Illustrative Examples

Example – 1 : Find the complex form of the Fourier series for f(x) = e^x in – $\pi < x < \pi$

Sol^n : The complex Fourier series of $f(x)$ having period 2π is given by $f(x)=\sum_{n=-\infty}^{\infty} C_n e^{inx}$ where

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cdot e^{-inx}dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}dx = \frac{1}{2\pi(1-in)}\left[e^{(1-in)x}\right]_{-\pi}^{\pi}$$

$$= \frac{(1+in)}{2\pi(1+n^2)}e^{(1-in)\pi} - e^{-(1-in)\pi}$$

$$= \frac{(1+in)}{2\pi(1+n^2)}e^{\pi}(\cos n\pi - i\sin n\pi) - e^{-\pi}(\cos n\pi + i\sin n\pi)$$

$$= \frac{(1+in)}{2\pi(1+n^2)}\left(e^{\pi}-e^{-\pi}\right)\cos n\pi = \frac{(1+in)2\sinh\pi(-1)^n}{2\pi(1+n^2)}$$

Thus $C_n = \dfrac{(1+in)(-1)^n \sinh\pi}{\pi(1+n^2)}$

The required complex form of Fourier series given by

$$f(x) = \frac{\sinh\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(1+in)(-1)^n}{1+n^2}e^{inx}$$

Example – 2 : Obtain the complex form of the Fourier series for the function

$$f(x) = \begin{cases} -k \text{ in } -\pi < x < 0 \\ +k \text{ in } 0 < x < \pi \end{cases}$$

Sol^n : $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ where,

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} -ke^{-inx}dx + \int_{0}^{\pi} ke^{-inx}dx\right\}$$

$$= \frac{k}{2\pi}\left\{\left[\frac{e^{-inx}}{in}\right]_{-\pi}^{0} + \left[\frac{e^{-inx}}{-in}\right]_{0}^{\pi}\right\} = \frac{k}{2\pi in}\left\{\left(1-e^{in\pi}\right)-\left(e^{-in\pi}-1\right)\right\}$$

$$= \frac{k}{2\pi in}\left\{2-\left(e^{in\pi}+e^{-in\pi}\right)\right\} = \frac{k}{2\pi in}(2-2\cos n\pi)$$

Thus $C_n = \dfrac{k\left\{1-(-1)^n\right\}}{\pi in}, (n \neq 0)$

The required complex form of the Fourier series is given by

$$f(x) = \frac{k}{i\pi}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\left\{1-(-1)^n\right\}}{n} e^{inx}$$

Example – 3 : Obtain the complex form of the Fourier series for f(x) = e^{-x} in (–1, 1)

Sol^n : Here $2l = 2$ or $l = 1$. Hence we have,

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x} \quad \text{where,} \quad C_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-in\pi x} dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-x} \cdot e^{-in\pi x} dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2}\cdot\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1} = \frac{e^{-(1+in\pi)} - e^{(1+in\pi)}}{-2(1+in\pi)} = \frac{e^{(1+in\pi)} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e(cos\, n\pi + i\, sin\, n\pi) - 1/e(cos\, n\pi - i\, sin\, n\pi)}{2(1+in\pi)}$$

$$= \frac{(e - 1/e)cos\, n\pi}{2(1+in\pi)} = \frac{e^1 - e^{-1}}{2}\cdot\frac{(-1)^n}{1+in\pi} \quad (\because sin\, n\pi = 0)$$

Thus $C_n = \dfrac{sinh\, 1(-1)^n (1-in\pi)}{1+n^2\pi^2}$

The required complex form of Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{sinh\, 1(-1)^n (1-in\pi)}{1+n^2\pi^2} e^{in\pi x}$$

Example – 4. Obtain the Fourier series in the complex form for f(x) = x, in $-\pi < x < \pi$.

Sol^n: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{mx}$ where

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, e^{-inx} dx$$

$$= \frac{1}{2\pi}\left[x \cdot \frac{e^{-inx}}{-in} - 1 \cdot \frac{e^{-inx}}{i^2 n^2}\right]_{-\pi}^{\pi}, \text{ by Bernoulli's rule,}$$

$$\frac{-1}{2\pi\, in}\left[x\, e^{-inx}\right]_{-\pi}^{\pi}$$

$$c_n = c_0 = \frac{a_0}{2} \quad \because \text{ the second term is zero.}$$

$$= \frac{-1}{2\pi in}\left(\pi e^{-in\pi} - (-\pi)e^{in\pi}\right) = \frac{-1}{2in}\left(e^{in\pi} + e^{-in\pi}\right)$$

Thus $c_n = \frac{i}{2n} 2\cos n\pi = \frac{i(-1)^n}{n}, \ (n \neq 0)$

The required complex form of the Fourier series is given by

$$f(x) = i \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{(-1)^n}{n} e^{inx}$$

Example – 5. Obtain the complex form of the Fourier series for the function

$$f(x) = \begin{cases} -a \text{ in } -\pi <, x < 0 \\ a \quad \text{in } 0 << x < \pi \end{cases}$$

Sol^n: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \ e^{-inx} dx$$

$$= \frac{1}{2\pi}\left\{ \int_{-\pi}^{0} -ae^{-inx} dx + \int_{0}^{\pi} ae^{-inx} dx \right\}$$

$$= \frac{a}{2\pi}\left\{ \left[\frac{e^{-inx}}{in}\right]_{-\pi}^{0} + \left[\frac{e^{-inx}}{-in}\right]_{0}^{\pi} \right\}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 - a^2} = \frac{\alpha}{asinha^{\alpha}}$$

$$= \frac{a}{2\pi in}\left\{ (l - e^{in\pi}) - (e^{-in\pi} - 1) \right\}$$

$$= \frac{a}{2\pi in}\left\{ 2 - (e^{in\pi} + e^{-in\pi}) \right\} = \frac{a}{2\pi in}(2 - 2\cos n\pi)$$

Thus $c_n = \frac{a\{1-(-1)^n\}}{\pi in}, \ (n \neq 0)$

The required complex form of the Fourier series is given by

$$f(x) = \frac{a}{i\pi} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\{1-(-1)^n\}}{n} e^{inx}$$

Example–6. Derive the complex form of the Fourier series for f(x) = e^{ax} in $-\pi<x<\pi$ given that a is a real constant. When a is constant other than an integer, deduce that

$$\textbf{(i)}\ \cos \alpha x = \frac{\sin\alpha\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{\alpha e^{inx}}{\alpha^2 - n^2} \ \textit{and (ii)} \ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{\alpha}{asinha\pi}$$

Sol^n: The complex form of the Fourier series for $f(x) = e^{ax}$ is given by $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} \, dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(a-in)x}}{a-in} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left[\frac{e^{(a-in)}\pi - e^{-a(a-in)\pi}}{a-in} \right]$$

$$= \frac{(-1)^n}{\pi(a-in)} \left(\frac{e^{a\pi} - e^{a\pi}}{2} \right) \left[\because \; e^{in\pi} = (-1)^n = e^{in\pi} \right]$$

$$= \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a-in)(a+in)} = \left(\frac{\sin \text{h}\, a\pi}{\pi} \right) \left[\frac{(-1)^n (a+in)}{a^2+n^2} \right]$$

$$\therefore \; e^{ax} = \sum_{n=-\infty}^{\infty} \left(\frac{\sinh\, a\pi}{\pi} \right) \left[\frac{(-1)^n (a+in)}{a^2+n^2} \right] e^{inx}$$

$$= \left(\frac{\sin h a\pi}{\pi} \right) \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n (a+in)}{a^2+n^2} \right] e^{inx} \qquad ...(1)$$

To deduce (i) we use the identity $\cos \alpha x = \frac{1}{2}\left(e^{i\alpha x} + e^{-i\alpha x}\right)$ ...(2)

Fourier series for $e^{i\alpha x}$ is got from (1) by replacing a by $i\alpha$.

$$\therefore \; e^{i\alpha x} = \frac{\sin h\,(i\alpha\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n i(\alpha+n) e^{inx}}{-\alpha^2+n^2}$$

$$= \frac{i \sin h\,(i\alpha\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (\alpha+n) e^{inx}}{\alpha^2-n^2}$$

$$= \left(\frac{\sin\, \alpha\pi}{\pi} \right) \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n (\alpha+n) e^{inx}}{\alpha^2-n^2} \right] \qquad ...(3)$$

[The relation $\sin\theta = -i \sin\, h(i\theta) \Rightarrow \sin \alpha\pi = i \sinh(i\alpha\pi)$]

Similarly, $e^{-i\alpha x} = \dfrac{\sin(-\alpha\pi)}{\pi} \displaystyle\sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n (-\alpha+n) e^{inx}}{\alpha^2-n^2} \right]$

$$= \left(\frac{\sin \alpha\pi}{\pi} \right) \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n (\alpha-n) e^{inx}}{\alpha^2-n^2} \right] \qquad ...(4)$$

Adding (3) and (4) we get

$$e^{i\alpha x} + e^{-i\alpha x} = \left(\frac{\sin \alpha\pi}{\pi} \right) \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n 2\alpha e^{inx}}{\alpha^2-n^2} \right]$$

$$\therefore \; \frac{e^{i\alpha x} + e^{-i\alpha x}}{2} = \left(\frac{\sin \alpha\pi}{\pi} \right) \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n \alpha e^{inx}}{\alpha^2-n^2} \right]$$

Hence by (2) $\cos\, \alpha x = \left(\dfrac{\sin \alpha\pi}{\pi} \right) \displaystyle\sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n \alpha e^{inx}}{\alpha^2-n^2} \right]$

To deduce (ii), we put $x = 0$ in (1)

$$1 = \left(\frac{\sin ha\pi}{\pi}\right) \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n (a+in)}{a^2+n^2}\right]$$

Equating real parts on both sides we get

$$1 = \left(\frac{\sin ha\,\pi}{\pi}\right) \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n a}{a^2+n^2}\right]$$

$$\therefore \quad \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n n}{a^2+n^2}\right] = \left(\frac{\pi}{a \sin h\ a\pi}\right)$$

Example–7. Find the complex form of the Fourier series of f(x) = sin ax where a is not an integer, in $-\pi<x<\pi$.

Sol^n: The complex form of the Fourier series for $f(x) = \sin ax$ is given by $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx \ = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\ ax\ e^{-inx} dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{a^2-n^2}(-in\ \sin\ ax - a\ \cos\ ax)\right]_{-\pi}^{\pi}$$

[using the formula $\int e^{ax} \sin\ bx\ dx = \frac{a^{ax}}{a^2+b^2}(a\ \sin\ bx - b\cos bx)$]

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{a^2-n^2}(-in\ \sin\ a\pi - a\cos a\pi - \frac{e^{in\pi}}{a^2-n^2}(in\ \sin\ a\pi - a\cos a\pi)\right]$$

$$= \frac{-1}{2\pi(a^2-n^2)}\left[in\ \sin\ a\pi(e^{-in\pi}+e^{in\pi}) + a\cos a\pi(e^{-in\pi}-e^{in\pi})\right]$$

$$= \frac{-1}{2\pi(a^2-n^2)}\left[2in\ \sin\ a\pi\ \cos n\pi - 2ia\cos a\pi \sin n\pi\ \right]$$

$$= \frac{-\sin a\pi[in(-1)^n]}{\pi(a^2-n^2)} = \frac{(-1)^{n+1}\sin a\pi\,(in)}{\pi(a^2-n^2)}$$

$\therefore$ The required Fourier series is $\sin ax = \frac{i\ \sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^{n+1} ne^{inx}}{a^2-n^2}\right]$

6.11 : *Practical Harmonic Analysis*

Harmonic analysis is the theory of expanding a given function $y = f(x)$ in terms of Fourier series. Some times we may not know the formula for a function and we may have only a table of values of $f(x)$. We know that the Fourier coefficients of Euler's formulae of a given series f(x) which can be expanded in terms of Fourier series with period 2π in the interval $(-\pi, \pi)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\ \cos\ nx + b_n \sin nx) \qquad ...(1)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$ ...(2)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad ...(3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin n\, x dx \quad ...(4)$$

When $f(x)$ is analytic, the integrals in the R.H.S of (2) (3) and (4) can be evaluated and the Fourier coefficient a_0, a_n and b_n can be determined completely. However some practical cases the function $f(x)$ is in tabulated form. In that cases practical harmonic analysis deals with the determination of the approximated values of the Fourier coefficients a_0, a_n and b_n.

In the Fourier series expansion the term $a_1 \cos x + b_1 \sin x$ is called the first harmonic of fundamental harmonic. The term $a_2 \cos 2x + b_2 \sin 2x$ is called the second harmonic and so on.

Suppose $f(x)$ is continuous on $[-\pi, \pi]$, tabulated $f(x)$ of 'n' equal parts of width 'h', where 'h' being space length or step length.

S.t $-\pi \le x_0 < x_1 < x_2 ......x_n \le \pi$ and subinterval of size $h = \frac{\pi - (-\pi)}{n} = \frac{2\pi}{n}$

Let $y_i = f(x_i)$ where $i = 0,1,2,....n$

Now the integrals on the R.H.S. of (2) (3) and (4) are approximately evaluated using rectangular formula (area = sum of n rectangles $= \sum_{i=1}^{n}$ width $h \times$ ordinate y_i). Then the Fourier coefficients of (2) (3) & (4) are determined approximately by the following manner.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\; dx = \frac{1}{\pi}\left[\sum_{i=1}^{n} h.y_i\right] = \frac{1}{\pi}.\frac{2\pi}{n}\sum_{i=1}^{n} y_i$$

or $a_0 = \frac{2}{n}\sum_{i=1}^{n} y_i$ ...(5)

also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\; \cos nx\; dx$

$$= \frac{1}{\pi}\left[\sum_{i=1}^{n} h.\, y_i\; \cos nx_i\right] = \frac{1}{\pi}.\frac{2\pi}{n}\sum_{i=1}^{n} y_i\; \cos nx_i$$

$$a_n = \frac{2}{n}\sum_{i=1}^{n} y_i\; \cos nx_i$$

Similarly $b_n = \frac{2}{n}\sum_{i=1}^{n} y_i\; \sin nx_i$ ...(7)

Illustrative Examples

Example – 1. Compute approximately the Fourier coefficients a_0, a_1, a_2, a_3 ***and*** b_1, b_2, b_3 ***in the Fourier series expansion of function tabulated as follows. Find the amplitude of the first harmonic calculate y(3).***

x:	***0***	***1***	***2***	***3***	***4***	***5***
y :	***9***	***18***	***24***	***28***	***26***	***20***

Solⁿ:

Here n = number of sub interval = 6. The interval $(0, 2\pi)$ is divided into 7 subintervals of size $\frac{2\pi - 0}{6} = \frac{2\pi}{6} = 60$

x	θ	**cos θ**	**cos 2θ**	**cos 3θ**	y	**y cos θ**	**y cos 2θ**	**y cos 3θ**
0	0°	1	1	1	9	9	9	9
1	60°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	18	9	–9	–18
2	120°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	24	–12	–12	24
3	180°	– 1	1	–1	28	–28	28	–28
4	240°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	26	–13	–13	26
5	300°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	20	10	–10	–20
				Total	**125**	**–25**	**–7**	**–7**

Now $a_0 = \frac{2}{n}\sum_{i=1}^{n} y_i = \frac{2}{6}(125) = 41.666$

$$a_1 = \frac{2}{n}\sum_{i=1}^{n} y_i \cos x_i = \frac{2}{6}(-25) = -8.333$$

$$a_2 = \frac{2}{n}\sum_{i=1}^{n} y_i \cos 2x_i = \frac{2}{6}(-7) = -2.333$$

$$a_3 = \frac{2}{n}\sum_{i=1}^{n} y_i \cos 3x_i = \frac{2}{6}(-7) = -2.333$$

Similarly,

x	θ	**sin θ**	**sin 2θ**	**sin 3θ**	y	**y sin θ**	**y sin 2θ**	**y sin 3θ**
0	0°	0	0	0	9	0	0	0
1	60°	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	18	$9\sqrt{3}$	$9\sqrt{3}$	0
2	120°	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	24	$12\sqrt{3}$	$-12\sqrt{3}$	0
3	180°	0	0	0	28	0	0	0
4	240°	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	26	$-13\sqrt{3}$	$13\sqrt{3}$	0
5	300°	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	20	$-10\sqrt{3}$	$-10\sqrt{3}$	0
				Total	**125**	$-2\sqrt{3}$	**0**	**0**

$$b_1 = \frac{2}{n}\sum_{i=1}^{n} y_i \sin x_i = \frac{2}{6}\left(-2\sqrt{3}\right) = -1.1547$$

$$b_2 = \frac{2}{n}\sum_{i=1}^{n} y_i \sin 2x_i = \frac{2}{6}(0) = 0$$

Similarly $b_3 = 0$

amplitude of the first harmonic

$\sqrt{a_1^2 + b_1^2} = \sqrt{(-8.333)^2 + (-1.1547)^2} = 8.4126$

The Fourier series of $y(x)$ containing the first 4 cosine terms and 3 sine term is

$$y = \frac{41.666}{2} + (-8.333)\cos x - 2.333\cos x - 2.337\cos 2x + (-1.1547)\sin x + 0 + 0$$

at $x = 3, \theta = \pi$

$$y(3) = y(\theta = \pi) = \frac{41.666}{2} + (-8.333)(-1) - 2.333(1) - (2.337)(-1) = 29.166$$

Exact value of $y(3) = 28$

Example– 2. Compute a_0, a_1, a_2, a_3 and b_1, b_2, b_3 in the Fourier cosine series for y which is tabulated below.

x:	***0***	***1***	***2***	***3***	***4***	***5***
y :	***4***	***8***	***15***	***7***	***6***	***2***

Find amplitude of first harmonic.

Sol^n: Here n = number of sub-intervals = 6, the interval $(0, 2\pi)$ is divided into 6 sub intervals of size

$$\frac{2\pi - 0}{6} = \frac{2\pi}{6} = 60°$$

x	θ	**cos θ**	**cos 2θ**	**cos 3θ**	y	**y cos θ**	**y cos 2θ**	**y cos 3θ**
0	0°	1	1	1	4	4	4	4
1	60°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	8	4	–4	–8
2	120°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	15	–7.5	-7.5	15
3	180°	–1	1	–1	7	–7	7	–7
4	240°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	6	–3	–3	6
5	300°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	2	1	–1	–2
				Total	**42**	**–8.5**	**–4.5**	**8**

Now $a_0 = \frac{2}{n}\sum_{i=1}^{n} y_i = \frac{2}{6}(42) = 14$ $a_1 = \frac{2}{n}\sum_{i=1}^{n} y_i \cos x_i = \frac{2}{6}(-8.5) = -2.8$

$a_2 = \frac{2}{n}\sum_{i=1}^{n} y_i \cos 2x_i = \frac{2}{6}(-4.5) = -1.5$ $\qquad a_3 = \frac{2}{n}\sum_{i=1}^{n} y_i \cos 3x_i = \frac{2}{6}(8) = 2.7$

Similarly,

x	θ	**sin θ**	**sin 2θ**	**sin 3θ**	y	**y sin θ**	**y sin 2θ**	**y sin 3θ**
0	0°	0	0	0	4	0	0	0
1	60°	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	8	$4\sqrt{3}$	$4\sqrt{3}$	0
2	120°	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	15	12.99	–12.99	0
3	180°	0	0	0	7	0	0	0
4	240°	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	6	$-3\sqrt{3}$	$3\sqrt{3}$	0
5	300°	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	2	$-\sqrt{3}$	$-\sqrt{3}$	0
				Total	**42**	**12.99**	**–2.597**	**0**

$$b_1 = \frac{2}{n}\sum_{i=1}^{n} y_i \sin x_i = \frac{2}{6}(12.99) = 4.33$$

$$b_2 = \frac{2}{n}\sum_{i=1}^{n} y_i \sin 2x_i = \frac{2}{6}(-2.597) = -0.865$$

$b_3 = 0$, amplitude of first harmonic $\sqrt{a_1^2 + b_1^2} = \sqrt{(-2.8)^2 + (4.33)^2} = 5.156$

Example– 3. Find an empirical formula of the form $y = a_0 + a_1 \cos x + b_1 \sin x$ for the following data given that f(x) is periodic with period 2π.

x:	***0***	***60°***	***120°***	***180°***	***240°***	***300°***
y :	***40.0***	***31.0***	***–13.7***	***20.0***	***3.7***	***–21.0***

Find complitude of first harmonic.

Sol^n: Here n = number of sub-intervals = 6 the interval (0, 2π) is divided into 6 subintervals of size

$$\frac{2\pi - 0}{6} = \frac{2\pi}{6} = 60°$$

θ	**cos θ**	y	**y cos θ**	**sin θ**	**y sin θ**
0°	1.0	40.0	40.00	0	0
60°	0.5	31.0	15.50	.866	26.846
120°	–0.5	–13.7	6.85	.866	–11.864
180°	–1.0	20.0	–20.00	0	0
240°	–0.5	3.7	–1.85	–.866	–3.204
300°	0.5	–21.0	–10.50	–.866	18.186
	Total	**60**	**30**		**29.964**

Now $a_0 = \frac{2}{n}\sum_{i=1}^{n} y_i = \frac{2}{6}(60) = 20$

$$a_1 = \frac{2}{n}\sum_{i=1}^{n} y_i \cos x_i = \frac{2}{6}(30) = 10$$

$$b_1 = \frac{2}{n}\sum_{i=1}^{n} y_i \sin x_i = \frac{2}{6}(29.964) = 9.988$$

$$\therefore y = 20 + 10\cos x + 9.988\sin x$$

Working procedure for problems

- We have to first write down the period of $y = f(x)$ from the given range of the values of x.
- If the period is 2π, depending on the harmonics required we prepare the relevant table along with the summations (Σ) of $y \cos x$, $y \cos 2x$...; $y \sin x$, $y \sin 2x$... and compute the harmonics rising the formulae derived by taking n =1, 2, ...
- If the period is not 2π we equate it with $2l$ to obtain the value of l.
- The summations of y; $y \cos \theta$, $y \cos 2\theta$,...; y sin, $y \sin 2\theta$... where $\theta = \pi x/l$ will be required to compute the desired harmonics.

Example – 4. Obtain the Fourier series of y up to the second harmonics for the following values.

x^o:	45^o	90^o	135^o	180^o	225^o	270^o	315^o	360^o
y :	4.0	3.8	2.4	2.0	–1.5	0	2.8	3.4

Sol^n. The interval of x is $0 < x \le 2\pi$ and period of $y = f(x)$ is 2π. We have to compute $a_0; a_1, b_1; a_2, b_2$

$$a_0 = \frac{2}{n}\Sigma y_i,\ a_1 = \frac{2}{n}\Sigma y_i \cos x_i,\ b_1 = \frac{2}{n}\Sigma y_i \sin x_i$$

$$a_2 = \frac{2}{n}\Sigma y_i \cos 2x_i, b_2 = \frac{2}{n}\Sigma y_i \sin 2x_i \quad \text{Here n=8;} \quad \frac{2}{n} = \frac{1}{4}$$

x^0	y	cos x	y cos x	sin x	y sin x
45	4.0	0.7071	2.8284	0.7071	2.8284
90	3.8	0	0	1	3.8
135	2.4	–0.7071	–1.69704	0.7071	1.69704
180	2.0	–1	–2.0	0	0
225	–1.5	–0.7071	1.06065	–0.7071	1.06065
270	0	0	0	–1	0
315	2.8	0.7071	1.97988	–0.7071	–1.97988
360	3.4	1	3.4	0	0
Total	**16.9**		**5.57189**		**7.40621**

cos 2x	y cos 2x	sin 2x	y sin 2x
0	0	1	4.0
–1	–3.8	0	0
0	0	–1	–2.4
1	2.0	0	0
0	0	1	–1.5
–1	0	0	0
0	0	–1	–2.8
1	3.4	0	0
	1.6		**–2.7**

From the table,

$\Sigma y = 16.9,\ \Sigma y\ \cos x = 5.57189,\ \Sigma y\ \sin x = 7.40621$

$\Sigma y\ \cos 2x = 1.6,\ \Sigma y\ \sin 2x = -2.7$

$a_0 = 1/4.(16.9) = 4.225, a_0/2 = 2.1125,$

$a_1 = 1/4.(5.57189) = 1.393.\ b_1 = 1/4.(7.4061) = 1.8516$

$a_2 = 1/4.(1.6) = 0.4 \qquad b_2 = 1/4.(-2.7) = -0.675$

The Fourier series of y upto the second harmonic is given by

$y = a_0/2 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$

Thus $y = 2.1125 + (1.393\cos x + 1.8516\sin x) + (0.4\cos 2x - 0.675\sin 2x)$

Example– 5. Obtain the constant term and the coefficients of the first cosine and sine terms in the Fourier expansion of y from the table.

x	***0***	***1***	***2***	***3***	***4***	***5***
y :	***9***	***18***	***24***	***28***	***26***	***20***

Sol^n: The values at 0, 1, 2, 3, 4, 5 are given ($n = 6$) and hence the interval of x should be $0 \le x < 6$. That is (0, 6)

$\therefore$ length of the interval is 6 – 0 = 6. Comparing with $2l$ we have $2l = 6$ or $l = 3$.

The Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Since $l = 3$, the series containing the first harmonics is

$$y = f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}$$

Writing $\frac{\pi x}{3} = \theta, y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta; n = 6$ and $\frac{2}{n} = \frac{1}{3}$

x	$\theta = \pi x/3$	y	$\cos\ \theta$	$y \cos\ \theta$	$\sin\ \theta$	$y \sin\ \theta$
0	0	9	1	9	0	0
1	60°	18	0.5	9	0.866	15.588
2	120°	24	–0.5	–12	0.866	20.784
3	180°	28	–1	–28	0	0
4	240°	26	–0.5	–13	–0.866	–22.516
5	300°	20	0.5	10	–0.866	–17.32
Total		**125**		**–25**		**–3.464**

$$a_0 = \frac{2}{n}\Sigma y \quad = \frac{1}{3}(125) \approx 41.67;\ \frac{a_0}{2} = 20.835$$

$$a_1 = \frac{2}{n}\sum y\cos\theta \quad = \frac{1}{3}(-25) \approx -8.333$$

$$b_1 = \frac{2}{n}\Sigma\ y\sin\theta \quad = \frac{1}{3}(-3.464) \approx -1.155$$

Constant term $= a_0 / 2 = 20.835$

Coefficient of the first cosine term $= a_1 = -8.333$

Coefficient of the first sine term $= b_1 = -1.155$

Example– 6. Express y as a Fourier series up to the third harmonics given the following values.

x	***0***	***1***	***2***	***3***	***4***	***5***
y	***4***	***8***	***15***	***7***	***6***	***2***

Solⁿ: As in the previous problem the interval of x is $(0,\ 6)$

$\therefore\ 2l = 6$ or $l = 3,\ n = 6;\ \dfrac{2}{n} = \dfrac{1}{3}$

Fourier series up to the third harmonics is given by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right)$$

$$+ \left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right), \text{ where } l = 3$$

$$\therefore\ y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right) + \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

Putting $\pi x / 3 = \theta$

$$y = a_0 / 2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

x	$\theta = \pi x / 3$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	0	4	4	4	4
1	60°	8	4	– 4	– 8
2	120°	15	– 7.5	– 7.5	15
3	180°	7	– 7	7	– 7
4	240°	6	– 3	– 3	6
5	300°	2	1	– 1	– 2
Total		**42**	**– 8.5**	**– 4.5**	**8**

y sin θ	y sin2 θ	y sin 3 θ
0	0	0
6.928	6.928	0
12.99	– 12.99	0
0	0	0
– 5.196	5.196	0
– 1.732	– 1.732	0
12.99	**– 2.598**	**0**

$a_0 = \dfrac{2}{n}\Sigma y = \dfrac{1}{3}(42) = 14;\ \dfrac{a_0}{2} = 7$ $\qquad a_1 = \dfrac{2}{n}\Sigma y \cos\theta = \dfrac{1}{3}(-8.5) = -2.833$

$b_1 = \dfrac{2}{n}\Sigma\ y \sin\theta = \dfrac{1}{3}(12.99) = 4.33$ $\qquad a_2 = \dfrac{2}{n}\Sigma y \cos 2\theta = \dfrac{1}{3}(-4.5) = -1.5$

$b_2 = \frac{2}{n}\Sigma\, y \sin 2\theta = \frac{1}{3}(-2.598) = -0.866$ $\qquad a_3 = \frac{2}{n}\Sigma\, y \cos 3\theta = \frac{1}{3}(8) = 2.667$

$b_3 = \frac{2}{n}\Sigma\, y \sin 3\theta = \frac{1}{3}(0) = 0$

The required Fourier series upto the third harmonics is given by

$$y = 7 - 2.833\cos\frac{\pi x}{3} + 4.33\sin\frac{\pi x}{3} - 1.5\cos\frac{2\pi x}{3} - 0.866\sin\left(\frac{2\pi x}{3}\right) + 2.667\cos\pi x$$

Exercise – 6.2

1. Compute approximately the Fourier coefficients a_0, a_1, a_2, a_3, and b_1, b_2, b_3 in the fourier series expansion of the function tabulated as follows. Find the amplitude of the first harmonic, calculated y (3).

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

2. Find the harmonics a_0, a_1, a_2, a_3, b_1, b_2, b_3 by the Fourier series of the following data.

x :	0	$\frac{\pi}{3}$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y :	1·0	1·4	1·9	1·7	1·5	1·2	1·0

3. Find the first three coefficients of cosine and two coefficients of sene terms in the Fourier series expansion of the following fucntion.

x :	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
y :	0	9·2	14·4	17·8	17·3	11·7

4. Find the complex Fourier series of the following functions
   (a) $f(x) = e^x$ is $-\pi < x < \pi$
   (b) $f(x) = x$ is $(-\pi < x < \pi)$
   (c) $f(x) = x$ is $(0 < x < 2\pi)$

Answers

1. $a_0 = 41{\cdot}666$, $a_1 = -8{\cdot}333$, $a_2 = -2{\cdot}333$, $a_3 = -2{\cdot}337$
$b_1 = -1{\cdot}1547$ $b_2 = 0$, $b_3 = 0$, amplitude = 8·4126
y(3) = 28

2. $a_0 = 2{\cdot}9$, $a_1 = -{\cdot}37$, $a_2 = -0{\cdot}1$, $a_3 = 0{\cdot}03$, $b_1 = 0{\cdot}17$, $b_2 = -0{\cdot}06$, $b_3 = 0$

3. $a_0 = 23{\cdot}46$, $a_1 = -7{\cdot}73$, $a_2 = -2{\cdot}83$, $b_1 = -1{\cdot}566$, $b_2 = +{\cdot}116$

4. **(a)** $\frac{2\,sinh\,\pi}{x}\left(\frac{1}{2} - \frac{1}{1+1^2}(cos\,x - sin\,x) + \frac{1}{1+2^2}(cos\,2x - 2\,sin\,2x) + \ldots\right)$

(b) $i\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} e^{inx}$ $\qquad$ **(c)** $\pi + i\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{inx}$