4.7 : Geometrical Applications of Double Integrals

We now consider the use of double integrals for computing areas of plane and curved surfaces, and volumes.

Plane area

Let us recall the expression $\int_A f(x,y)dA = \iint_R f(x,y)dxdy = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)dydx$.

For $f(x,y)=1$ this expression becomes.

$$\int_A dA = \iint_R dxdy = \int_a^b \int_{y_1(x)}^{y_2(x)} dy\, dx = \int_c^d \int_{x_1(y)}^{x_2(y)} dx\, dy \qquad ...(1)$$

The integral $\int_A dA$ represents the total area A of the plane region R over which the repeated integral area taken. Thus (1) may be used to compute the area A. We note that *dxdy* is the plane area element *dA* in the Cartesian form.

We find the following formula for area in polar coordinates, bounded by the curves $r=f_1(\theta)$, $r = f_2(\theta)$, and the line $\theta = \alpha$, $\theta = \beta$ is given by

$$\iint_R dx\, dy = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta$$

We observe that $r\, dr\, d\theta$ is the plane area elements in polar form.

Illustrative Examples

Example – 1 : Find the area bounded between the parabolas $y^2=4ax$ and $x^2=4ay$.

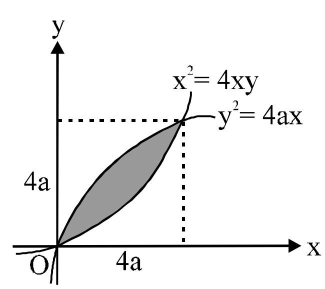
Solution : Solving the given equations we find that the two parabolas intersect at the points (0,0) and (4*a*, 4*a*). Therefore, in the region between these parabolas, *x* varies from 0 to 4*a* and, for each *x*,*y* varies from a point on the parabola $x^2=4ay$ to a point on the parabola $y^2 = 4ax$; that from $y=x^2/4a$ to $y = 2\sqrt{ax}$. *(fig. 4.6)*

Hence the required area is

$$A = \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$



(Fig. 4.6)

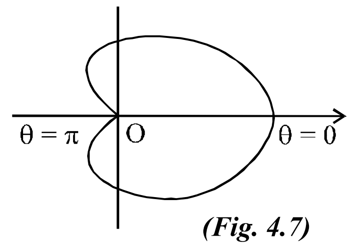
Example – 2 : Find the areas enclosed by the cardioid : $r = a\,(1+\cos\theta)$ between $\theta = 0$ and $\theta = \pi$.

Solution : Here, θ varies from 0 to π *(fig.4.7)* and, for each θ, r varies from 0 to $a(1+\cos\theta)$. Therefore, the required area is

$$A = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi} \left\{ \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta \right\} = \frac{a^2}{2} \int_0^{\pi} (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} 1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\, d\theta \;\; = \frac{3a^2}{4}$$



(Fig. 4.7)

Example – 3 : Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution : The two curves intersect at points whose abscissae are given by $4x - x^2 = x$ or $x^2 - 3x = 0$ i.e. $x = 0, 3$

Using vertical strips, the required area lies between $x=0$, $x=3$ and $y=x$, $y=4x-x^2$.*(fig. 4.8)*

$$\therefore \text{ Required area } = \int_0^3 \int_x^{4x-x^2} dy\,dx$$

$$= \int_0^3 [y]_x^{4x-x^2}\, dx$$

$$= \int_0^3 \left(4x - x^2 - x\right) dx = \int_0^3 \left(3x - x^2\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - 9 = 4.5$$

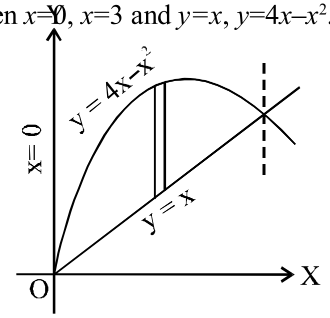


Fig. 4.8

4.8 : *Change of Variables*

Sometimes an integral (double or triple) with its present form may not be simple to evaluate. By choice of an appropriate co-ordinate system the given integral can be transformed into a simple integral involving the new variables.

Change in Triple Integral

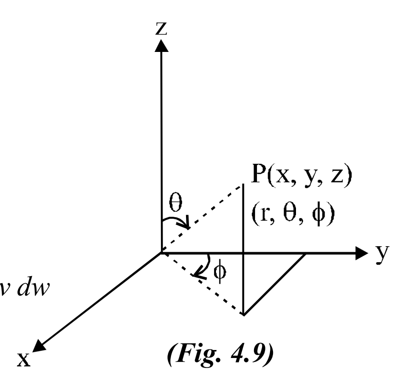
Let $(x,y,z) \to (u, v, w)$ and the relations are as follows :

$x = \phi_1\,(u, v, w)$, $y = \phi_2\,(u, v, w)$, $z = \phi_3\,(u, v, w)$

The Jacobian for this transformation $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Then $\displaystyle\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(\phi_1, \phi_2, \phi_3)\,J\,du\,dv\,dw$

The region V in (x,y,z) is covered by the limits of u,v and w and is denoted as V′. *(fig.4.9)*



(Fig. 4.9)

(a) Change of variable from Cartesian to spherical polar co-ordinate system.
$(x,y,z) \to (r,\theta,\phi)$ and the relations are as follows :
$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$
So that $x^2 + y^2 + z^2 = r^2$

$$\text{and } J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & \theta \end{vmatrix} = r^2 \sin\theta$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

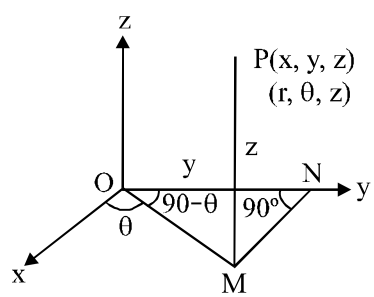
The region V in (x,y,z) is to be covered by the limits of r, θ, ϕ and is denoted as V′.

(b) Change of Variables from Cartesian to Cylindrical Co-ordinate System *(fig.4.10)*
$(x,y,z) \to (r,\theta,z)$ and the relations are as follows :
$x = r\cos\theta,\ y = r\sin\theta,\ z = z$ (remains, same)

$$\text{and } J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r$$



(Fig. 4.10)

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

The region V in (x,y,z) is to be covered by the limits of r, θ, z and is denoted as V′.

Illustrative Examples

Example – 1 : Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ ***by changing to spherical Polar Co-ordinates.***

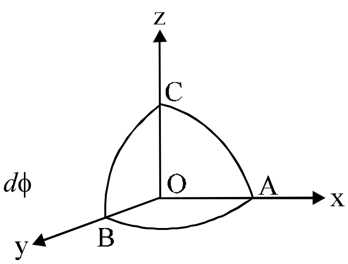
Solution : The region of integration is bounded by *(fig. 4.10)*

$z = 0,\ z = \sqrt{1-x^2-y^2}$, i.e. $x^2 + y^2 + z^2 = 1,\ y = 0,\ y = \sqrt{1-x^2}$
i.e. $x^2 + y^2 = 1,\ x = 0,\ x = 1$
Which is the volume of the sphere
$x^2 + y^2 + z^2 = 1$ in the positive Octant.
The same region in spherical polar co-ordinates will be as follows :

$r : 0 \to 1,\ \theta : 0 \to \pi/2,\ \phi : 0 \to \pi/2$ and $x^2 + y^2 + z^2 = r^2$

$$\therefore I = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \frac{r^2 \sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}} \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left[\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right] dr \sin\theta\, d\theta\, d\phi$$



(Fig. 4.10)

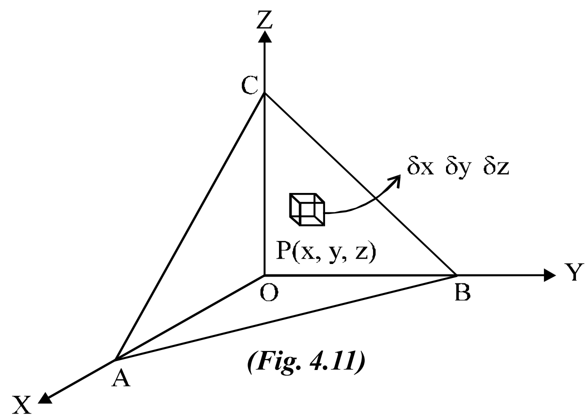
$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[\sin^{-1} r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1} r\right)\right]_0^1 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\pi}{2} - \frac{1}{2}.\frac{\pi}{2}\right] \sin\theta\, d\theta\, d\phi = \frac{\pi}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta\, d\theta\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} (-\cos\theta)_0^{\pi/2} d\phi = \frac{\pi}{4} \int_0^{\pi/2} d\phi = \frac{\pi}{4}.\frac{\pi}{2} = \frac{\pi^2}{8}$$

4.9 : Volumes of Solids

Volume as the triple integral : Divide the given solid by planes parallel to the co-ordinate planes into rectangular parallelopipeds of volume $\delta x\ \delta y\ \delta z$ *(fig. 4.11)*



(Fig. 4.11)

$\therefore$ the total volume $\iiint dx\, dy\, dz$ with appropriate limits of integration.

$\delta v = \delta x\ \delta y\ \delta z$, Note. (i) Mass = volume × density

$$= \iiint \rho\, dx\, dy\, dz$$

where ρ is density.

Illustrative Examples

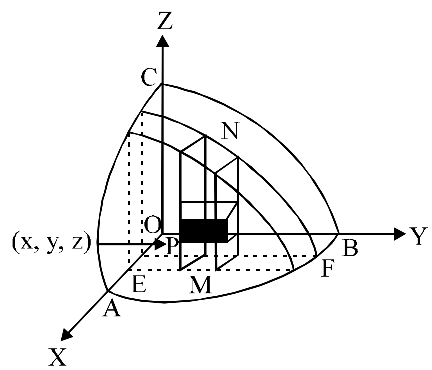
Example – 1 : Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution : Let OABC be the positive octant of the given ellipsoid which is bounded by the planes

OAB(z=0), OBC (x=0), OCA(y=0) and the surface ABC i.e. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Divide this region R into rectangular parallelopiped of volume $\partial x\ \partial y\ \partial z$. Consider such an element at P(x,y,z).

$\therefore$ the required volume $=8\iiint_R dx\ dy\ dz$



In this region R,

(i) z varies from 0 to MN where

$MN=c\sqrt{\left(1-x^2/a^2-y^2/b^2\right)}$

(ii) y varies from 0 to EF, where

$EF=b\sqrt{\left(1-x^2/a^2\right)}$ from the equation of the ellipse OAB, i.e. $x^2/a^2+y^2/b^2=1$

(iii) x varies from 0 to OA = a.

Hence the volume of the whole ellipsoid. *(fig.4.12)* ***(Fig. 4.12)***

$$=8\int_0^a\int_0^{b\sqrt{\left(1-x^2/a^2\right)}}\int_0^{c\sqrt{\left(1-x^2/a^2-y^2/b^2\right)}}dx\ dy\ dz$$

$$=8\int_0^a dx\int_0^{b\sqrt{\left(1-x^2/a^2\right)}}dy[z]_0^{c\sqrt{\left(1-x^2/a^2-y^2/b^2\right)}}$$

$$=8\int_0^a dx\int_0^{b\sqrt{\left(1-x^2/a^2\right)}}\sqrt{\left(1-x^2/a^2-y^2/b^2\right)}dy$$

$$=\frac{8c}{b}\int_0^a dx\int_0^{\rho}\sqrt{\left(\rho^2-y^2\right)}dy \text{ when } \rho=b\sqrt{\left(1-x^2/a^2\right)}$$

$$=\frac{8c}{b}\int_0^a dx\left[\frac{y\sqrt{\left(\rho^2-y^2\right)}}{2}+\frac{\rho^2}{2}sin^{-1}\frac{y}{\rho}\right]_0^{\rho}=\frac{8c}{b}\int_0^a\frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}dx$$

$$=2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a=\frac{4\pi abc}{3}$$

Example – 2 : Find the volume of the solid enclosed by the surface $x^2+y^2=cz$, $x^2+y^2=2ax$, $z=0$

Solution : Volume $=\iiint dx\ dy\ dz$ integration to be carried over the given region above the XY plane enclosed by the circular cylinder and the paraboloid of revolution.

Limits for z vary from 0 to $\frac{x^2+y^2}{c}$; for y, the limits are $-\sqrt{\left(2ax-x^2\right)}$ to $\sqrt{\left(2ax-x^2\right)}$ and for x are 0 to $2a$.

$$\therefore V=\int_0^{2a}\left\{\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}\left[\int_0^{\frac{x^2+y^2}{c}}dz\right]dy\right\}dx$$

$$= \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} \frac{x^2+y^2}{c} dx\, dy \; = 2\int_0^{2a}\left[\frac{x^2 y}{c}+\frac{y^3}{3c}\right]_0^{\sqrt{2ax-x^2}} dx$$

$$= \frac{2}{3c}\int_0^{2a}\left[3a^2\sqrt{2ax-x^2}+\left(2ax-x^2\right)^{3/2}\right]dx$$

$$= \frac{2}{3a}\int_0^{2a}\left[3x^{5/2}(2a-x)^{1/2}+x^{3/2}(2a-x)^{3/2}\right]dx$$

$$= \frac{2}{3c}\int_0^{\frac{\pi}{2}}\left[3(2a)^3 sin^5 t\left(1-sin^2 t\right)^{1/2}+(2a)^3 sin^3 t\left(1-sin^2 t\right)^{3/2}\right]4a\, sin\, t\, cos\, t\, dt$$

where x=2a $sin^2 t$.

$$= \frac{64a^4}{3c}\int_0^{\frac{\pi}{2}}\left(3 sin^6 t\, cos^2 t + sin^4 t\, cos^4 t\right)dt$$

$$= \frac{64a^4}{3c}\left[\frac{3.5.3.1.1}{8.6.4.2}\frac{\pi}{2}+\frac{3.1.3.1}{8.6.4.2}\frac{\pi}{2}\right]=\frac{3\pi a^4}{2c}$$

Example – 3 : Find the volume of the solid enclosed between the two surface $z = x^2 + 3y^2$ and $z = 8–x^2 –y^2$.

Solution : Volume $= \int\int\int dx\, dy\, dz$ integration is to be carried over the region where z varies from (x^2+3y^2) to (8–x^2–y^2). Eliminating z from the two equations, the orthogonal projection of the volume on the XY plane is the ellipse x^2+2y^2=4, which determines the limits for x and y.

$$V = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\, dx\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}\left[\left(8-x^2-y^2\right)-\left(x^2+3y^2\right)\right]dx\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}\left(8-2x^2-4y^2\right)dx\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}}\left[\left(8-4y^2\right)x-\frac{2x^3}{3}\right]_0^{\sqrt{4-2y^2}} dy$$

$$= 4\int_0^{\sqrt{2}}\frac{4}{3}\left(4-2y^2\right)^{3/2} dy, \text{ put } y=\sqrt{2}\, sin\, t,\; dy=\sqrt{2}\, cos\, t\, dt$$

$$= \frac{128\sqrt{2}}{3}\int_0^{\pi/2} cos^4 t\, dt = \frac{128\sqrt{2}}{3}\left(\frac{3.1.\pi}{4.2.2}\right)=8\pi\sqrt{2}$$

4.10 : *Computation of Volume by Triple Integrals*

Recall expression $V = \iiint_V dx\, dy\, dz$. In the particular case where $f(x,y,z) \equiv 1$, this expression becomes.

$$\int_V dV = \iiint_R dx\, dy\, dz = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} dz\, dy\, dx \qquad ...(1)$$

The integral $\int_V dV$ represents the volume V of the region R.

Thus, expression (1) may be used to compute V.
If (x,y,z) are changed to (u, v, w), we obtain in the following expression for the volume.

$$\int_V dV = \iiint_R dx\, dy\, dz = \iiint_R J\, du\, dv\, dw \qquad ...(2)$$

Taking $(u, v, w) = (R, \phi, z)$ in the above expression, we obtain the following expression for volume in terms of cylindrical polar co-ordinates.

$$\int_V dV = \iiint_R R\, dR\, d\phi\, dz \qquad ...(3)$$

Similarly, we obtain the following expression for volume in terms of spherical polar co-ordinates.

$$\int_V dV = \iiint_R r^2 \sin\theta\, dr\, d\theta\, d\phi \qquad ...(4)$$

Observe that $R\, dR\, d\phi\, dz$ is the volume element in cylinderical polar co-ordinates, and $r^2 \sin\theta\, dr\, d\theta\, d\phi$ is the volume elements in spherical polar co-ordinates.

Illustrative Examples

Example – 1 : Find the volume bounded by the surface $z=a^2-x^2$ and the planes x = 0, y = 0, z = 0 and y = b

Solution : Here, z varies from z=0 to $z=a^2-x^2$. For $z = 0$, x varies from 0 to a, and y varies from 0 to b. Therefore the required volumes is

$$V = \iiint_R dV = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{a^2-x^2} dz\, dy\, dx$$

$$= \int_0^a \int_0^b (a^2 - x^2) dy\, dx = \int_0^a b(a^2 - x^2) dx = b\left(a^3 - \frac{a^3}{3}\right) = \frac{2}{3} ba^3$$

Example – 2 : Find the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$.

Solution : Here, it is convenient to employ cylindrical polar co-ordinates (R, ϕ, z). In terms of these co-ordinates, the equations of the given sphere and the given cylinder read $R^2 + z^2 = a^2$ and $R = a \cos\phi$ respectively.

From *(fig.4.13)* we observe that in the given region, z varies from $-\sqrt{a^2-R^2}$ to $+\sqrt{a^2-R^2}$, R varies from 0 to a $cos\ \phi$, and ϕ varies from 0 to π. Therefore, the required volume is

$$V = \int_{\phi=0}^{\pi}\int_{R=0}^{a\cos\phi}\int_{z=-\sqrt{a^2-R^2}}^{\sqrt{a^2-R^2}} R\,dz\,dR\,d\phi$$

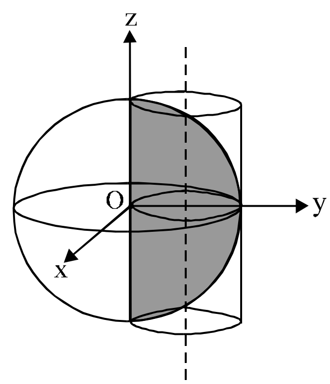
$$= 2\int_{\phi=0}^{\pi}\int_{R=0}^{a\cos\phi} R\sqrt{a^2-R^2}\,dR\,d\phi$$

$$= 2\int_{\phi=0}^{\pi}\left\{\left[-\frac{1}{3}\left(a^2-R^2\right)^{3/2}\right]_0^{a\cos\phi}\right\}d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi}\left(1-sin^3\phi\right)\phi d\phi = \frac{2a^3}{3}\left\{\pi - \int_0^{\pi} sin^3\phi\, d\phi\right\}$$

$$= \frac{2a^3}{3}\left[\pi - \left\{\int_0^{\pi/2} sin^3\phi d\phi + \int_0^{\pi/2} sin^3(\pi-\phi)d\phi\right\}\right]$$

$$= \frac{2a^3}{3}\left\{\pi - 2\int_0^{\pi/2} sin^3\phi\, d\phi\right\} = \frac{2a^3}{3}\left\{\pi - 2.\frac{2}{3}\right\} = \frac{2a^3}{9}(3\pi-4)$$



(Fig. 4.13)

Example – 3 : Evaluate $\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\,dy\,dx}{\sqrt{(x^2+y^2+z^2)}}$

Solution : We change to spherical polar co-ordinates (r,θ,ϕ), so that

$x = r\ sin\theta\ cos\ \phi$, $y = r\ sin\ \theta\ sin\phi$, $z = r\ cos\ \theta$

and $J = r^2\ sin\theta$, $x^2 + y^2 + z^2 = r^2$.

The region of integration is common to the cone $z^2=x^2+y^2$ and the cylinder $x^2+y^2=1$ bounded by the plane z=1 in the positive octant *(Fig.4.14)*

Hence θ varies from 0 to $\pi/4$, r varies from 0 to $sec\theta$ and ϕ varies from 0 to $\pi/2$.

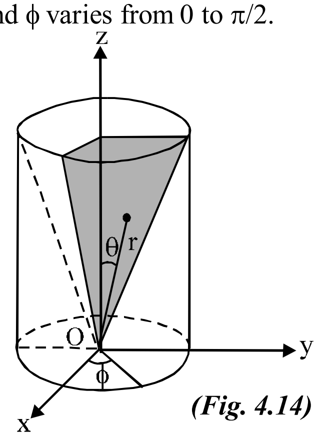
$\therefore$ Given integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{sec\theta}\frac{1}{r}.r^2\ sin\theta\ dr\ d\theta\ d\phi$$

$$= \int_0^{\pi/2}d\phi\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{sec\theta} sin\theta d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/4}\frac{sec^2\theta}{2}sin\theta\ d\theta = \frac{\pi}{4}\int_0^{\pi/4} sec\theta\ tan\theta\ d\theta$$

$$= \frac{\pi}{4}[sec\theta]_0^{\pi/4} = \frac{\left(\sqrt{2}-1\right)\pi}{4}$$



(Fig. 4.14)

Example – 4 : Find by double integration, the volume common to the cylinders $x^2 + y^2 = 2ax$ and $z^2=2ax$.

Solution : Here the cylinder are given by

$x^2+y^2=2ax$ .....(i)

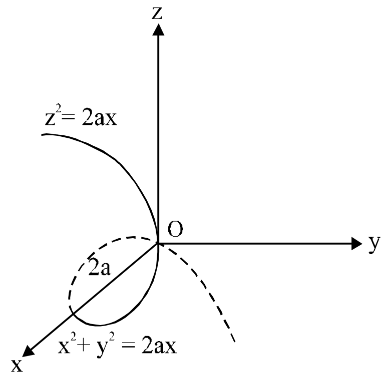
$z^2 = 2ax$ ....(ii)

$\Rightarrow z = \pm\sqrt{2ax}$ .....(iii)

$\therefore$ Each of the surface is symmetrical about the *xy*- plane *(fig. 4.15)*

$\therefore V = 2\iint_R f(x,y)dx\,dy$ ..... (iv)

where (iii) $z=\sqrt{2ax}$



(Fig. 4.15)

and R is the region bounded by the circle given in (i) on the xy-plane.

i.e. $0 \le x \le 2a$ and $-\sqrt{2ax-x^2} \le y \le \sqrt{2ax-x^2}$

$$\therefore V = 2\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} \sqrt{2ax}\ dx\,dy = 2\int_0^{2a}\sqrt{2ax}\,[y]_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}\,dx$$

$$= 4\sqrt{2a}\int_0^{2a}\sqrt{x}.\sqrt{2ax-x^2}\,dx$$, Let $x=2a\,sin^2\,\theta$, $dx = 4a\,sin\theta\,cos\,\theta$

$$= 4\left(\sqrt{2a}\right)^2 8a^2\int_0^{\pi/2} sin^3\theta\,cos^2\theta\,d\theta = 64a^3\,\frac{2.1}{5.3.1} = \frac{128}{15}a^3.$$

Example – 5 : Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

Solution : The problem can be solved in cylinderical co-ordinate system.

Here, $z = 4 - y = 4 - r\,cos\theta$.

r : $0 \to 2$, $\theta : 0 \to 2\pi$

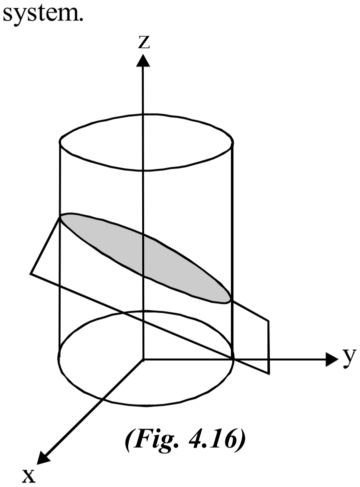
z : $0 \to (4 - r\,cos\theta)$ *(fig. 4.16)*

$$\text{Required Volume} = \int_0^{2\pi}\int_0^{2}\int_0^{4-r\cos\theta} r dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r.[z]_0^{4-r\cos\theta}\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} r(4 - r\cos\theta)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\left(4r - r^2\cos\theta\right)dr\,d\theta = \int_0^{2\pi}\left(8 - \frac{8}{3}\cos\theta\right)d\theta$$

$$= \left[8\theta - \frac{8}{3}\sin\theta\right]_0^{2\pi} = 16\pi$$

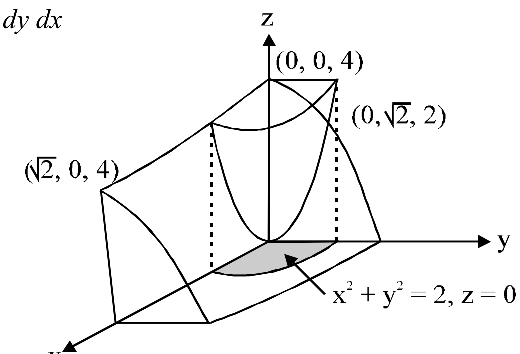


(Fig. 4.16)

Example – 6 : Find the volume bounded by the paraboloid $z=2x^2+y^2$ and the parabolic cylinder $z = 4-y^2$.

Solution : Consider the volume in the positive Octant. The projection of the intersection region is obtained by equating the equations of paraboloid and the parabolic cylinder which is a circle $x^2+y^2=2$, $z = 0$. *(fig. 4.17)*

$$\therefore \text{ Required volume } = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} dz\,dy\,dx$$



(Fig. 4.17)

$$= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} [z]_{2x^2+y^2}^{4-y^2}\,dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} \left(4-2x^2-2y^2\right) dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\left[\left(4-2x^2\right)^2 y - \frac{2y^3}{3}\right]_0^{\sqrt{2-x^2}} dx$$

$$= 4\int_0^{\sqrt{2}}\left[2\left(2-x^2\right)\sqrt{2-x^2} - \frac{2}{3}\left(2-x^2\right)^{3/2}\right]dx = \frac{16}{3}\int_0^{\sqrt{2}}\left(2-x^2\right)^{3/2} dx$$

$$= \frac{16}{3}\int_0^{\sqrt{2}} 2^{3/2}.\cos^3\theta\, 2^{1/2}\cos\theta\, d\theta \quad (\text{Let } x = \sqrt{2}\sin\theta, dx = \sqrt{2}\cos\theta d\theta)$$

$$= \frac{16}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta = \frac{64}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = 4\pi$$

4.11: Transformation of Double Integrals Into Line Integrals – Green's Theorem

Introduction : We may transform double integrals over a plane region, under suitable conditions, into line integrals over the boundary of a region and conversely. This transformation is of practical interest because it makes the evaluation of an integral easier. It also helps in the theory whenever we want to switch from one type of integral to other.

This transformation can be done by means of a theorem known as **Green's Theorem** which is due to English mathematician, George Green (1793-1841). We shall now state this theorem.

Green's Theorem in the plane : (Tangential form)

If S is a plane surface in the xy-plane bounded by a simple closed curve consists of finitely many smooth curves. Let M and N are continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in the regions R.

Then $\oint_C (Mdx+Ndy) = \iint_S\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \oint_C F\cdot\vec{n}ds$

the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the directions of integration.

Proof : Consider the region S bounded by a single closed curve C which is cut by any line parallel to the axis at the most in two points.

Let the equation of the curve C_1 i.e.AEB be $y = \phi_1(x)$ and the equation of the curve C_2 i.e. BFA be $y = \phi_2(x)$ i.e. C is divided in to two curves C_1 & C_2

If the region S is bounded by C, we have *(fig 4.18)*

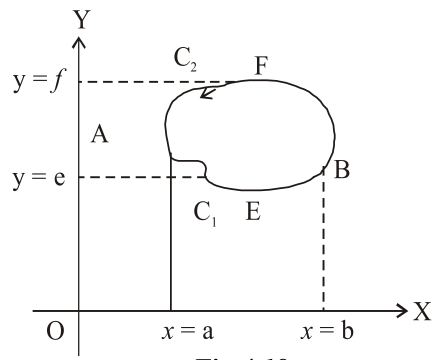


Fig 4.18

$$\iint_S \frac{\partial M}{\partial y}\, dx\, dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial M}{\partial y}\, dy\, dx = \int_a^b [M(x, y)]_{\phi_1(x)}^{\phi_2(x)} dx$$

$$= \int_a^b M(x,\phi_2(x))\, dx - \int_a^b M(x,\phi_1(x))\, dx$$

$$= -\int_b^a M(x, \phi_2(x))\, dx - \int_a^b M(x, \phi_1(x))\, dx$$

$$= -\int_{C_2} M(x\ \ y)\, dx - \int_{C_1} M(x\ \ y)\, dx = -\left[\int_{C_2} M(x\ \ y)\, dx + \int_{C_1} M(x\ \ y)\, dx\right] = -\oint_C M(x\ \ y)\, dx$$

$$\Rightarrow \oint_C M(x, y)\, dx = -\iint_S \frac{\partial M}{\partial y}\, dx\, dy \qquad \text{..... (i)}$$

Similarly, the equation of the curve EAF be $x = \psi_1(y)$ and the equation of the curve EBF be $x = \psi_2(y)$. Then

$$\iint_S \frac{\partial N}{\partial x}\, dx\, dy = \int_e^f \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial N}{\partial x}\, dx\, dy$$

$$= \int_e^f [\mathrm{N}(\psi_2(y), y - \mathrm{N}(\psi_1(y), y]\, dy = \int_e^f \mathrm{N}(\psi_2(y), y)dy + \int_f^e \mathrm{N}(\psi_1(y), y)dy = \oint_C N(x,y)\, dy$$

$$\text{i.e., } \oint_C N(x,y)\, dy = \iint_S \frac{\partial N}{\partial x}\, dx\, dy \qquad \text{..... (ii)}$$

Adding (i) and (ii) we get

$$\oint_C (M\, dx + N\, dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Note : 1. The above theorem is also true for the curves C for which the line parallel to the coordinate axes cut C in more than two points.

2. The above theorem is also true for the multiply connected region S. Proofs are not considered in this text.

Green's Theorem (Normal Form)

If *'S'* is a plane surface in xy plane bounded by a simple closed curve *'C'*, M and N are continuous function of x and y having continuous derivative in the regions then.

$$\oint_C (Mdy - Ndx) = \iint_S \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \oint_C \vec{F}\,\vec{n}\,ds$$

Divergence Integral

The outward flux of a field $F = M\hat{i} + N\hat{j}$ across a simple closed curve C equals the double integral of $div\ \vec{F}$ over the region R enclosed by C.

Illustrative Examples

Example – 1 : Find by Green's theorem $\oint_C (x^2 ydx + ydy)$ ***along the closed curve*** C ***formed by*** $y^2 = x$ ***and*** $y = x$ ***between (0, 0) and (1, 1).***

Solution : We can use either form of Green's Theorem to change the line integral into a double integral.

(1) **(With Tangential form)**

Taking $M = x^2y$ and $N = y$.

By Green's Theorem, we have

$$\oint_C (Mdx + Ndy) = \iint_S \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

$$\oint_C (x^2 ydx + ydy) = \iint_S \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x^2 y)\right] dx\,dy$$

Where S is the region bounded by the arc of parabola $y^2 = x$ and the line $y = x$ between their point of intersection (0, 0) and (1, 1).

$$\therefore \oint_C (x^2 ydx + ydy) = \int_0^1 \int_y^{y^2} (-x^2) dxdy$$

$$= -\int_0^1 \left[\frac{x^3}{3}\right]_y^{y^2} dy = -\frac{1}{3}\int_0^1 (y^6 - y^3) dy$$

$$= \frac{1}{3}\left[\frac{y^4}{4} - \frac{y^7}{7}\right]_0^1 = \frac{1}{28}$$

(Normal form)

$$\oint_C (Mdy - Ndx) = \int_S \int_S \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

Here $M = y, N = -x^2y$.

By Green's Theorem, we have

$$\oint_C ydy + \left(-x^2 y\right)dx = \iint_S \left(\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}\left(-x^2 y\right)\right)$$

$$\int_0^{-1}\int_y^{y^2}\left(-x^2\right)dx\,dy = -\int_0^1\left[\frac{x^3}{3}\right]_0^{y^2} dy = -\frac{1}{3}\int_0^1\left(y^6 - y^3\right)dy = \frac{1}{3}\left[\frac{y^4}{4} - \frac{y^7}{7}\right]_0^1 = \frac{1}{28}.$$

Here both Tangential and normal forms are same.

Example – 2 : Evaluate $\oint_C\left[\left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy\right]$ ***around the boundary C of the region*** $y^2 = 8x, x = 2.$

Solution : The boundary C of the region $y^2 = 8x$, $x = 2$ is the curve OABO and area S enclosed is shown as shaded *(fig 4.19)*

Using Green's theorem

$$\oint_C\left[\left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy\right]$$

$$= \iint_S\left[\frac{\partial}{\partial x}\left(x^2 y + 3\right) - \frac{\partial}{\partial y}\left(x^2 - 2xy\right)\right]dx\,dy$$

$$= \int_{x=0}^{2}\left[\int_{y=-2\sqrt{2x}}^{+2\sqrt{2x}}(2xy + 2x)dy\,dx\right]$$

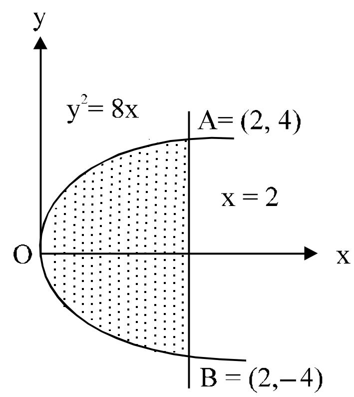


Fig 4.19

$$= \int_{x=0}^{2}\left[xy^2 + 2xy\right]_{-2\sqrt{2x}}^{2\sqrt{2x}}$$

$$= \left[-\int_{x=0}^{2} x.\left(2\sqrt{2x}\right)^2 + 2x\cdot 2\sqrt{2x} - x\left(-2\sqrt{2x}\right)^2 - 2x\left(-2\sqrt{2x}\right)\right]dx$$

$$= \int_{x=0}^{2} 4x.2\sqrt{2x}\;dx = 8\sqrt{2}\int_0^2 x^{3/2}dx$$

$$= 8\sqrt{2}\left[\frac{x^{5/2}}{5/2}\right]_0^2 = \frac{16\sqrt{2}}{5}\cdot 4\sqrt{2} = \frac{128}{5}$$

Example – 3 : Verify Green's theorem in the plane for $\oint_C\left[(3x^2 - 8y^2)dx + (4y - 6xy)dy\right]$ ***where C is the boundary of the region defined by x = 0, y = 0, x + y = 1.***

Solution : Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$$

$$\text{R.H.S.} = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_0^{1-x} (-6y + 16y)\,dxdy$$

$$= 10 \int_0^1 \int_0^{1-x} y\,dy\,dx = 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = 5 \int_0^1 (1-x)^2\,dx = 5\left[\frac{-(1-x)^3}{3}\right]_0^1 = \frac{5}{3}$$

Now C consists of the paths OA, OB and BO.

On OA, $y = 0$, $x : 0 \to 1$, $dy = 0$

$$\therefore \quad I_1 = \int_0^1 3x^2 dx = 3\left[\frac{x^3}{3}\right]_0^1 = 1$$

On AB, $x + y = 1$, $\Rightarrow x = 1 - y$, $dx = -dy$ and $y : 0 \to 1$ *(fig 4.20)*

$$I_2 = \int_0^1 \{3(1-y)^2 - 8y^2\}(-dy) + \{4y - 6(1-y)y\}\,dy = \int_0^1 (11y^2 + 4y - 3)dy$$

$$= \left[11\frac{y^3}{3} + 4\,.\,\frac{y^2}{2} - 3y\right]_0^1 = \frac{8}{3}$$

on BO, $x = 0$, $dx = 0$, $y : 1 \to 0$

$$I_3 = \int_1^0 4y\,dy = 4\,.\left[\frac{y^2}{2}\right]_1^0 = -2$$

$\therefore$ L.H.S $= I_1 + I_2 + I_3 = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

$\therefore$ L.H.S. = R.H.S.

Hence Green's theorem in plane is verified.

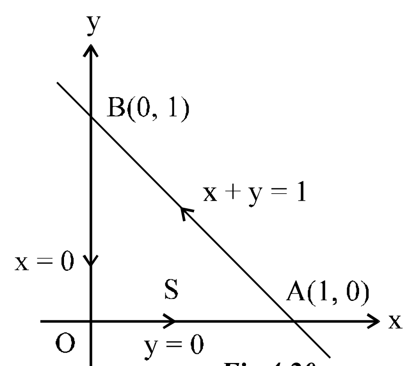


Fig 4.20

Example – 4 : Verify Greens theorem for $\oint_C \left(xy + y^2\right)dx + x^2dy$***, where C is the boundary of the closed region bounded by y = x and y = x².***

Solution : Evaluation by Green's theorem,

$$\oint_C \left(xy + y^2\right)dx + x^2dy \text{ with } \oint_C Mdx + Ndy$$

Here M = $xy + y^2$, N = x^2

$$\frac{\partial M}{\partial y} = x + 2y,\ \frac{\partial N}{\partial x} = 2x \quad \textit{(fig 4.21)}$$

By Green's theorem

$$\oint_C Mdx + Ndy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

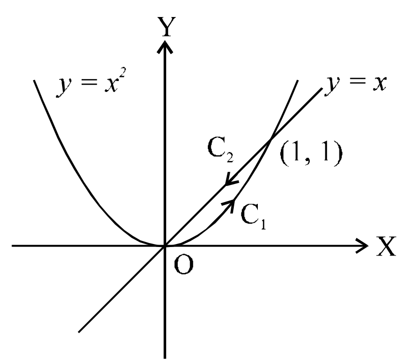


Fig 4.21

$$\therefore \quad \oint_C \left(xy + y^2\right)dx + x^2 dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\, dy\, dx = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y) dy\, dx$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx = \int_0^1 (-x^3 + x^4) dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{-1}{4} + \frac{1}{5} = \frac{-1}{20} \qquad \text{....(1)}$$

(Evaluation by line integral)

$$\oint_C \left(xy + y^2\right)dx + x^2 dy = \oint_{C_1} \left(xy + y^2\right)dx + x^2 dy + \oint_{C_2} \left(xy + y^2\right)dx + x^2 dy$$

Along C_1 ; $y = x^2 \quad \therefore\ dy = 2x\, dx$ and $0 \le x \le 1$
Along C_2 ; $y = x \quad \therefore\ dy = dx$; $1 \le x \le 0$

$$= \int_0^1 \left(x^3 + x^4\right) dx + x^2 .\ 2xdx + \int_1^0 \left(x^2 + x^2\right) dx + x^2 dx$$

$$= \int_0^1 \left(3x^3 + x^4\right) dx + \int_1^0 3x^2 dx = \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[x^3\right]_1^0$$

$$= \frac{3}{4} + \frac{1}{5} - 1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20} \text{ which is same as (1)}$$

Hence Green's theorem is verified.

Example – 5 : Verify Green's theorem for $\oint_C e^{-x}(\sin y\, dx + \cos y\, dy)$ ***where C is the boundary of a rectangle whose vertices are (0, 0), (π, 0),*** $\left(\pi, \frac{\pi}{2}\right)$ ***and*** $\left(0, \frac{\pi}{2}\right)$.

Solution : Boundary of the region i.e., that of a rectangle is given by $y = 0$, $x = \pi$, $y = \frac{\pi}{2}$, $x = 0$.

By Green's theorem,

$$\oint_C M\, dx + N\, dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Compare $\oint M\, dx + N\, dy$ with

$\oint_C e^{-x} \sin y\, dx + e^{-x} \cos y\, dy$, we have

$$M = e^{-x} \sin y;\ N = e^{-x} \cos y,\ \frac{\partial M}{\partial y} = e^{-x} \cos y\ ;\ \frac{\partial N}{\partial x} = -e^{-x} \cos y$$

$$\therefore \quad \oint_C e^{-x}(\sin y\, dx + \cos y\, dy) = \int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^{\pi} \left(-e^{-x} \cos y - e^{-x} \cos y\right) dx\, dy \quad \textit{(fig 4.22)}$$

$$= -2 \int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^{\pi} e^{-x} \cos y\, dx\, dy = -2 \int_0^{\frac{\pi}{2}} \cos y\, dy \left[\int_0^{\pi} e^{-x} dx\right]$$

$$= -2 \int_{y=0}^{\frac{\pi}{2}} \left(\cos y\, dy \left[-e^{-x}\right]_0^{\pi}\right) = -2 \int_0^{\frac{\pi}{2}} \left(e^{-\pi} + 1\right) \cos y\, dy$$

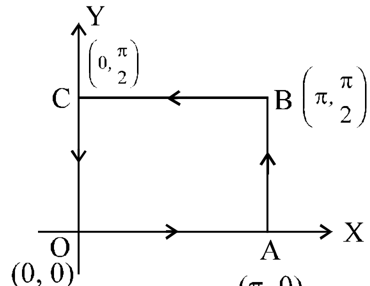


Fig. 4.22

$$= 2 \int_0^{\frac{\pi}{2}} \left(e^{-\pi} - 1\right) \cos y\, dy = 2\left(e^{-\pi} - 1\right)\left[\sin y\right]_0^{\frac{\pi}{2}} = 2\left(e^{-\pi} - 1\right) \quad .... (1)$$

[Now find $\oint_C M\, dx + N\, dy = \int_{OABC} e^{-x}(\sin y\, dx + \cos y\, dy)$ by line integral]

$$\therefore \quad \int_{OABC} e^{-x}(\sin y\, dx + \cos y\, dy)$$

$$= \int_{OA} e^{-x}(\sin y\, dx + \cos y\, dy) + \int_{AB} e^{-x}(\sin y\, dx + \cos y\, dy)$$

$$+ \int_{BC} e^{-x}(\sin y\, dx + \cos y\, dy) + \int_{CO} e^{-x}(\sin y\, dx + \cos y\, dy) \quad .... (2)$$

Along OA ; $y = 0$ $\therefore$ $dy = 0$ and x varies from 0 to π

$$\therefore \quad \int_{OA} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_0^{\pi} e^{-x}(0 + 0) = 0 \quad .... (3)$$

Along AB ; $x = \pi$, $dx = 0$ and y varies from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \int_{AB} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_0^{\frac{\pi}{2}} e^{-\pi} \cos y\, dy = e^{-\pi}\left[\sin y\right]_0^{\frac{\pi}{2}} = e^{-\pi} \quad .... (4)$$

Along BC ; $y = \frac{\pi}{2}$ $\therefore$ $dy = 0$ and x varies from π to 0

$$\therefore \quad \int_{BC} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_{\pi}^{0} e^{-x}(1.dx) = \left[-e^{-x}\right]_{\pi}^{0} = e^{-\pi} - 1 \quad .... (5)$$

Along CO ; $x = 0$ $\therefore$ $dx = 0$ and y varies from $\frac{\pi}{2}$ to 0

$$\therefore \int_{CO} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_{\frac{\pi}{2}}^{0} (0 + \cos y)\, dy = [\sin y]_{\frac{\pi}{2}}^{0} = -1 \qquad .... (6)$$

Substituting (3), (4), (5) and (6) in (2), we get

$$\oint_C e^{-x}(sin\ y\ dx + cos\ y\ dy) = 0 + e^{-\pi} + e^{-\pi} - 1 - 1 = 2(e^{-\pi} - 1)$$

which is same as (1)

$\therefore$ Green's theorem is verified

Example – 6 :Verify Green's theorem for $\oint_C (x^2 - cos\, h\ y)dx + (y + sin\ x)\ dy$ ***where C is the boundary of a rectangle whose vertices as O (0, 0), A (π, 0), B(π, 1), C(0, 1).***

Solution : Here boundary of the region is that of a rectangle given by y = 0, x = π, y = 1, x = 0

Substituting (3), (4), (5) and (6) in (2), we get

By Green's theorem

$$\oint_C M\ dx + N\ dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy \quad \textit{(fig 4.23)}$$

Compare $\oint_C M\ dx + N\ dy$ to $\oint_C (x^2 - cosh\ y)\ dx + (y + sin\ x)\ dy$

M = x^2 – cosh y; N = y + sin x

$\therefore$ $\frac{\partial M}{\partial y}$ = –sinh y ; $\frac{\partial N}{\partial x}$ = cos x

$$\therefore \oint_C (x^2 - \cosh y)\ dx + (y + \sin x)\ dy \because$$

$$= \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{\pi} (cos\ x + sinh\ y)\ dx\ dy$$

$$= \int_0^1 [-\sin x + x \sin h\ y]_0^{\pi} dy = \int_0^1 (0 + \pi \sin h\ y)\ dy$$

$$= [\pi \cos h\ y]_0^1 = 0 - \pi = -\pi \quad ..... (1) \quad [\because \cos h1 = 0]$$

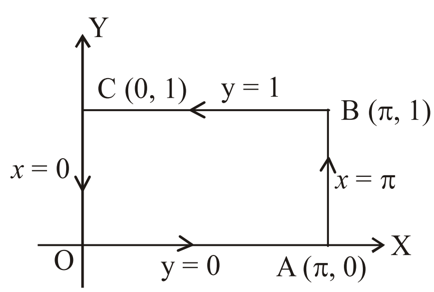


Fig 4.23

Now, find $\oint_C (x^2 - cos h\ y)dx + (y + sin\ x)\ dy$ by line integral

$$\therefore \oint_C (x^2 - cos h\ y)dx + (y + sin\ x)\ dy = \oint_{OABC} (x^2 - cos h\ y)dx + (y + sin\ x)\ dy$$

$$= \oint_{OA} \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy + \oint_{AB} \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy$$

$$+ \oint_{BC} \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy + \oint_{CO} \left(x^2\ \cos h\ y\right)dx + \left(y + \sin x\right) dy \quad .... (2)$$

Along OA; $y = 0$ $\because$ $dy = 0$ and $0 \le x \le \pi$

$$\therefore \quad \oint_{OA} \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy = \int_0^{\pi} \left(x^2 - 1\right)dx = \left[\frac{x^3}{3} - x\right]_0^{\pi} = \frac{\pi^3}{3} - \pi \quad .... (3)$$

Along AB; $x = \pi$ $\therefore$ $dx = 0;\ 0 \le y \le 1$

$$\therefore \quad \oint_{AB} \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy = \int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} \quad .... (4)$$

Along BC; $y = 1$ $\therefore$ $dy = 0$; $\pi \le x \le 0$

$$\therefore \quad \oint_{BC} \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy = \int_{\pi}^{0} x^2 dx = \left[\frac{x^3}{3}\right]_{\pi}^{0} = -\frac{\pi^3}{3} \quad .... (5)$$

Along CO; $x = 0$ $\therefore$ $dx = 0;\ 1 \le y \le 0$

$$\therefore \quad \oint_{CO} \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy = \int_1^0 y\ dy = \left[\frac{y^2}{2}\right]_0^1 = -\frac{1}{2} \quad .... (6)$$

Substituting (3), (4), (5) and (6) in (2), we get

$$\oint_C \left(x^2 - \cos h\ y\right)dx + \left(y + \sin x\right) dy = \frac{\pi^3}{3} - \pi + \frac{1}{2} - \frac{\pi^3}{3} - \frac{1}{2} = -\pi$$

Which is same as (1).

$\therefore$ Green's theorem is verified.

Exercise – 4

1. Show that $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is conservative.

2. If $\vec{F} = \left(2x + 6y^2\right)\hat{i} - 10z\hat{j} - x^2 z\hat{k}$, evaluate $\int_C \vec{F}.d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the path $x =$ t, y = t^2, z = t^3.

3. Compute $\oint_C \vec{F}.d\vec{r}$, where $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ and C is the circle $x^2 + y^2 = 1$ transversed counter clockwise.

4. Show that $\int_C x^2 y^2 ds = \frac{\pi a^5}{4}$ around the circle $x^2 + y^2 = a^2$.

5. Find the volume of the region enclosed by the surfaces $z = x^2 + 3y^2$, $z = 8 - x^2 - y^2$.

6. A triangular prism is formed by the planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of this prism between the plane $z = 0$ and the surface $z = c + xy$.

7. Verify both forms of Green's Theorem for the field $F(x, y) = (x - y)\, i + xj$ and the region bounded by the unit circle $r(\vec{t}) = (\cos t)\hat{i} + (\sin t)\hat{j}$,$0 \le t \le 2\pi$

8. Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2}\int x\,dy - y\,dx$. Hence find the area of the ellipse whose semi major and minor axes are of lengths a and b.

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