1) wet get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\frac{a_n}{2}(e^{inx}+e^{-inx})-\frac{b_n i}{2}(e^{inx}-e^{-inx})\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\left(\frac{a_n-ib_n}{2}\right)e^{inx}+\left(\frac{a_n+ib_n}{2}\right)e^{-inx}\right]$$

Let $c_0=\frac{a_0}{2}$, $c_n=\frac{a_n-ib_n}{2}$, $c_n'=\frac{a_n+ib_n}{2}$ ...(2)

$$\therefore\quad f(x)=c_0+\sum_{n=1}^{\infty}\left(c_ne^{inx}+c_n'e^{-inx}\right) \qquad ...(3)$$

Integrating (3) w.r t x between $(\alpha,\ \alpha+2\pi)$ we have

$$\int_{\alpha}^{\alpha+2\pi} f(x)\, dx = c_o \int_{\alpha}^{\alpha+2\pi} dx + \sum_{n=1}^{\infty}\left[c_n \int_{\alpha}^{\alpha+2\pi} e^{inx} dx + c_n^{'} \int_{\alpha}^{\alpha+2\pi} e^{inx} dx \right]$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\, dx = c_0 [x]_{\alpha}^{\alpha+2\pi} + \sum_{n=1}^{\infty}\left[c_n .0 + c_n^{'} .0 \right]$$

$$\left[\therefore \int_{\alpha}^{\alpha+2\pi} e^{inx} dx = \int_{\alpha}^{\alpha+2\pi} e^{-inx} dx = 0 \right]$$

$$= \int_{\alpha}^{\alpha+2\pi} (\cos nx + i \sin nx)\, dx = 0$$

$$\therefore \int_{\alpha}^{\alpha+2\pi} f(x)\, dx = c_0\, 2\pi \qquad \therefore c_0 = \frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\, dx$$

Now from equation (3) multiplying e^{-inx} and integrate within the limits $\propto$ to $\propto +2\pi$, we get

$$= = \int_{\alpha}^{\alpha+2\pi} e^{-inx} f(x)\, dx = c_0 \int_{\alpha}^{\alpha+2\pi} e^{-inx} dx + \sum_{n=1}^{\infty}\left[c_n \int_{\alpha}^{\alpha+2\pi} e^{inx} e^{-inx} dx + c_n^{1} \int_{\ell}^{\ell+2\pi} e^{-inx} . e^{-inx} dx \right]$$

$$= \int_{\alpha}^{\alpha+2\pi} e^{-inx} f(x)\, dx = c_0 .0 + c_n \int_{\alpha}^{\alpha+2\pi} dx + c_n^{'} .0$$

$$= \int_{\alpha}^{\alpha+2\pi} f(x) e^{-inx} dx = c_{\pi} .2\pi$$

$$\therefore c_n = \frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x) e^{-inx} dx$$

Now from equation (3) multiplying e^{inx} and integrate within the limits α to $\alpha+2\pi$, we get

$$\int_{\alpha}^{\alpha+2\pi} e^{inx} f(x)\, dx = c_0 \int_{\alpha}^{\alpha+2\pi} e^{inx} dx + \sum_{n=1}^{\infty}\left[c_n \int_{\alpha}^{\alpha+2\pi} e^{inx} . e^{inx} + c_n^{'} \int_{\alpha}^{\alpha+2\pi} e^{inx} . e^{-inx} \right]$$

$$= c_0 .0 + c_n .0 + c_n{'} .2\pi$$

$$\therefore \quad c_n^{'} = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} e^{inx} f(x)\, dx$$

We have $c_n = \dfrac{a_n - ib_n}{2}$ for $n = 1$ to ∞ and $\dfrac{a_n + ib_n}{2}$ for the range $n = -\infty$ to -1

$c_n = c_0 = \dfrac{a_0}{2}$ also for $n = 0$

$\therefore$ Equation can be represented on the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

where $c_n = \dfrac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x) e^{-inx}\, dx$

n is positive negative or zero

This is called complex form of Fourier series or the exponential form of Fourier series and the c_n are called **complex Fourier coefficients**.

Also complex form of Fourier series of $f(x)$ having arbitrary period $2l$ where

$l < x < l + 2l$, $f(x) = \sum_{n=\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$, $c_n = \dfrac{1}{2l}\int_{l}^{l+2l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx$