Illustrative Examples

Example – 1 : Find the divergence of the vector $\vec{v} = x^2 y\hat{i} + 2yz\hat{k} - 2xz\hat{j}$.

Solution : From definition, div $\vec{v} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz)$

$= 2xy + 0 + 2y = 2y\,(x + 1)$

Example – 2 : Show that the vector $\vec{v} = (x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$ ***is solenoidal.***

Solution : We know that A is solenoidal if div $\vec{v} = 0$.

Now div $\vec{v} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 3z) + \frac{\partial}{\partial z}(x - 2z) = 1 + 1 - 2 = 0$.

Hence $\vec{v}$ is solenoidal vector.

Example – 3 : Show that div (grad r^n***) = n (n + 1)*** r^{n-2}.

Solution : Let r be the distance of a point P (x, y, z) from a fixed point A (x_0, y_0, z_0).

$$\therefore r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$$\therefore r^n = \left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right]^{\frac{n}{2}}$$

Now grad $(r^n) = \sum \hat{i}\,\frac{\partial}{\partial x}\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right]^{\frac{n}{2}}$

$$= \sum \hat{i}\left[\frac{n}{2}\left\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right\}^{\frac{n}{2}-1} \cdot 2(x - x_0)\right]$$

$$\therefore div\left(grad\ r^n\right) = \sum \frac{\partial}{\partial x}\left[n\left\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right\}^{\frac{n}{2}-1}(x - x_0)\right]$$

$$= \Sigma n\left[\left(\frac{n}{2} - 1\right)\left\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right\}^{\frac{n}{2}-1-1} \cdot 2(x - x_0)^2 + \left\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right\}^{\frac{n}{2}-1} \cdot 1\right]$$

$$= \sum n(n - 2)r^{2\left(\frac{n}{2}-2\right)}(x - x_0)^2 + \sum n\, r^{2\left(\frac{n}{2}-1\right)}$$

$= n\,(n - 2)\, r^{n-4}\,[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2] + 3\,n\,r^{n-2}$

$= r^{n-2}\,[n\,(n - 2) + 3n] = r^{n-2}\,[n^2 - 2n + 3n] = r^{n-2}[n^2 - 2n + 3n] = n\,(n + 1)\,r^{n-2}$

Hence the result.

Example – 4 : If r = $x\hat{i} + y\hat{j} + z\hat{k}$, show that

(a) div r = 3

(b) div (rϕ) = 3ϕ + r. grad ϕ, where ϕ is a scalar function of x, y, z.

(c) $div(\hat{r}) = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$

Solution : Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and div $\vec{F} = \nabla \cdot \vec{F} = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right) \cdot \vec{F}$

(a) $div\,\vec{r} = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 1 + 1 + 1 = 3$

(b) $div\,(\vec{r}\,\phi) = \nabla \cdot \left(x\phi\hat{i} + y\phi\hat{j} + z\phi\hat{k}\right) = \dfrac{\partial}{\partial x}(x\phi) + \dfrac{\partial}{\partial y}(y\phi) + \dfrac{\partial}{\partial z}(z\phi)$

$$= \phi + x\frac{\partial \phi}{\partial x} + \phi + y\frac{\partial \phi}{\partial y} + \phi + z\frac{\partial \phi}{\partial z} = 3\phi + x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}$$

$$= 3\phi + \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) = 3\phi + \mathrm{r} \cdot \mathrm{grad}\ \phi$$

(c) Here $|r| = \sqrt{x^2 + y^2 + z^2}$, $\hat{r} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

$$\therefore div\,\vec{r} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$= \left[\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}} + x \cdot \left(-\frac{1}{2}\right)(2x)\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}}\right] + - - - - + -$$

$$\left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}} - x^2\left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}} + \left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}} - y^2\left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}}$$

$$+ \left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}} - z^2\left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}}$$

$$= 3\left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}} - \left(x^2 + y^2 + z^2\right)^{\frac{-3}{2}}\left(x^2 + y^2 + z^2\right)$$

$$3\left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}} - \left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}}$$

$$= 2\left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}} = \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

Example – 5 : If $\vec{V_1}$ and $\vec{V_2}$ be vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable pint (x, y, z). Prove that div $(\vec{V_1} \times \vec{V_2}) = 0$.

Solution : $\vec{V_1} = (x-x_1)\, i + (y-y_1)\, j + (z-z_1)\, k,\ \vec{V_2} = (x-x_2)\, i + (y-y_2)\, j + (z-z_2)\, k$

$$\therefore\ \vec{V_1} \times \vec{V_2} = \begin{vmatrix} i & j & k \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix}$$

$= \Sigma i\, \{(y-y_1)(z-z_2) - (y-y_2)(z-z_1)\}$

Now div $(\vec{V_1} \times \vec{V_2}) = \nabla \cdot (\vec{V_1} \times \vec{V_2}) = \Sigma i \dfrac{\partial}{\partial x} \{(y-y_1)(z-z_2) - (y-y_2)(z-z_1)\}$

$= \Sigma \dfrac{\partial}{\partial x} \{(y-y_1)(z-z_2) - (y-y_2)(z-z_1)\}$

$= 0 + 0 + 0 = 0$

Thus div $(\vec{V_1} \times \vec{V_2}) = 0$

3.18 : *Curl of a Vector Field*

The curl of a continuously differentiable vector point function $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ is denoted by curl $\vec{F}$ and is defined as

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$curl\ \vec{F} = \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}$$

Note that (i) $\nabla \times \vec{F}$ is a vector point function and in the expansion of the determinant, the operators $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$ must preceed $\vec{F_1}, \vec{F_2}\ \vec{F_3}$.

(ii) Curl of a vector point function is also called rotation of a vector point function.

Remark:

It must be noted that cure $\vec{F}$ is a vector quantity. Thus the curl of vector point function is a vector point function.

Physical Interpretation of Curl

Consider a rigid body rotating about a fixed axis through O. Let the uniform angular velocity be

$\vec{\Omega} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ (w_1, w_2 and w_3 are constants)

The velocity $\vec{V}$ of any point P (x, y, z) on the body is given by $\vec{V} = \vec{\Omega} \times \vec{r}$, where $\vec{r}$ is the position vector of P.

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} = (w_2 z - w_3 y)\vec{i} + (w_3 x - w_1 z)\vec{j} + (w_1 y - w_2 x)\vec{k}$$

$$\therefore \quad Curl\,\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= (w_1 + w_1)\hat{i} + (w_2 + w_2)\vec{j} + (w_3 + w_3)\vec{k} = 2w_1\vec{i} + 2w_2\vec{j} + 2w_3\vec{k}$$

$$= 2\left(w_1\vec{i} + w_2\vec{j} + w_3\vec{k}\right) = 2\vec{\Omega} \qquad \therefore \quad \vec{\Omega} = \frac{1}{2}\text{curl }\vec{V}$$

Thus, the angular velocity at any point is equal to half the curl of the linear velocity at that point of the body.

Irrotational Vector

If curl $\vec{V} = 0$, then the fluid $\vec{F}$ is said to be **irrotational** otherwise, **rotational**.

Conservative Field:

If $\vec{V}$ represents force field then curl $\vec{V} = 0$ implies that the **force field is conservative**.

Cor. If curl $\vec{V} = \vec{0}$, then the field V is termed irrotational.

For example Let $\vec{V} = \dfrac{\vec{r}}{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}, \quad \vec{V} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}$$

$$Curl\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{\sqrt{x^2 + y^2 + z^2}} & \dfrac{y}{\sqrt{x^2 + y^2 + z^2}} & \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \end{vmatrix}$$

$$= \sum \hat{i} \cdot \left\{ \frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right) \right\}$$

$$= \sum \hat{i} \cdot \left\{ z\left(-\frac{1}{2}\right)\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}}(2y) - y\left(-\frac{1}{2}\right)\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}}(2z) \right\}$$

$$= \sum \hat{i}\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}}(-yz + yz) = \vec{0} \qquad \therefore \vec{V} = \frac{\vec{r}}{r} \text{ is irrotational vector.}$$

3.19 : Properties of grad div and curl

Theorem – 1 : **grad $(\phi\psi) = \phi$ grad $\psi + \psi$ grad ϕ**

or, $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

Proof : grad $(\psi\phi)$ $= \frac{\partial}{\partial x}(\phi\psi) + \frac{\partial}{\partial y}(\phi\psi) + \frac{\partial}{\partial z}(\phi\psi)$

$$= \phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x} + \phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y} + \phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}$$

$$= \phi\left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial z}\right) + \psi\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\right) = \phi \text{ grad } \psi + \psi \text{ grad } \phi$$

Theorem – 2 : $\boldsymbol{div\left(\vec{A} + \vec{B}\right) = div\,\vec{A} + div\,\vec{B}}$

or $\nabla \cdot \left(\vec{A} + \vec{B}\right) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

Proof : $\nabla \cdot \left(\vec{A} + \vec{B}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{A} + \vec{B}\right)$

$$= \vec{i} \cdot \frac{\partial}{\partial x}\left(\vec{A} + \vec{B}\right) + \vec{j} \cdot \frac{\partial}{\partial y}\left(\vec{A} + \vec{B}\right) + \vec{k} \cdot \frac{\partial}{\partial z}\left(\vec{A} + \vec{B}\right)$$

$$= \vec{i} \cdot \left(\frac{\partial\vec{A}}{\partial x} + \frac{\partial\vec{B}}{\partial x}\right) + \vec{j} \cdot \left(\frac{\partial\vec{A}}{\partial y} + \frac{\partial\vec{B}}{\partial y}\right) + \vec{k} \cdot \left(\frac{\partial\vec{A}}{\partial z} + \frac{\partial\vec{B}}{\partial z}\right)$$

$$= \left(\vec{i} \cdot \frac{\partial\vec{A}}{\partial x} + \vec{j} \cdot \frac{\partial\vec{A}}{\partial y} + \vec{k} \cdot \frac{\partial\vec{A}}{\partial z}\right) + \left(\vec{i} \cdot \frac{\partial\vec{B}}{\partial x} + \vec{j} \cdot \frac{\partial\vec{B}}{\partial y} + \vec{k} \cdot \frac{\partial\vec{B}}{\partial z}\right) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$$

Theorem – 3 : $\boldsymbol{curl\left(\vec{A} + \vec{B}\right) = curl\,\vec{A} + curl\,\vec{B}}$

or, $\nabla \times \left(\vec{A} + \vec{B}\right) = \nabla \times \vec{A} + \nabla \times \vec{B}$

Proof : curl $\left(\vec{A} + \vec{B}\right) = \nabla \times \left(\vec{A} + \vec{B}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\vec{A} + \vec{B}\right)$

$$= \hat{i} \times \frac{\partial}{\partial x}\left(\vec{A} + \vec{B}\right) + \hat{j} \times \frac{\partial}{\partial y}\left(\vec{A} + \vec{B}\right) + \hat{k} \times \frac{\partial}{\partial z}\left(\vec{A} + \vec{B}\right)$$

$$= \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}\right) + \hat{j} \times \left(\frac{\partial \vec{A}}{\partial y} + \frac{\partial \vec{B}}{\partial y}\right) + \hat{k} \times \left(\frac{\partial \vec{A}}{\partial z} + \frac{\partial \vec{B}}{\partial z}\right)$$

$$= \hat{i} \times \frac{\partial \vec{A}}{\partial x} + \hat{i} \times \frac{\partial \vec{B}}{\partial x} + \hat{j} \times \frac{\partial \vec{A}}{\partial y} + \hat{j} \times \frac{\partial \vec{B}}{\partial y} + \hat{k} \times \frac{\partial \vec{A}}{\partial z} + \hat{k} \times \frac{\partial \vec{B}}{\partial z}$$

$$= \left(\hat{i} \times \frac{\partial A}{\partial x} + \hat{j} \times \frac{\partial A}{\partial y} + \hat{k} \times \frac{\partial A}{\partial z}\right) + \left(\hat{i} \times \frac{\partial B}{\partial x} + j \times \frac{\partial B}{\partial y} + \hat{k} \times \frac{\partial B}{\partial z}\right)$$

$$\phi\left(\nabla \times \vec{A}\right) + \left(\nabla\phi\right) \times \vec{A} = \phi\, curl\, \vec{A} + (grad\,\phi) \times \vec{A}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{A} + \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{B} \quad = \nabla \times \vec{A} + \nabla \times \vec{B}$$

$$= \text{curl } \vec{A} + \text{curl } \vec{B}$$

Hence curl $(\vec{A} + \vec{B}) =$ curl $\vec{A}$ + curl $\vec{B}$

***Theorem – 4 :* If $\vec{A}$ is a vector function and ϕ is a scalar function, then**

$$\textbf{div } \left(\phi\vec{A}\right) = \phi\, \boldsymbol{div}\vec{A} + \left(\boldsymbol{grad}\ \phi\right)\vec{A}$$

or $\quad \nabla \cdot \left(\phi\vec{A}\right) = \phi\left(\nabla \cdot \vec{A}\right) + \left(\nabla\phi\right) \cdot \vec{A}$

Proof : $\nabla \cdot \left(\phi\vec{A}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\phi\vec{A}\right)$

$$= \phi\left\{i\frac{\partial}{\partial x} + \hat{j}.\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}.\vec{A} + \left\{i\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right\}\vec{A}$$

$$= \vec{i} \cdot \frac{\partial}{\partial x}\left(\phi\vec{A}\right) + \vec{j} \cdot \frac{\partial}{\partial y}\left(\phi\vec{A}\right) + \vec{k} \cdot \frac{\partial}{\partial z}\left(\phi\vec{A}\right) = \sum \vec{i} \cdot \frac{\partial}{\partial x}\left(\phi\vec{A}\right) = \sum \vec{i}\left(\phi\frac{\partial\vec{A}}{\partial x} + \frac{\partial\phi}{\partial x}\vec{A}\right)$$

$$= \phi\sum \hat{i}\frac{\partial}{\partial x}\vec{A} + \sum\left(\hat{i}\frac{\partial\phi}{\partial x}\right).\vec{A}$$

$$= \phi\left\{i\frac{\partial}{\partial x}\vec{A} + \hat{j}.\frac{\partial}{\partial y}\vec{A} + \hat{k}\frac{\partial}{\partial z}\vec{A}\right\} + \left\{\left(i\frac{\partial\phi}{\partial x}\right)\vec{A} + \left(\hat{j}\frac{\partial\phi}{\partial y}\right)\vec{A} + \left(\hat{k}\frac{\partial\phi}{\partial z}\right)\vec{A}\right\}$$

$$= \phi\left(\nabla \cdot \vec{A}\right) + \left(\nabla\phi\right) \cdot \vec{A}$$

$$\therefore div\left(\phi\vec{A}\right) = \phi\left(div\vec{A}\right) + \left(grad\,\phi\right).\vec{A}$$

Theorem – 5 : **curl** $(\phi\vec{A}) = (\textbf{\textit{grad}}\ \phi)\times\vec{A} + \phi\,\textbf{\textit{curl}}\ \vec{A}$ (if ϕ is a scalar field and $\vec{F}$ is a vector field)

or $\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$

Proof : $\nabla\times(\phi\vec{A}) = \left(\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right)\times(\phi\vec{A}) = \vec{i}\times\dfrac{\partial}{\partial x}(\phi\vec{A}) + \vec{j}\times\dfrac{\partial}{\partial y}(\phi\vec{A}) + \vec{k}\times\dfrac{\partial}{\partial z}(\phi.\vec{A})$

$$= \vec{i}\times\left(\phi\frac{\partial\vec{A}}{\partial x} + \frac{\partial\phi}{\partial x}.\vec{A}\right) + \vec{j}\times\left(\phi\frac{\partial\vec{A}}{\partial y} + \frac{\partial\phi}{\partial y}.\vec{A}\right) + \vec{k}\times\left(\phi\frac{\partial\vec{A}}{\partial z} + \frac{\partial\phi}{\partial z}.\vec{A}\right)$$

$$= \sum\vec{i}\times\frac{\partial}{\partial x}(\phi\vec{A}) = \sum\vec{i}\times\left(\frac{\partial\phi}{\partial x}\vec{A} + \phi\frac{\partial\vec{A}}{\partial x}\right) = \sum\vec{i}\times\frac{\partial\phi}{\partial x}\vec{A} + \phi\sum\vec{i}\times\frac{\partial\vec{A}}{\partial x}$$

$$= \sum\left(\frac{\partial\phi}{\partial x}\vec{i}\right)\times\vec{A} + \phi\sum\vec{i}\times\frac{\partial\vec{A}}{\partial x}$$

$$= \phi\left(\vec{i}\frac{\partial\vec{A}}{\partial x} + \vec{j}\times\frac{\partial\vec{A}}{\partial y} + \vec{k}\times\frac{\partial\vec{A}}{\partial z}\right) + \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)\times\vec{A}$$

$$= \nabla\phi\times\vec{A} + \phi(\nabla\times\vec{A})$$

Theorem – 6 : $\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$

$$\textbf{\textit{grad}}\left(\vec{A}\cdot\vec{B}\right) = \vec{A}\times\textbf{\textit{curl}}\,\vec{B} + \vec{B}\times\textbf{\textit{curl}}\,\vec{A} + \left(\vec{A}\cdot\nabla\right)\cdot\vec{B} + \left(\vec{B}\,\nabla\right)\cdot\vec{A}$$

Proof : $\nabla(\vec{A}\cdot\vec{B}) = \sum\hat{i}\dfrac{\partial}{\partial x}(\vec{A}\cdot\vec{B}) = \sum\hat{i}\left\{\vec{A}\dfrac{\partial\vec{B}}{\partial x} + \dfrac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right\} = \sum\left(\vec{A}\cdot\dfrac{\partial\vec{B}}{\partial x}\right)\vec{i} + \sum\left(\vec{B}\cdot\dfrac{\partial\vec{A}}{\partial x}\right)\hat{i}$

Now, we know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

$\therefore (\vec{a}\cdot\vec{b})\,\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$

$$\therefore \left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = (\vec{A}\cdot\vec{i})\frac{\partial\vec{B}}{\partial x} - \vec{A}\times\left(\frac{\partial\vec{B}}{\partial x}\times\vec{i}\right) = (\vec{A}\cdot\vec{i})\frac{\partial\vec{B}}{\partial x} + \vec{A}\times\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$\therefore \sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = \left(\vec{A}\cdot\Sigma\vec{i}\frac{\partial}{\partial x}\right)\vec{B} + \vec{A}\times\Sigma\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right) = (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B})$$

Similarly, $\Sigma\left(\vec{B}\cdot\dfrac{\partial\vec{A}}{\partial x}\right)\hat{i} = (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A})$ (By interchanging A and B)

$$\therefore \nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B}) + (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A})$$

$$= (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$$

Theorem – 7 : $\nabla\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$

or, $div\left(\vec{A}\times\vec{B}\right)=\vec{B}\ curl\ \vec{A}-\vec{A}\cdot curl\ \vec{B}$

Proof : $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\Sigma\hat{i}\cdot\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)=\Sigma\hat{i}\cdot\left\{\vec{A}\times\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right\}$

$$=\Sigma\left\{\hat{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right\}+\Sigma\left\{\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right\}=\Sigma\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)-\Sigma\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)$$

$$=\Sigma\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B}-\Sigma\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A}=\left(\nabla\times\vec{A}\right)\cdot\vec{B}-\left(\nabla\times\vec{B}\right)\cdot\vec{A}\quad\left(\because\vec{a}\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right).\vec{c}\right)$$

$$=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$$

Or, $div\left(\vec{A}\times\vec{B}\right)=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\left(\vec{A}\times\vec{B}\right)$

$$=\vec{i}\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)+\vec{j}\frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right)+\vec{k}\frac{\partial}{\partial z}\left(\vec{A}\times\vec{B}\right)$$

$$=\vec{i}\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)+\vec{j}\left(\vec{A}\times\frac{\partial\vec{B}}{\partial y}+\frac{\partial\vec{A}}{\partial y}\times\vec{B}\right)+\vec{k}\left(\vec{A}\times\frac{\partial\vec{B}}{\partial z}+\frac{\partial\vec{A}}{\partial z}\times\vec{B}\right)$$

$$=\vec{i}.\vec{A}\times\frac{\partial\vec{B}}{\partial x}+\vec{j}.\vec{A}\times\frac{\partial\vec{B}}{\partial y}+\vec{k}.\vec{A}\times\frac{\partial\vec{B}}{\partial z}+\vec{i}\frac{\partial\vec{A}}{\partial x}\times\vec{B}+\vec{j}\frac{\partial\vec{A}}{\partial y}\times\vec{B}+\vec{k}\frac{\partial\vec{B}}{\partial z}\times\vec{B}$$

$$=-\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}\vec{A}+\vec{j}\times\frac{\partial\vec{B}}{\partial y}.\vec{A}+\vec{k}\times\frac{\partial\vec{B}}{\partial z}.\vec{A}\right)+\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}.\vec{B}+\vec{j}\times\frac{\partial\vec{A}}{\partial y}.\vec{B}+\vec{k}\times\frac{\partial\vec{A}}{\partial z}\vec{B}\right)$$

$$=-\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}+\vec{j}\times\frac{\partial\vec{B}}{\partial y}+\vec{k}\times\frac{\partial\vec{B}}{\partial z}\right).\vec{A}+\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}+\vec{j}\times\frac{\partial\vec{A}}{\partial y}+\vec{k}\times\frac{\partial\vec{A}}{\partial z}\right).\vec{B}$$

$$=\vec{B}.\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}+\vec{j}\times\frac{\partial\vec{A}}{\partial y}+\vec{k}\times\frac{\partial\vec{A}}{\partial z}\right)-\vec{A}.\left(\vec{i}\times\frac{\partial\vec{B}}{\partial x}+\vec{j}\times\frac{\partial\vec{B}}{\partial y}+\vec{k}\times\frac{\partial\vec{B}}{\partial z}\right)$$

$$=\vec{B}.\left(\nabla\times\vec{A}\right)-\vec{A}\left(\nabla\times\vec{B}\right)$$

$$=\vec{B}\,curl\,\vec{A}-\vec{A}\,curl\,\vec{B}$$

Theorem – 8 : $\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

or $curl\left(\vec{A} \times \vec{B}\right) = \left(div\,\vec{B}\right)\vec{A} - \left(div\,\vec{A}\right)\vec{B} + \left(\vec{B} \cdot \nabla\right) \cdot \vec{A} - \left(\vec{A} \cdot \nabla\right)\vec{B}$

Proof : $\nabla \times (\vec{A} \times \vec{B}) = \Sigma\hat{i} \times \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \Sigma\hat{i} \times \left\{\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x}\right\}$

$$= \Sigma\hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) + \Sigma\hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \Sigma\left\{(\hat{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right\} + \Sigma\left\{\left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - (\hat{i} \cdot \vec{A})\frac{\partial \vec{B}}{\partial x}\right\}$$

$$\left(\because \vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} . \vec{C}\right) . \vec{B} - \left(\vec{A} . \vec{B}\right)\vec{C}\right)$$

$$= \Sigma(\vec{B} \cdot \hat{i})\frac{\partial \vec{A}}{\partial x} - \left\{\Sigma\hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right\}\vec{B} + \left\{\Sigma\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right\}\vec{A} - \Sigma(\vec{A} \cdot \hat{i})\frac{\partial \vec{B}}{\partial x}$$

$$= \left(\vec{B} \cdot \Sigma\hat{i}\frac{\partial}{\partial x}\right)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - \left\{\vec{A} \cdot \Sigma\hat{i} \cdot \frac{\partial}{\partial x}\right\}\vec{B}$$

$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B} \;= (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

REPEATED OPERATIONS BY ∇

Before starting with the repeated operations by ∇, student are advised to note the following :

If ϕ (x, y, z) and $\vec{V}$ (x, y, z) be scalar and vector point functions respectively.

then (i) Since ϕ is scalar we can take its gradient only.

(ii) Since grad ϕ and $\vec{V}$ are both vector functions we can take their divergence as well as curl.

(iii) Since div $\vec{V}$ is a scalar function we can take its gradient only.

Theorem – 9 : **div (grad ϕ) = $\nabla^2\phi$, where** $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Proof : Div (grad ϕ) = $\nabla \cdot (\nabla\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Note. ∇^2 is called Laplacian Operator and $\nabla^2\phi = 0$ is called Laplace equation.

Theorem – 10 : **Curl (grad ϕ) = $\nabla \times (\nabla\phi) = \vec{0}$**

Proof : Curl (grad ϕ) = $\nabla \times (\nabla\phi)$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix} = \Sigma\hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right\} = \Sigma\hat{i}\left\{\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right\} = \vec{0}$$

$$\left(\because \frac{\partial^2\phi}{\partial y\partial z}=\frac{\partial^2\phi}{\partial z\,\partial y}\right)$$

Hence curl (grad ϕ) = $\vec{0}$

Provided we suppose that ϕ has continuous second order partial derivatives so that order of differentiation is immaterial.

Theorem – 11 : **div.** $\left(\textbf{Curl }\vec{\mathbf{V}}\right)=\nabla\cdot\left(\nabla\times\vec{\mathbf{V}}\right)=\mathbf{0}$

Proof : $\text{div}\left(\text{Curl }\vec{V}\right)=\nabla\cdot\left(\nabla\times\vec{V}\right)$

Let $\vec{V}=V_1\hat{i}+V_2\hat{j}+V_3\hat{k}$

$$\therefore \nabla\times\vec{V}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_1 & V_2 & V_3\end{vmatrix}=\Sigma\hat{i}\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)$$

$$\therefore div\left(Curl\vec{V}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\Sigma\hat{i}\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)\right)$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{\hat{i}\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)+\hat{j}\left(\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\right)+\hat{k}\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)\right\}$$

$$=\frac{\partial}{\partial x}\left\{\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right\}+\frac{\partial}{\partial y}\left\{\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\right\}+\frac{\partial}{\partial z}\left\{\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right\}$$

$$=\frac{\partial^2 V_3}{\partial x\partial z}-\frac{\partial^2 V_2}{\partial x\partial z}+\frac{\partial^2 V_1}{\partial y\partial z}-\frac{\partial^2 V_3}{\partial y\partial x}+\frac{\partial^2 V_2}{\partial z\partial x}-\frac{\partial^2 V_1}{\partial z\partial y}=0$$

Here $\text{Div}\cdot\left(\text{Curl }\vec{V}\right)=0$

assceming that $\vec{V}$ has continous second order partial derivatives.

Theorem – 12 : $\mathbf{Curl}\left(\mathbf{Curl}\,\vec{\mathbf{V}}\right) = \mathbf{grad}\left(\mathbf{div}\;\vec{\mathbf{V}}\right) - \nabla^2\,\vec{\mathbf{V}}$

Or $\mathbf{Curl}\left(\mathbf{Curl}\,\vec{\mathbf{V}}\right) = \nabla\left(\nabla\cdot\vec{\mathbf{V}}\right) - (\nabla\cdot\nabla)\vec{\mathbf{V}}$ $\left[\nabla\times\left(\nabla\times\vec{V}\right) = \nabla\cdot\left(\nabla\cdot\vec{V}\right) - \nabla^2\,\vec{V}\right]$

Proof : Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\text{Curl }\vec{V} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\text{Curl }\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V_3}{\partial y} - \dfrac{\partial V_2}{\partial z} & \dfrac{\partial V_1}{\partial z} - \dfrac{\partial V_3}{\partial x} & \dfrac{\partial V_2}{\partial x} - \dfrac{\partial V_1}{\partial y} \end{vmatrix}$$

$$= \Sigma\hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right\} = \Sigma\hat{i}\left\{\frac{\partial^2 V_2}{\partial y\partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} + \frac{\partial^2 V_3}{\partial z\partial x}\right\}$$

$$= \Sigma\hat{i}\left\{\frac{\partial^2 V_2}{\partial y\partial x} + \frac{\partial^2 V_3}{\partial z\partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2}\right\}$$

Add and subtract $\Sigma\hat{i}\;\dfrac{\partial^2 V_1}{\partial x^2}$

$$= \Sigma\hat{i}\left\{\left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial x\partial y} + \frac{\partial^2 V_3}{\partial x\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right\}$$

$$= \Sigma\hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right]$$

$$= \Sigma\hat{i}\left\{\frac{\partial}{\partial x}\left(\nabla\cdot\vec{V}\right) - \nabla^2 V_1\right\} = \Sigma\hat{i}\;\frac{\partial}{\partial x}\left(\nabla\cdot\vec{V}\right) - \nabla^2\Sigma\hat{i}\,V_1$$

$$= \nabla\left(\nabla\cdot\vec{V}\right) - \nabla^2\,\vec{V} = \text{grad}\left(\text{div}\,\vec{V}\right) - \nabla^2\,\vec{V}$$

Theorem –13 : Prove that grad (div $\vec{V}$) is a vector

Proof : Let $\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

grad (div $\vec{V}$) = $= \nabla.\left(\nabla.\vec{\nabla}\right)$

$$= \nabla\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right) = \text{vector}$$

Cor. From above result we can also deduce

$$\text{grad}\left(\text{div}\,\vec{V}\right) = \text{Curl}\left(\text{Curl}\,\vec{V}\right) + \nabla^2\vec{V}$$

or $$\nabla\left(\nabla\cdot\vec{V}\right) = \nabla\times\left(\nabla\times\vec{V}\right)+\nabla^2\vec{V}$$

Note. For application in questions, the results of repeated application of ∇ can easily be written down (treating ∇ as a vector.)

(i) $\nabla\cdot\nabla\phi = \nabla^2\phi$ i.e. div (grad ϕ) $\quad \because \vec{a}\cdot\vec{a} = a^2$

(ii) $\nabla\times\nabla\phi = \vec{0}$ i.e., curl (grad ϕ) $\quad \because \vec{a}\times\vec{a} = \vec{0}$

(iii) $\nabla\cdot\left(\nabla.\vec{V}\right) = 0$ i.e., grad (div $\vec{V}$)

(iv) $(\nabla.\nabla)\ \vec{V}$ i.e. $\nabla^2\vec{V}$

(v) $\nabla\cdot\left(\nabla\times\vec{V}\right) = 0$ i.e., div (curl $\vec{V}$) $\because$ in scalar triple product $\vec{a}\cdot\left(\vec{a}\times\vec{b}\right) = 0$

(vi) $\nabla\times\left(\nabla\times\vec{V}\right)$ i.e., curl (curl $\vec{V}$) $=\left(\nabla\cdot\vec{V}\right)\nabla - \nabla^2 V$

$$\because \vec{a}\times\left(\vec{b}\times\vec{c}\right) = (\vec{a}\cdot\vec{c})\,\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

***Theorem–14 :* If $\phi(x, y, z)$ is a harmonic function, show that grad ϕ is both solenoidal and irrotational.**

Proof. $\phi\ (x, y, z)$ is a harmonic function implies that it satisfies the Laplace's equation $\nabla^2\phi = 0$.

$$\therefore \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0 \quad ....(1)$$

We have do show that div (grad ϕ) = 0 and curl (grad ϕ) = $\vec{0}$

$$div(grad\ \phi) = \nabla\cdot\nabla\phi = \left(\Sigma\frac{\partial}{\partial x}i\right)\cdot\left(\Sigma\frac{\partial\phi}{\partial x}i\right) = \Sigma\frac{\partial^2\phi}{\partial x^2}$$

$$= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

By using (1) we have div (grad ϕ) = 0 and hence grad ϕ is solenoidal. Also, establishing curl (grad ϕ) = $\vec{0}$ is nothing but establishing.

We know $\nabla\times\left(\nabla\times\vec{V}\right) = \left(\nabla\cdot\vec{V}\right)\nabla - \nabla^2 V \qquad \because \vec{a}\times\left(\vec{b}\times\vec{c}\right) = (\vec{a}\cdot\vec{c})\,\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$

***Theorem – 15 :* Prove that if $\phi(x, y, z)$ is a scalar than $\phi\nabla\phi$ is irrotational.**

Proof. We have to prove that curl $(\phi\nabla\phi) = \vec{0}$

i.e., to prove that $\nabla \times (\phi\nabla\phi) = \vec{0}$.

We have the vector identify

$$\nabla \times \left(\phi \vec{A}\right) = \nabla\left(\phi \times \vec{A}\right) + \nabla\phi \times \vec{A} \quad \text{(can be assumed)}$$

Taking $\vec{A} = \nabla\phi$ we have

$$\nabla \times (\phi\nabla\phi) = \phi\nabla \times (\nabla\phi) + \nabla\phi \times \nabla\phi$$

$\nabla \times (\phi\nabla\phi) = \vec{0} + \vec{0} = \vec{0}$, because the first term is zero by the vector identify and the second term is zero since $\vec{V} \times \vec{V}$ is $\vec{0}$ for any vector $\vec{V}$.

Thus $\nabla \times (\phi\nabla\phi) = \vec{0} \Rightarrow \phi\nabla\phi$ is irrotational.

***Theorem – 16 :* If $\vec{F}_1$ and $\vec{F}_2$ are irrotational, prove that $\vec{F}_1 \times \vec{F}_2$ is solenoidal.**

Proof. $\vec{F}_1$ and $\vec{F}_2$ are irrotational by data.

$\Rightarrow$ $curl\,\vec{F}_1 = \vec{0}$ and $curl\,\vec{F}_2 = \vec{0}$ ....(1)

We have to prove that $div\left(\vec{F}_1 \times \vec{F}_2\right) = 0$

We have the vector identity

$$div\left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot curl\,\vec{A} - \vec{A} \cdot curl\,\vec{B} \quad \text{(assumed)}$$

$$\therefore \quad div\left(\vec{F}_1 \times \vec{F}_2\right) = \vec{F}_2 \cdot curl\,\vec{F}_1 - \vec{F}_1 \cdot curl\,\vec{F}_2$$

i.e., $div\left(\vec{F}_1 \times \vec{F}_2\right) = \vec{F}_2 \cdot \vec{0} - \vec{F}_1 \cdot \vec{0} = 0$ by using (1)

$$\therefore \quad div\left(\vec{F}_1 \times \vec{F}_2\right) = 0 = \vec{F}_1 \times \vec{F}_2 \text{ is solenoidal.}$$

Illustrative Examples

***Example – 1* : *Prove that div* $\left(\dfrac{\vec{r}}{r^3}\right) = 0$.**

Solution : Where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad } \vec{r} = \vec{i} \cdot \frac{\partial r}{\partial y} + \vec{j}\frac{\partial r}{\partial z} + \vec{k}\frac{\partial r}{\partial z} = \frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{\vec{r}}{r} \text{ and } div\,\vec{r} = 3$$

We have property, div $\left(\phi\vec{A}\right) = \phi\left(\text{div}\vec{A}\right) + \vec{A}$ grad ϕ

$$\text{div}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\left(\text{div }\vec{r}\right) + \vec{r}\cdot\text{grad}\left(\frac{1}{r^3}\right)$$

$$= 3r^{-3} + \vec{r}\cdot\left(-3r^{-4}\text{ grad r}\right) = 3r^{-3} + \vec{r}\cdot\left(-3r^{-4}\cdot\frac{\vec{r}}{r}\right) \quad \left[\because div\,\vec{r} = 3\right]$$

$$= 3r^{-3} - 3r^{-5}\left(\vec{r}\cdot\vec{r}\right) = 3r^{-3} - 3r^{-5}\cdot r^2 = 0$$

Example – 2 : ***Show that the vector field given by***

$$\vec{A} = 3x^2y\vec{i} + \left(x^3 - 2yz^2\right)\vec{j} + \left(3z^2 - 2y^2z\right)\vec{k}$$

is irrotational but not solenoidal. Also find its scalar ϕ (x, y, z)

Solution : $div\ \vec{A} = \frac{\partial}{\partial x}\left(3x^2y\right) + \frac{\partial}{\partial y}\left(x^3 - 2yz^2\right) + \frac{\partial}{\partial z}\left(3z^2 - 2y^2z\right) = 6xy - 2z^2 + 6z - 2y^2 \neq 0$

$\therefore\ \vec{A}$ is not solenoidal.

$$Curl\ \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 - 2yz^2 & 3z^2 - 2y^2z \end{vmatrix} = \left(-4yz + 4yz\right)\vec{i} + \left(0 - 0\right)\vec{j} + \left(3x^2 - 3x^2\right)\vec{k} = 0$$

$\Rightarrow \vec{A}$ is irrotational.

Since curl (grad ϕ) = 0 (by property 9)

$\therefore\quad \vec{A}$ = grad ϕ

$$\Rightarrow 3x^2y\vec{i} + \left(x^3 - 2yz^2\right)\vec{j} + \left(3z^2 - 2y^2z\right)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ we get

$$\frac{\partial\phi}{\partial x} = 3x^2y \text{ ....(i)}, \quad \frac{\partial\phi}{\partial y} = x^3 - 2yz^2 \text{ ....(ii)}$$

$$\frac{\partial\phi}{\partial z} = 3z^2 - 2y^2z \text{ ....(iii)}$$

Integrating (i) w.r.t. to x we get

$\phi = x^3y + f_1(y, z)$ where $f_1(y, z)$ is constant.

Integrating (ii) w.r. to y we get

$\phi = x^3y - y^2z^2 + f_2(x, z)$ where $f_2(x, z)$ is constant.

Integrating (iii) w.r. to z we get

$\phi = z^3 - y^2z^2 + f_3(x, y)$ where $f_3(x, y)$ is constant.

Now to select f_1 (y, z), $f_2(x, z)$ and f_3 (x, y) such that we get a unique ϕ.

$\therefore$ By inspecting, let $f_1(y, z) = -y^2z^2 + z^3$,

$f_2(x, z) = z^3$ and f_3 $(x, y) = x^3y$

$\therefore\ \phi = x^3y - y^2z^2 + z^3$ which is the required scalar potential.