Clearly coordinate of any point on the curve depends upon arc length s.

So $x = f_1(s),\ y = f_2(s),\ z = f_3(s)$

$\therefore \vec{r} = f_1(s)\hat{i} + f_2(s)\hat{j} + f_3(s)\hat{k}$

$$\boxed{\vec{r} = \vec{f}(s)}$$

This is the general equation of space curve when length is taken as independent parameter.

3.15 : Scalar and Vector Field

Point Function: A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

Point functions are of two types:

(i) Scalar Point Function

(ii) Vector Point Function

A function $\phi\ (x, y, z)$ is called a scalar point function if it associates a scalar point function if it associates a scalar with every point in Region R of a space.

Scalar Point Function : If to each point P (x ,y, z) of a region R in space, there corresponds a unique scalar ϕ (P), then is ϕ called a scalar point function and we say that a scalar field ϕ has been defined in R.

e.g. : (i) The temperature at any point within or on the surface of earth at a certain time defines a scalar field :

(ii) $\phi\ (x, y, z) = x^2\, y^3 - 3z^2$ defines a scalar field.

Vector Point functions. If to eac point P (x, y, z) of a region R in space, there corresponds a unique vector f (P), then f is called a vector point function and we say that a vector field **f** has been defined in R.

e.g. : (i) If the velocity at any point (x, y, z) of a particle moving in a curve is known at a certain time then a vector field is defined.

(ii) $f\,(x, y, z) = xy^2\, i + 3yz^3\mathbf{j} + 2x^2z\mathbf{k}$ defines a vector field.

Level surface :

Let a scalar point function $f\,(x, y, z)$ be defined in a certain region of space and consider those points of the field for which ϕ has a fixed value c. The totality of points satisfying the equation $\phi\ (x, y, z) = c$ defines, in general, a surface. Such a surface is called a level surface, since at every point of the surface ϕ has a constant value c. For different values of c, different level surfaces are obtained and no two level surfaces will intersect.

e.g. : $\phi\ (x, y, z) = x^2 + y^2 + z^2 = r^2$, represents a sphere of radius **r**, which is a level surface. Different values of r will give concentric spheres, which are different level surfaces.

Let $\vec{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ be a vector function. Then for various values of t we get a set of constant vectors. The sum of constant vectors form, a space or region which is known as vector field.

The vector differential operation del (∇)

Coaxial cylinders are also examples of level surfaces.

Let $\phi = \phi\,(x, y, z)$ be a scalar function. Then for various values of x, y, z we get a set of points or scalars. The sum of these scalars form a space or region which is known as scalar field.

Vector operation ∇ (read as del.) is defined by the equation

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

This operation has a great role in vector calculus. Laplacian operator ∇^2 is defined as follows:

$$\nabla^2 = \nabla\cdot\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

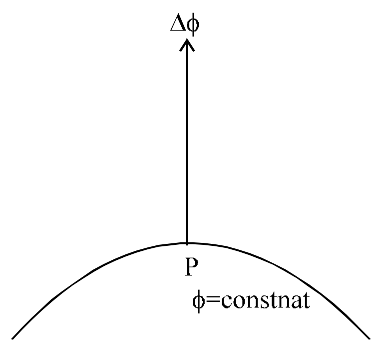
It is to be noted that the vector function and scalar function are sometimes termed as vector point function and scalar point and scalar point function since they depend upon the position of point.

Gradient – The vector function $\nabla\phi$ is defined as the gradient of the scalar function $\phi = \phi\,(x, y, z)$, i.e., $\phi\,(x, y, z)$ by any continuously differentiable scalar function. The gradient of scalar function ϕ is mathematically defined as $grad\ \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$

$$= \nabla\phi$$

Where $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}$ is called del or enable.

Geometrically, $\nabla\phi$ represents a normal at any point P to the surface $\phi\,(x, y, z)$ = constant and has a magnitude equal to the rate of change of $\phi\,(x, y, z)$ along this normal. ***(fig 3.19)***



(Fig 3.19)

Physical Interpretation.

In this scalar field let there be two level surfaces S_1 and S_2 very close together characterised by the scalar functions ϕ and $\phi + d\phi$ respectively. Consider two points P and R on the level surfaces S_1 and S_2 respectively. Let $\vec{r}$ and $\vec{r} + d\vec{r}$ be the position vectors of P and R respectively relative to any arbitrary origin, then $PR = d\vec{r}$.

If co-ordinates of P and R are (x, y, z) and $(x + dx, y+dy, z + dz)$ respectively, then

$$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz \text{ ....(2)}$$

As the values of scalar function at P (x, y, z) and R $(x + dx, y + dy, z + dz)$ are $\phi + d\phi$, we may write. ***(fig 3.20)***

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \qquad \text{....(3)}$$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{dy} + \hat{k}\frac{\partial f}{dz}\right)\times\left(\hat{i}\ dx + \hat{j}\ dy + \hat{k}\ dz\right)$$

$$= (\nabla\phi)\cdot d\vec{r} \text{ ....(4) Using (1) and (2)}$$

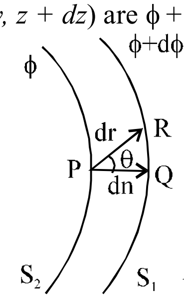


Fig 3.20

In particular if we consider that the point R (i.e., dr) lies on the level surface S_1, then $d\phi = 0$, so that $(\nabla\phi) \cdot \vec{dr} = 0$ ....(5)

thereby showing that the vector $\nabla\phi$ is normal to the level surface S_1.

If dr represents the distance along the normal from point P to the surface S_2, then

$$\vec{dr} = PQ = \vec{dr} \cos\theta = \hat{n} \cdot \vec{dr} \quad .....(6)$$

Where $\hat{n}$ is a unit vector normal to the surface S_1 and P. If we proceed from P to Q the value of scalar function ϕ increases by an amount $d\phi$; therefore, we may write

$$d\phi = \frac{\partial\phi}{\partial n} dn = \frac{\partial\phi}{\partial n} \hat{n} \cdot dr \quad \text{(using (6))} \quad .....(7)$$

Comparing (4) and (7), we get

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial n} \hat{n}. \qquad ....(8)$$

Thus, gradient of a scalar function ϕ at any point is a vector whose magnitude is equal to the rate of change of scalar function ϕ along the normal to the level surface and whose direction is normal to the level surface at that point.

As $\frac{\partial\phi}{\partial n} \hat{n}$ gives the greatest rate of increase of ϕ with respect to space variables, therefore grad ϕ may be defined as follows :

The gradient of scalar function ϕ is a vector whose magnitude is equal to maximum rate of change of scalar function ϕ with respect to space variables and whose direction is along that change.

If u and v are scalar differentiable functions, then it may be easily seen that

$$\text{grad } (u + v) = \text{grad } u + \text{grad } v \qquad .....(9)$$

$$\text{grad } (uv) = u \text{ grad } v + v \text{ grad } u. \qquad ......(10)$$

Equation of the tangent plane to the surface ϕ (x, y, z) = c at a point P (x_1, y_1, z_1) can be derived from the gradient vector at that point.

Since the gradient vector at a point P (x_1, y_1, z_1) on the surface ϕ (x, y, z) = c represents normal to the surface at that point. If we take R (x, y, z) be any point on the tangent plane at (x_1, y_1, z_1) then the vector.

$(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$ will be perpendicular to the normal vector at (x_1, y_1, z_1).

$$\therefore (\nabla\phi)_{(x_1, y_1, z_1)} \left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] = 0$$

i.e. $$\left(\frac{\partial\phi}{\partial x}\right)_{(x_1, y_1, z_1)} (x - x_1) + \left(\frac{\partial\phi}{\partial y}\right)_{(x_1, y_1, z_1)} (y - y_1) + \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)} (z - z_1) = 0$$

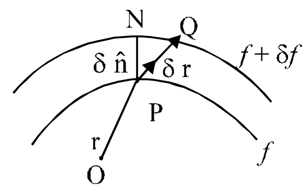
Which is the equation of the tangent place at (x_1, y_1, z_1) to ϕ (x, y, z) = 0.

Equation of normal at P (x_1, y_1, z_1) to the surface ϕ (x, y, z) = C

Let R(x, y, z) be any variable point on the normal to the surface ϕ (x, y, z) = 0. Then $\overrightarrow{PR}$ is parallel to $\nabla\phi$.

$\therefore\ \overrightarrow{PR} \times \nabla\phi = 0$, i.e. $\left[(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}\right]\times\left[\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right]=\vec{0}$

$$\therefore \begin{vmatrix} i & j & k \\ (x-x_1) & (y-y_1) & (z-z_1) \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \vec{0}$$



(fig 3.21)

$$\Rightarrow \hat{i}\left[(y-y_1)\frac{\partial f}{\partial z}-(z-z_1)\frac{\partial f}{\partial y}\right]-\hat{j}\left[(x-x_1)\frac{\partial f}{\partial z}-(z-z_1)\frac{\partial f}{\partial x}\right]+\hat{k}\left[(x-x_1)\frac{\partial f}{\partial y}-(y-y_1)\frac{\partial f}{\partial x}\right]$$

$$= 0\hat{i}+0\hat{j}+0\hat{k}$$

Comparing coefficients of $\hat{i}, \hat{j}$ & $\hat{k}$ on both sides we get

$$(y-y_1)\frac{\partial\phi}{\partial z}-(z-z_1)\frac{\partial\phi}{\partial y}=0..........(1)$$

$$\therefore (x-x_1)\frac{\partial\phi}{\partial z}-(z-z_1)\frac{\partial\phi}{\partial x}=0..........(2)$$

and $(x-x_1)\frac{\partial\phi}{\partial y}-(y-y_1)\frac{\partial\phi}{\partial x}=0..........(3)$

Again from (1) $(y-y_1)\frac{\partial\phi}{\partial z}=(z-z_1)\frac{\partial\phi}{\partial y}$

$$\Rightarrow \frac{y-y_1}{\frac{\partial\phi}{\partial y}}=\frac{z-z_1}{\frac{\partial\phi}{\partial z}}$$

Similarly from (2) and (3) we get

$$\frac{x-x_1}{\frac{\partial\phi}{\partial x}}=\frac{z-z_1}{\frac{\partial\phi}{\partial z}} \text{ and } \frac{x-x_1}{\frac{\partial\phi}{\partial x}}=\frac{y-y_1}{\frac{\partial\phi}{\partial y}}$$

Comparing all three, we get the equation of the normal at A (x_1, y_1, z_1)

$$\frac{x-x_1}{\frac{\partial\phi}{\partial x}}=\frac{y-y_1}{\frac{\partial\phi}{\partial y}}=\frac{z-z_1}{\frac{\partial\phi}{\partial z}}$$

Thus the gradient of a scalar field f is a vector normal to the surface f = (constant) and having magnitude equal to the rate of change of f along this normal.

3.16 : Properties of Gradient

Theorem – 1 : **If ϕ is a constant scalar point function then $\nabla\phi = \vec{0}$.**

Proof : Given ϕ (x, y, z) = c is a constant scalar point funtion

$$\therefore \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f \;= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 0\cdot\hat{i} + 0\cdot\hat{j} + 0\cdot\hat{k} = \vec{0}$$

Theorem – 2 : **If ϕ_1 and ϕ_2 are two scalar point functions then**

(a) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$ (b) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1 and c_2 are constants.

(c) $\nabla(\phi_1\cdot\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$ (d) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\cdot\nabla\phi_1 - \phi_1\cdot\nabla\phi_2}{\phi_2^2}, \phi_2 \neq 0$

Proof : **(a)** $\nabla(\phi_1 \pm \phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi_1 \pm \phi_2)$

$$= \hat{i}\frac{\partial}{\partial x}(\phi_1 \pm \phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1 \pm \phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1 \pm \phi_2)$$

$$= \hat{i}\left(\frac{\partial\phi_1}{\partial x} \pm \frac{\partial\phi_2}{\partial x}\right) + \hat{j}\left(\frac{\partial\phi_1}{\partial y} \pm \frac{\partial\phi_2}{\partial y}\right) + \hat{k}\left(\frac{\partial\phi_1}{\partial z} \pm \frac{\partial\phi_2}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}\right) \pm \left(\hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}\right)$$

$$= \nabla\phi_1 \pm \nabla\phi_2$$

(b) $\nabla(c_1\phi_1 + c_2\phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(c_1\phi_1 + c_2\phi_2)$

$$= \hat{i}\frac{\partial}{\partial x}(c_1\phi_1 + c_2\phi_2) + \hat{j}\frac{\partial}{\partial y}(c_1\phi_1 + c_2\phi_2) + \hat{k}\frac{\partial}{\partial z}(c_1\phi_1 + c_2\phi_2)$$

$$= c_1\hat{i}\frac{\partial\phi_1}{\partial x} + c_2\hat{i}\frac{\partial\phi_2}{\partial x} + c_1\hat{j}\frac{\partial\phi_1}{\partial y} + c_2\hat{j}\frac{\partial\phi_2}{\partial y} + c_1\hat{k}\frac{\partial\phi_1}{\partial z} + c_2\hat{k}\frac{\partial\phi_2}{\partial z}$$

$$= c_1\left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}\right) + c_2\left(\hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}\right) = c_1\nabla\phi_1 + c_2\nabla\phi_2$$

(c) $\nabla(\phi_1\phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi_1\phi_2)$

$$= \hat{i}\frac{\partial}{\partial x}(\phi_1\phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1\phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1\phi_2)$$

$$= \hat{i}\left(\phi_1 \cdot \frac{\partial \phi_2}{\partial x} + \phi_2 \cdot \frac{\partial \phi_1}{\partial x}\right) + \hat{j}\left(\phi_1 \cdot \frac{\partial \phi_2}{\partial y} + \phi_2 \cdot \frac{\partial \phi_1}{\partial y}\right) + \hat{k}\left(\phi_1 \cdot \frac{\partial \phi_2}{\partial z} + \phi_2 \cdot \frac{\partial \phi_1}{\partial z}\right)$$

$$= \phi_1\left(\hat{i}\frac{\partial \phi_2}{\partial x} + \hat{j}\frac{\partial \phi_2}{\partial y} + \hat{k}\frac{\partial \phi_2}{\partial z}\right) + \phi_2\left(\hat{i}\frac{\partial \phi_1}{\partial x} + \hat{j}\frac{\partial \phi_1}{\partial y} + \hat{k}\frac{\partial \phi_1}{\partial z}\right) = \phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1$$

(d) $$\nabla\left(\frac{\phi_1}{\phi_2}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi_1}{\phi_2}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_2}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{\phi_1}{\phi_2}\right)$$

$$= \hat{i}\left(\frac{\phi_2 \frac{\partial}{\partial x}\phi_1 - \phi_1 \frac{\partial \phi_2}{\partial x}}{\phi_2^2}\right) + \hat{j}\left(\frac{\phi_2 \frac{\partial \phi_1}{\partial y} - \phi_1 \frac{\partial \phi_2}{\partial y}}{\phi_2^2}\right) + \hat{k}\left(\frac{\phi_2 \frac{\partial \phi_1}{\partial z} - \phi_1 \frac{\partial \phi_2}{\partial z}}{\phi_2^2}\right)$$

$$= \frac{1}{\phi_2^2}\left\{\left[\hat{i}\phi_2 \frac{\partial \phi_1}{\partial x} + \hat{j}\phi_2 \frac{\partial \phi_1}{\partial y} + \hat{k}\phi_2 \frac{\partial \phi_1}{\partial z}\right] - \left[\hat{i}\phi_1 \frac{\partial \phi_2}{\partial x} + \hat{j}\phi_1 \frac{\partial \phi_2}{\partial y} + \hat{k}\phi_1 \frac{\partial \phi_2}{\partial z}\right]\right\}$$

$$= \frac{\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2}{\phi_2^2} = \frac{\phi_2\left(\vec{i}\,\frac{\partial \phi_1}{\partial x} + \vec{j}\,\frac{\partial \phi_1}{\partial y} + \vec{k}\,\frac{\partial \phi_1}{\partial z}\right) - \phi_1\left(\vec{i}\,\frac{\partial \phi_2}{\partial x} + \vec{j}\,\frac{\partial \phi_2}{\partial y} + \vec{k}\,\frac{\partial \phi_2}{\partial z}\right)}{\phi_2^2}$$

$$\therefore \quad \nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2}{\phi_2^2}, \ \phi_2 \neq 0$$

***Theorem – 3 :* Prove that grad** $\left\{(\vec{\mathbf{r}} \times \vec{\mathbf{a}}) \cdot (\vec{\mathbf{r}} \times \vec{\mathbf{b}})\right\} = \vec{\mathbf{b}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{a}}) + \vec{\mathbf{a}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{b}})$

Proof : Grad $\left\{(\vec{r} \times \vec{a}) \cdot (\vec{r} \times \vec{b})\right\} = \text{grad}\left[\vec{b} \cdot \{(\vec{r} \times \vec{a}) \times \vec{r}\}\right] = \textit{grad}\left[-\vec{b} \cdot \{\vec{r} \times (\vec{r} \times \vec{a})\}\right]$

$$= \text{grad}\left[-\vec{b} \cdot \{(\vec{r} \cdot \vec{a})\vec{r} - r^2\vec{a}\}\right] = \text{grad}\left[r^2(\vec{a} \cdot \vec{b}) - (\vec{r} \cdot \vec{a})(\vec{r} \cdot \vec{b})\right]$$

$$= \text{grad}\left[r^2(\vec{a} \cdot \vec{b})\right] - \text{grad}\left[(\vec{r} \cdot \vec{a})(\vec{r} \cdot \vec{b})\right]$$

$$= (\nabla r^2)(\vec{a} \cdot \vec{b}) + r^2\nabla(\vec{a} \cdot \vec{b}) - (\vec{r} \cdot \vec{a})\nabla(\vec{r} \cdot \vec{b}) - (\vec{r} \cdot \vec{b})\nabla(\vec{r} \cdot \vec{a})$$

$$= 2\,\vec{r}(\vec{a} \cdot \vec{b}) + 0 - (r \cdot \vec{a})\left[\vec{r} \times (\nabla \times \vec{b}) + \vec{b} \times (\nabla \times \vec{r}) + (\vec{r} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{r}\right] -$$

$$(\vec{r} \cdot \vec{b}) + \left[\vec{r} \times (\nabla \times \vec{a}) + \vec{a} \times (\nabla \times \vec{r}) + (\vec{r} \cdot \nabla)\vec{a} + (\vec{a} \cdot \nabla)\vec{r}\right]$$

$$= 2(\vec{a} \cdot \vec{b})\vec{r} - (\vec{r} \cdot \vec{a})\left[\vec{b}\right] - (\vec{r} \cdot \vec{b})(\vec{a})$$

(Since all other terms vanish)

$$= 2(\vec{a} \cdot \vec{b})\vec{r} - (\vec{r} \cdot \vec{a})\vec{b} - (\vec{r} \cdot \vec{b})\vec{a}$$

$$= (\vec{a} \cdot \vec{b})\vec{r} - (\vec{a} \cdot \vec{r})\vec{b} + (\vec{a} \cdot \vec{b})\vec{r} - (\vec{b} \cdot \vec{r})\vec{a} = \vec{b} \times (\vec{r} \times \vec{a}) + \vec{a} \times (\vec{r} \times \vec{b})$$

Theorem – 4 : **Prove that Grad $\nabla^2 f = \nabla^2$ grad f**

Proof : $\nabla^2 grad\ f = \nabla^2\left(\frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j} + \frac{df}{dz}\hat{k}\right)$

$$= \sum \frac{d^2}{dx^2}\left(\frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j} + \frac{df}{dz}\hat{k}\right) = \sum\left[\frac{\delta^2}{\delta x^2}\left(\frac{\delta f}{\delta x}\right) + \frac{\delta^2}{\delta y^2}\left(\frac{\delta f}{\delta x}\right) + \frac{\delta^2}{\delta z^2}\left(\frac{\delta f}{\delta x}\right)\right]\hat{i}$$

$$= \sum\left[\frac{\delta}{\delta x}\left(\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2}\right)\right]\hat{\mathrm{i}} = \sum\left(\frac{\delta}{\delta x}\nabla^2 f\right)\hat{\mathrm{i}} = \mathrm{grad}\nabla^2 f$$

Directional Derivative

The component of $\nabla\phi$ in the direction of a unit vector $\vec{a}$ is $\nabla\phi.\ \vec{a}$ is called the directional derivative of ϕ in the direction $\vec{a}$.

Physically, the directional derivative implies the rate of change of ϕ at (x, y, z) along $\vec{a}$ (or in the direction $\vec{a}$). Thus, the directional derivative is maximum in the direction $\nabla\phi$ and the magnitude of this maximum is equal to $|\nabla\phi|$.

If $\phi(x, y, z)$ is a scalar function and $\vec{d}$ is a given direction then $\nabla\ \phi.\ \hat{n}$ where $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$ is called as the directional derivative of ϕ along $\hat{n}$.

Theorem – 5 : The directional derivative of a scalar field ϕ at a point P (x, y, z) in the direction of a unit vector **a** is given by

$$\frac{d\phi}{ds} = \nabla\phi.\boldsymbol{a}$$

Proof : Let ϕ (x, y, z) define a scalar field in the region R, Let **r** denote the position vector of any point P (x, y, z) in this region where $\mathbf{r} = x\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k}$. If s denotes the distance of P from some fixed point A in the direction of **a**, then Δs denotes a small element at P in the direction of **a**. Therefore $\frac{d\boldsymbol{r}}{ds}$ is a unit vector at P in this direction.

i.e. $\frac{d\boldsymbol{r}}{ds} = \boldsymbol{a}$

But $\mathbf{r} = x\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k}$.

$$\therefore \quad \frac{d\boldsymbol{r}}{ds} = \frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j} + \frac{dz}{ds}\boldsymbol{k}$$

Now, directional derivative of ϕ in the direction **a**.

$$\frac{d\phi}{ds} = \frac{\partial\phi}{dx}.\frac{dx}{ds} + \frac{\partial\phi}{dy}.\frac{dy}{ds} + \frac{\partial\phi}{dz}.\frac{dz}{ds}$$

$$= \left(\boldsymbol{i}\frac{\partial\phi}{dx} + \boldsymbol{j}\frac{\partial\phi}{dy} + \boldsymbol{k}\frac{\partial\phi}{dz}\right).\left(\frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j} + \frac{dz}{ds}\boldsymbol{k}\right)$$

$$= \nabla\phi.\frac{d\boldsymbol{r}}{ds} = \nabla\phi.\boldsymbol{a}$$

Remark –1 : If θ be the angle between the direction $\nabla\phi$ and **a**, then

$$\frac{d\phi}{ds} = \nabla\phi.\boldsymbol{a} = |\nabla\phi||\boldsymbol{a}|\cos\theta = |\nabla\phi|\cos\theta \qquad [\because |a| = 1]$$

Hence $\frac{d\phi}{ds}$ is maximum when $\theta = 0$, i.e., when $\nabla\phi$ and $\boldsymbol{a}$ have the same direction and the corresponding maximum value of $\frac{d\phi}{ds}$ is $|\nabla\phi|$.

Thus, the directional derivative $\frac{d\phi}{ds}$ is maximum when it is in the direction of $\nabla\phi$ and the maximum value is $|\nabla\phi|$..

Remark – 2 : If $\phi\,(x, y, z) = c$, be a level surface through the point P, then

$$\frac{d\phi}{ds} = 0\,.$$

Hence $\nabla\phi.\frac{d\boldsymbol{r}}{ds} = 0$ ... (1)

$\frac{d\boldsymbol{r}}{ds}$ acts in the direction of PQ where P and Q are neighbouring points on the level surface.

Hence as $\Delta s \to 0$, PQ becomes the tangent at P.

$\therefore$ $\frac{d\boldsymbol{r}}{ds}$ is the unit tangent vector at P.

Then from result (1), we see that $\nabla\phi$ acts in a direction perpendicular to the direction of $\frac{d\boldsymbol{r}}{ds}$, i.e., along the normal to the level surface at P.

Thus $\nabla\phi$ at P is in the direction of the normal at P to the level surface $\phi\,(x, y, z) = c$ through P.

Gradient in Polar Co-ordinates :

If $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ be the unit vectors parallel and perpendicular to the position vector **r** i.e., in the radial and transverse sense of any point P, on the path of a particle where **r** is the position vector of the point P, then gradient of a scalar function ϕ in polar coordinates is defined by

$$\nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{\boldsymbol{r}}\frac{\partial\phi}{\partial r}e_\theta \qquad \text{... (1)}$$

If three dimensional polar (or spherical polar) coordinates are considered, then gradient of a scalar function f is given by

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{\boldsymbol{r}}\frac{\partial f}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}e_\phi. \quad \text{... (2)}$$

Example – 1 : ***If ϕ (x, y, z) = $3x^2y - y^3z^2$, find grad ϕ at the point (1, –2, –1)***

Solution : grad $\phi = \nabla\phi = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\left(3x^2y - y^3z^2\right)$

$$= \boldsymbol{i}\frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right) + \boldsymbol{j}\frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right) + \boldsymbol{k}\frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)$$

$= \boldsymbol{i}\,(6xy) + \boldsymbol{j}\,(3x^2 - 3y^2z^2) + \boldsymbol{k}\,(-2y^3z)$

$= 6xy\,\boldsymbol{i} + (3x^2 - 3y^2z^2)\,\boldsymbol{j} - 2y^3z\,\boldsymbol{k}$

Putting $x = 1$, $y = -2$, $z = -1$, we get

grad ϕ at (1, –2, –1) = $-12\hat{i} - 9\hat{j} - 16\hat{k}$

Example – 2 : ***If $\nabla\phi = (y + y^2 + z^2)\, i + (x + z + 2xy)\, j + (y + 2zx)\, k$, find ϕ such that ϕ (1, 1, 1)***

Solution : We have $\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$ ... (1)

Given $\nabla\phi = (y + y^2 + z^2)\,\boldsymbol{i} + (x + z + 2xy)\,\boldsymbol{j} + (y + 2xz)\,\boldsymbol{k}$ ... (2)

From (1) and (2) $\frac{\partial\phi}{\partial x} = y + y^2 + z^2$ ... (3)

$$\frac{\partial\phi}{\partial y} = x + z + 2xy \quad \text{... (4)}$$

$$\frac{\partial\phi}{\partial z} = y + 2zx \quad \text{... (5)}$$

Integrating (3) with respect of x, we get

$\phi = xy + xy^2 + xz^2 + f(y, z)$ ... (6)

where the arbitrary constant of integration, $f(y, z)$, is free from x.

Differentiating (6) partially with respect to y and using (4), we get

$$\frac{\partial\phi}{\partial y} = x + 2xy + \frac{\partial}{\partial y}f(y,z) = x + z + 2xy$$

$\therefore$ $\frac{\partial}{\partial y}f(y,z) = z.$ ... (7)

Now integrating (7) with respect to y, we get

$f(y, z) = yz + g(z)$

So (6) is $\phi = xy + xy^2 + xz^2 + yz + g(z)$ ... (8)

Differentiating (8) partially with respect to z and using (5), we get

$$\frac{\partial\phi}{\partial z} = 2zx + y + \frac{\partial}{\partial z}g(z) = y + 2zx$$

$\therefore \qquad \dfrac{\partial}{\partial z} g(z) = 0$ or $g(z) = C$.

So from (8), $\qquad \phi = xy + xy^2 + xz^2 + yz + C$.

Here ϕ (1, 1, 1) = 4 + C. But ϕ (1, 1, 1) is given to be 3. So 4 + C = 3 or C = –1

Hence $\qquad \phi = xy + xy^2 + xz^2 + yz - 1$.

Illustrative Examples

Example – 1 : Find the directional derivative of $\phi = 2xy + z^2$ at the point (1, –1, 3) in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$.

Solution : Here $\phi(x, y, z) = 2xy + z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2y\vec{i} + 2x\vec{j} + 2z\vec{k}$$

$\therefore \quad \nabla\phi$ at (1, –1, 3) = $-2\vec{i} + 2\vec{j} + 6\vec{k} = 2(-\hat{i} + \hat{j} + 3\hat{k})$

The unit vector in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$ is

$$\hat{a} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{1+4+4}} = \frac{1}{3}\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)$$

Then the directional derivative = $\nabla\phi . \hat{a}$

$$= \frac{2}{3}(-1 + 2 + 6) = \frac{14}{3}$$

(Since this is positive, ϕ is increasing in this direction.)

Example – 2 : Find the angle between the tangent planes to the surfaces $x \ln z = y^2 - 1$, $x^2 y = 2 - z$ at the point (1, 1, 1).

Solution : Let $\phi_1 = x\ln z = y^2 - 1$, $\phi_2 = x^2 y + z = 2$

$$\nabla\phi_1 = \ln z\,\vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}$$

$$N_1 = -2\vec{j} + \vec{k} \text{ at}(1, 1, 1)$$

Tangent plane to the surface ϕ_1 = constant is

$$\left[\left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \left(\vec{i} + \vec{j} + \vec{k}\right)\right]\cdot\left(-2\vec{j} + \vec{k}\right) = 0$$

$$\Rightarrow -2(y - 1) + (z - 1) = 0 \qquad \text{........(i)}$$

$$\nabla\phi_2 = 2xy\vec{i} + x^2\vec{j} + \vec{k}$$

$$N_2 = 2\vec{i} + \vec{j} + \vec{k} \text{ at } (1, 1, 1)$$

Tangent plane to the surface ϕ_2 = constant

$$\left[\left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \left(\vec{i} + \vec{j} + \vec{k}\right)\right]\cdot\left(2\vec{i} + \vec{j} + \vec{k}\right) = 0$$

$$\Rightarrow 2(x - 1) + (y - 1) + (z - 1) = 0 \qquad \text{.......(ii)}$$

Let θ be the angle between the tangent planes (i) and (ii), then

$$\cos\theta = \frac{0\cdot 2+(-2)\cdot 1+1\cdot 1}{\sqrt{0+4+1}\sqrt{4+1+1}} = -\frac{1}{\sqrt{30}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

Example – 3 : Find a unit vector perpendicular to the surface $x^2 + y^2 - z^2 = 11$ at the point (4, 2, 3).

Solution : The given surface is $x^2 + y^2 - z^2 = 11$, i.e., the level surface is characterised by $\phi = x^2 + y^2 - z^2$ = constant. We know that the gradient of scalar function ϕ is perpendicular to the level surface. Therefore, grad ϕ will be perpendicular to $\phi = x^2 + y^2 - z^2 = 11$ = constant.

$$\therefore \quad grad\ \phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(x^2+y^2-z^2\right)$$

$$\hat{i}\cdot 2x+\hat{j}\cdot 2y-\hat{k}\cdot 2z = 2\left(x\hat{i}+y\hat{j}-z\hat{k}\right)$$

The coordinates of the point are (x, y, z) = (4, 2, 3).

$\therefore$ (grad ϕ) at point (4, 2, 3) = $2\left(4\hat{i}+2\hat{j}-3\hat{k}\right)$

$\therefore$ Required unit vector perpendicular to given surface is

$$\hat{n} = \pm\frac{2\left(4\hat{i}+2\hat{j}-3\hat{k}\right)}{\left|2\left(4\hat{i}+2\hat{j}-3\hat{k}\right)\right|} = \frac{4\hat{i}+2\hat{j}-3\hat{k}}{\sqrt{4^2+2^2+(-3)^2}} = \frac{4\hat{i}+2\hat{j}-3\hat{k}}{\sqrt{29}}$$

Example – 4 : Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point (2, –1, 2).

Solution : The level surfaces are characterised by

$\phi_1 = x^2 + y^2 + z^2 = 9$ (constant)

$\phi_2 = x^2 + y^2 - z = 3$ (constant).

We know that the gradient of scalar function is perpendicular to the level surface. Therefore, grad ϕ_1 and grad ϕ_2 will be perpendicular to given surfaces.

$$\therefore \quad \text{grad } \phi_1 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(x^2+y^2+z^2\right) = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

grad ϕ_1 at point (2, – 1, 2) $= 4\hat{i}-2\hat{j}+4\hat{k}$

$$\text{grad } \phi_2 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(x^2+y^2-z\right) = 2x\hat{i}+2y\hat{j}-\hat{k}$$

grad ϕ_2 at point (2, –1, 2) $= 4\hat{i}-2\hat{j}-\hat{k}$.

The angle between surfaces at any point is same as angle between the normal to the surfaces at that point. If θ is the angle between surface, then this will be the angle between grad ϕ_1 and grad ϕ_2.

$\therefore$ $(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2| \cos\theta$

i.e., $$\cos\theta = \frac{(\nabla\phi_1)\cdot(\nabla\phi_2)}{|\nabla\phi_1|\ |\nabla\phi_2|}$$

$$= \frac{\left(4\hat{i}-2\hat{j}+4\hat{k}\right)\cdot\left(4\hat{i}-2\hat{j}-\hat{k}\right)}{\left|4\hat{i}-2\hat{j}+4\hat{k}\right|\;\left|4\hat{i}-2\hat{j}-\hat{k}\right|} = \frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}} = \frac{16}{\sqrt{36}\sqrt{21}} = \frac{8\sqrt{21}}{63}$$

$$\therefore \quad \theta = \cos^{-1}\left\{\frac{8}{3\sqrt{21}}\right\}$$

Example – 5 : If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (–1, 1, 2) has a maximum magnitude of 32 units in the direction parallel to y-axis find a, b, c.

Solution : Maximum directional derivative is along $\nabla\phi$ and in the direction parallel to y-axis the magnitude is given to be 32 units.

$\therefore\ \nabla\phi\cdot j = 32$ at (–1, 1, 2)

We have $\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

i.e., $\nabla\phi = (ay^2 + 3x^2cz^2)i + (2axy + bz)j + (by + 2cx^3z)k$

$[\nabla\phi]_{(-1,1,2)} = (a + 12c)i + (-2a + 2b)j + (b - 4c)k$

Now $\nabla\phi\cdot j = -2a + 2b = 32$ by using (1) or $-a + b = 16$

Also since $\nabla\phi$ is parallel to the y-axis we must have $a + 12c = 0$ and $b - 4c = 0$

Thus by solving the three equations :

$-a + b = 16$, $a + 12c = 0$, $b - 4c = 0$ we obtain

$a = -12$, $b = 4$, $c = 1$

Example – 6 :(a) Find the unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).

(b) Find a unit of vector normal to the surface of $x^3y^3z^2 = 4$ at the point (–1, –1, 2)

Solution : ***(a)*** Let $\phi = x^3 + y^3 + 3xyz - 3$

A vector normal to the surface is

(grad ϕ) = $\left(3x^2 + 3yz\right)\hat{i} + \left(3y^2 + 3xz\right)\hat{j} + 3xy\hat{k}$

At (1, 2, –1)

(grad ϕ)$_{(1,2,-1)}$ = $\left(3\cdot1^2 + 3\cdot2(-1)\right)\hat{i} + \left(3\cdot2^2 + 3\cdot1\cdot(-1)\right)\hat{j} + (3\cdot1\cdot2)\hat{k} = -3\hat{i} + 9\hat{j} + 6\hat{k}$

Also $\left|(grad\ \phi)_{(1,2,-1)}\right| = \sqrt{9+81+36} = \sqrt{126} = 3\sqrt{14}$

Hence a unit vector normal to the surface $= \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}\left(-\hat{i}+3\hat{j}+2\hat{k}\right)$

(b) Proceeding as above as (a).

Example – 7 : Find grad r^m, where 'r' is the distance of any point from origin.

Solution : Here $r = \sqrt{x^2 + y^2 + z^2}$, where (x, y, z) is any point in space.

$\therefore r^m = \left(x^2 + y^2 + z^2\right)^{\frac{m}{2}}$

Hence grad $r^m = \frac{\partial r^m}{\partial x}\hat{i} + \frac{\partial r^m}{\partial y}\hat{j} + \frac{\partial r^m}{\partial z}\hat{k}$

$$= \frac{m}{2}\left(x^2+y^2+z^2\right)^{\frac{m}{2}-1}\left[2x\hat{i}+2y\hat{j}+2z\hat{k}\right] = m\, r^{2\left(\frac{m}{2}-1\right)}\left[x\hat{i}+y\hat{j}+z\hat{k}\right] = m \cdot r^{m-2} \cdot r$$

Example – 8 : Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t$, $y = t^2$ and $z = t^3$ at (1, 1, 1) and at (–1, 1, –1).

Solution : Here parametric representation of the curve is

$$r(\vec{t}) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\therefore \vec{r} = \hat{i} + 2t\hat{j} + 3t^2\hat{k} = \frac{d\vec{r}}{dt}$$

If $\hat{a}$ is the unit tangent to the curve, then $\hat{a} = \dfrac{\hat{i}+2t\hat{j}+3t^2\hat{k}}{\sqrt{1+4t^2+9t^4}}$

At (1, 1, 1) for the given curve t = 1

$$\therefore (\hat{a})_{(1,1,1)} = (\hat{a})_{t=1} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$

At (–1, 1, –1) for the given curve t = –1

$$\therefore (\hat{a})_{(-1,1,-1)} = (\hat{a})_{t=-1} = \frac{\hat{i}-2\hat{j}+3\hat{k}}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}\left(\hat{i}-2\hat{j}+3\hat{k}\right)$$

Now $\phi = xy^2 + yz^2 + zx^2$

$$\therefore (grad\ \phi) = \left(y^2+2xz\right)\hat{i} + \left(z^2+2xy\right)\hat{j} + \left(x^2+2yz\right)\hat{k}$$

$$\therefore (grad\ \phi)_{(1,1,1)} = 3\left(\hat{i}+\hat{j}+\hat{k}\right)$$

and $(\text{grad } \phi)_{(-1,\,1,\,-1)} = 3\hat{i} - \hat{j} - \hat{k}$

$\therefore$ Directional derivative at (1, 1, 1) along the tangent

$$= 3\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\frac{1}{\sqrt{14}}\left(\hat{i}+2\hat{j}+3\hat{k}\right) = \frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}$$

Example – 9 : Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, –1, 2). Also write the equation of the normal at (1, –1, 2).

Solution : Let $\phi(x, y, z) = 2xz^2 - 3xy - 4x - 7$

$$\frac{\partial\phi}{\partial x} = 2z^2 - 3y - 4,\quad \frac{\partial\phi}{\partial y} = -3x,\quad \frac{\partial\phi}{\partial z} = 4xz$$

At (1, –1, –2)

$$\frac{\partial\phi}{\partial x} = 8+3-4 = 7,\quad \frac{\partial\phi}{\partial y} = -3,\quad \frac{\partial\phi}{\partial z} = 8$$

$\therefore$ Equation of the tangent plane at (1, – 1, 2) is $(x-1)(7) + (y+1)(-3) + (z-2)\,8 = 0$

or $7x - 3y + 8z - 26 = 0$

Equation of the normal at (1, – 1, 2) is

$$\frac{x-1}{\frac{\partial\phi}{\partial x}} = \frac{y+1}{\frac{\partial\phi}{\partial y}} = \frac{z-2}{\frac{\partial\phi}{\partial z}}$$

i.e., $\dfrac{x-1}{7} = \dfrac{y+1}{-3} = \dfrac{z-2}{8}$

Example – 10 : If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ ***, show that***

if $\vec{r}$ ***is the position vector of a variable point (x, y, z) and*** $|\vec{r}| = r$

(i) ***grad*** $r = \dfrac{\vec{r}}{r}$ ***(ii) grad*** $\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$

(iii) $\nabla r^n = n\, r^{n-2}\vec{r}$ ***(iv)*** $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ ***where*** $\vec{a}$ ***is a constant vector.***

Solution : **(i)** $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ *or* $r^2 = x^2+y^2+z^2$

Differentiate partially w.r.t. *x, y* and *z* respectively, we get $2r\dfrac{\partial r}{\partial x} = 2x,\ \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$.

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r},\ \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Now, grad $r = \nabla\ r = \left(\vec{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(r)$

$$= \vec{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \vec{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{\vec{r}}{r}$$

Hence grad $r = \dfrac{\vec{r}}{r}$

(ii) grad $\dfrac{1}{r} = \nabla\left(\dfrac{1}{r}\right) = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right)\left(\dfrac{1}{r}\right)$

$$= \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= -\frac{1}{r^2}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right) = -\frac{1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = -\frac{\vec{r}}{r^3}$$

(iii) $\nabla(r^n) = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)r^n = \hat{i}\,nr^{n-1}\dfrac{\partial r}{\partial x} + \hat{j}nr^{n-1}\dfrac{\partial r}{\partial y} + \hat{k}nr^{n-1}\dfrac{\partial r}{\partial z}$

$$= nr^{n-1}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] = nr^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = nr^{n-2}\vec{r}$$

(iv) $\nabla(\vec{a}\cdot\vec{r})$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$\vec{a}\cdot\vec{r} = xa_1 + ya_2 + za_3$. *Since* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$$\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xa_1 + ya_2 + za_3)$$

$$= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

Hence $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

Example – 11 : In what direction from (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum.

Solution : Directional derivative of $\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^2z^4)$

$$= \hat{i}(2xy^2z^4) + \hat{j}(2x^2yz^4) + \hat{k}(4x^2y^2z^3)$$

Directional derivative at (3, 1, –2) $= 96\hat{i} + 288\hat{j} - 288\hat{k}$

Directional derivative is maximum in the direction given by $\nabla\phi$ at (3, 1, 2) $= 96\hat{i} + 288\hat{j} - 288\hat{k}$

In any other direction the magnitude of the directional derivative will be less than its maximum value which is $= \sqrt{(96)^2 + (288)^2 + (288)^2} = 96\sqrt{1+9+9} = 96\sqrt{19}$.

DIVERGENCE OF A VECTOR FIELD

3.17 *:* Introduction

We have developed a tool for determining the rate of change of a scalar field in space and time in the form of directional derivative and gradient. A natural question arises – How fast a vector field varies in a given region ? Another question, which may be asked is – can we extend the analysis of gradient to a vector field ? The answer to the second question is that it is not possible to extend the analysis of gradient to a vector field.

Rate of change of components in directions other than their own, called the **curl.** Let us first study the concept of divergence of a vector field.

DIVERGENCE : Derivatives involving rate of change of a vector component in its own direction, called the **divergence.**

We have seen that starting with a scalar point function *f*, we can construct a vector point function grad *f*. But, what happens if we start with a vector point function. If the constructed function is a scalar point function, then we call it the divergence of a vector point function. More precisely, we give the following definition :

Definition : If $\vec{V}$ (x, y, z) be any given continuously differentiable vector point function, then the funtion

$$\hat{i}\cdot\frac{\partial\vec{v}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{v}}{\partial y}+\hat{k}\frac{\partial\vec{v}}{\partial z}.$$

is called the divergence of $\vec{v}$ or divergence of the vector field defined by $\vec{v}$ and is denoted by div $\vec{v}$.

In terms of operator ∇, we write $div\,\vec{v}=\hat{i}\cdot\frac{\partial f}{\partial x}+\hat{j}\cdot\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}=\left(\hat{i}\cdot\frac{\partial}{\partial x}+\hat{j}\cdot\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{f}$

$=\nabla\cdot\vec{f}$

The symbol "∇." is pronounced as "del".

Let us consider a Cartesians coordinate system Oxyz. Let $\vec{v}$ have the scalar field components v_1, v_2, v_3 along the directions of x, y, z axes respectively, so that

$$\vec{v}=iv_1+jv_2+kv_3$$

$$\therefore div\,\vec{v}=\nabla\cdot\vec{v}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}v_1+\hat{j}v_2+\hat{k}v_3\right)=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

PHYSICAL INTERPRETATION OF DIVERGENCE : Let us consider the case of a fluid flow. Consider a small rectangular parallelopiped of dimension dx, dy, dz parallel to X-axis, Y-axis, Z-axis respectively.

Let $\vec{V}=V_x\hat{i}+V_y\hat{j}+V_z\hat{k}$ be the velocity of fluid at P (x, y, z) where (V_x, V_y, V_z) are components of $\vec{V}$ parallel to X-axis, Y-axis, Z-axis respectively. ***(fig 3.22)***

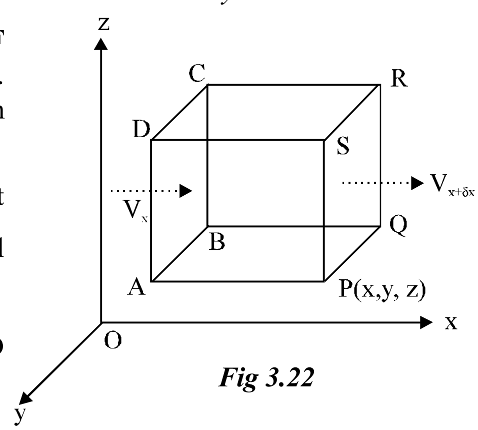


Fig 3.22

Mass of the fluid flowing in through the face ABCD per unit time = Velocity × area of the face = V_x(dy dz)

$\therefore$ Mass of the fluid flowing out across the face PQRS per unit time = $V_{x+\delta x}$(dy dz).

$$V_{x+\delta x}=V_x+\delta x\frac{\partial V_x}{\partial x}+\ldots \text{ by Taylor's theorem.}$$

$$\therefore V_{x+\delta x}(dydz)=\left(V_x+\frac{\partial V_x}{\partial x}dx\right)(dydz)$$

Decrease in mass of fluid in the parallelopiped corresponding to the flow along x-axis per unit time $=V_x dy\,dz-\left(V_x+\frac{\partial V_x}{\partial x}dx\right)dy\,dz=-\frac{\partial V_x}{\partial x}dx\,dy\,dz$ (–ve sign shows decrease)

Similarly the decrease in mass of fluid to the flow along y-axis $= \dfrac{\partial V_y}{\partial y}\, dx\, dy\, dz$ and decrease in mass along z-axis $= \dfrac{\partial V_z}{\partial z}\, dx\, dy\, dz$.

Total decrease in mass of fluid per unit time $= \left(\dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}\right) dx\, dy\, dz$

The rate of loss of fluid per unit volume $= \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i} V_x + \hat{j} V_y + \hat{k} V_z\right) = \nabla \cdot \vec{V} = div\, \vec{V} \quad \text{.............(1)}$$

Equation (1) is also called the equation of continuity or conservation of mass.

Cor. If div $\vec{\nabla} = 0$ everywhere in some region R of space, then $\vec{V}$ is called **Solenoidal Vector Point Function.**

For example let $\vec{V} = \dfrac{\vec{r}}{r^3}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

then div $\vec{V} = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right) \cdot \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\right)$$

$$= \sum \frac{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} \cdot 1 - x\frac{3}{2}\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}} \cdot 2x}{\left(x^2 + y^2 + z^2\right)^3}$$

$$= \sum \frac{\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}\left[x^2 + y^2 + z^2 - 3x^2\right]}{\left(x^2 + y^2 + z^2\right)^3} = \sum \frac{y^2 + z^2 - 2x^2}{\left(x^2 + y^2 + z^2\right)^{\frac{5}{2}}}$$

$$= \frac{y^2 + z^2 - 2x^2 + z^2 + x^2 - 2y^2 + x^2 + y^2 - 2z^2}{\left(x^2 + y^2 + z^2\right)^{\frac{5}{2}}} = 0$$

$\therefore \quad \vec{V} = \dfrac{\vec{r}}{r^3}$ is solenoidal vector.

Definition : A vector field $\vec{V}$ is called divergence free or solenoidal in a given region if for all points in that region. $\nabla \cdot \vec{V} = 0$

Thus magnetic field or velocity of a steady flow of compressible fluid are examples of solenoidal vector fields.

We now apply the concepts discussed in this section to some examples.

Laplacian : If $\phi(x, y, z)$ is a continuously differentiable scalar function $\vec{v}(x, y, z)$ is a continuously differentiable vector function we can define the Laplacian for ϕ as well as $\vec{v}$ for as follows.

$$\text{Laplacian } \phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{Laplacian } \vec{v} = \nabla^2 \vec{v} = \frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2}$$

The operator that results taking the dot product of the del operator with itself is denoted by ∇^2 and is called the laplacian operator. The operator has the form

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Note : If ϕ is a scalar function the equation $\nabla^2\phi = 0$ is called Laplace's equation and a function which satisfies Laplace's equation is called a harmonic function.

And $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ is called Laplace's equation in two dimensions.

Obviously Laplacian of a scalar function is a scalar quantity and Laplacian of a vector function is a vector quantity.

Remark : 1. If $f(x, y, z)$ is a scalar function then we have

$$grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\therefore \quad div\,(grad\ \phi) = \nabla \cdot \nabla\phi$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial y}k\right) \cdot \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial y}k\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$$

Thus div (grad ϕ) = $\nabla^2\phi$ or $\nabla \cdot \nabla\phi = \nabla^2\phi$

2. The computation of gradient, divergence, curl and laplcian is combination of the concepts of vector algebra and partial differentiation.