CHAPTER – 7

Fourier Transforms

7.0 : Introduction

We have already discussed with the solution of higher order ordinary differential equations with initial condition, (initial value problems) using Laplace transforms. In this chapter we introduce another well known integral transform called Fourier transform which is very useful in obtaining solutions of partial differential equation.

7.1 : *Finite Fourier Transforms and Inverse Finite Fourier Transforms*

We have already discussed the concept of *Half Range Fourier Series (Cosine and sine)* in the previous chapter. With this background we develop *finite Fourier cosines and sine transforms.*

Let $f(x)$ be a function defined in a finite interval $o < x < l$, i.e., when the range of one of the variable say 'x' is finite. Suppose $f(x)$ is neither periodic nor even nor odd. Now by redefining $f(x)$ as an even function in $-l < x < l$, the cosine half range Fourier series of $f(x)$ over the interval, $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad ..(i)$$

where, $a_0 = \dfrac{2}{l}\displaystyle\int_0^l f(x)\, dx$ ...(ii)

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \qquad ...(iii)$$

We can as well say that (iii) is valid for $n = 0, 1, 2, 3, ...$

Then the finite fourier cosine transform of $f(x)$ in $o < x < l$ is defined as

Denoting, $F_c(s) = \displaystyle\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$ ...(iv)

we have $a_n = \dfrac{2}{l} F_c(s)$ and in particular

$$a_0 = \frac{2}{l} F_c(o) \text{ or } \frac{a_0}{2} = \frac{1}{l} F_c(o)$$

Hence (i) become, $f(x) = \dfrac{1}{l} F_c(o) + \dfrac{2}{l} \displaystyle\sum_{n=1}^{\infty} F_c(s) \cos\left(\frac{n\pi x}{l}\right)$ ...(v)

For a given function $f(x)$ in $(0, l)$ (iv) is called finite Fourier cosine transform of $f(x)$ also denoted by $F.\ F_c[f(x)]$

Further the function $f(x)$ given by (v) is called the inverse finite Fourier cosine transform of $F_c(x)$.

Now by redefining *f(x)* as an odd function in $-l < x < l$,
The half range Fourier sine series of $f(x)$ over internal (0, l) is

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right) dx \qquad \text{...(vi)}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx \qquad \text{...(vii)}$$

Then the finite Fourier sine transform of $f(x)$ in $0 < x < 1$ is defined as which is a function of n, an integer.

$$F_s(x) = \int_0^l f(x) \sin\frac{n\pi x}{l} dx \qquad \text{...(viii)}$$

$$b_n = \frac{2}{l} F_s(s) \qquad \text{...(ix)}$$

The inverse finite Fourier sine transform of $f_s(x)$ is given by

Hence (vi) becomes $f(x) = \dfrac{2}{l}\displaystyle\sum_{n=1}^{\infty} F_s(s) \sin\frac{n\pi x}{l}$ ...(x)

For given function $f(x)$ in (0, l) (viii) is called finite Fourier sine transform of *f(x)* also denoted by $F.\ F_s\ [f(x)]$.

Illustrative Examples

Example–1. Find the finite Fourier cosine transform f(x) = e^{ax} in $0 < x < \pi$.

Sol^n: $F_c(s) = \displaystyle\int_0^\pi f(x)\cos s\ x\ dx = \int_0^\pi e^{ax}\cos sx\ dx$

$$= \left[\frac{e^{ax}}{a^2+s^2}(a\cos sx + s\sin sx)\right]_0^\pi, \text{ by a standard formula}$$

$$= \frac{a}{a^2+s^2}[e^{ax}\cos sx]_0^\pi = \frac{a}{a^2+s^2}(e^{a\pi}\cos s\pi - 1)$$

Thus $F_c(s) = \dfrac{a}{a^2+s^2}\left\{e^{a\pi}(-1)^s - 1\right\},\ s = 0,1,2,3,...$

Example– 2. Find finite Fourier cosine transform of the given function f(x) = $\begin{cases} -1, & 0 < x < 1/2 \\ 1, & 1/2 < x < 1 \end{cases}$

Sol^n: $F_c(s) = \displaystyle\int_0^l f(x) \cos\left(\frac{s\pi x}{l}\right) dx$

Here the interval of x is (0,1) and hence $l = 1$

$$F_c(s) = \int_0^1 f(x)\cos(s\pi x)\ dx$$

$$= \int_0^{1/2} -1.\cos(s\pi x)dx + \int_{1/2}^1 1.\cos(s\pi x)\ dx$$

$$= \left[\frac{-\sin\,(s\pi x)}{s\pi}\right]_0^{1/2} + \left[\frac{\sin(s\pi x)}{s\pi}\right]_{1/2}^{1}$$

$$= \frac{1}{s\pi}\{-\sin\,(s\pi/2) - \sin(s\pi/2)\}$$

$$F_c(s) = \frac{-2}{s\pi}\sin\,(s\pi/2),\ s \neq 0$$

Example –3. Find the finite Fourier sine transform of the given function $f(x) = \begin{cases} x, & 0 < x < 1/2 \\ 1 - x, & 1/2 < x < 1 \end{cases}$

Sol^n: $F_s(s) = \int_0^l f(x)\,\sin\,\frac{s\pi x}{l}\,dx$

Here the interval of x is [0, 1] and hence $l = 1$

$$F_s(s) = \int_0^1 f(x)\,\sin\,(s\pi x)\,dx$$

$$= \int_0^{1/2} x\,\sin\,(s\pi x)\,dx + \int_{1/2}^{1}(1-x)\sin\,(s\pi x)\,dx$$

$$= \left[x.-\frac{\cos(s\pi x)}{s\pi} - 1.-\frac{\sin(s\pi x)}{s^2\pi^2}\right]_0^{1/2} + \left[(1-x).\frac{-\cos(s\pi x)}{s\pi} - (-1).-\frac{\sin(s\pi x)}{s^2\pi^2}\right]_{1/2}^{1}$$

$$= \frac{-1}{s\pi}\left\{\frac{1}{2}\cos(s\pi/2)\right\} + \frac{1}{s^2\pi^2}\{\sin(s\pi/2)\}$$

$$\frac{-1}{s\pi}\left\{\frac{-1}{2}\cos(s\pi/2)\right\} - \frac{1}{s^2\pi^2}\{-\sin(s\pi/2)\}$$

Thus $F_s(s) = 2/s^2\pi^2.\sin(s\pi/2)$.

Example –4. Find the finite Fourier sine transform of the following function f(x) = cos kx (k is non integer) in (0,π)

Sol^n: $F_s(s) = \int_0^{\pi} f(x)\,\cos\,sx\,dx = \int_0^{\pi}\cos\,kx.\,\cos\,sx\,dx$

$$= \frac{1}{2}\int_0^{\pi} 2\cos\,kx\,.\,\cos\,sx\,dx = \frac{1}{2}\int_0^{\pi}\sin\,(s+k)\,x + \sin(s-k)x$$

$$= \frac{1}{2}\left[-\frac{\cos(s+k)x}{s+k} - \frac{\cos(s-k)x}{s-k}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\frac{-1}{s+k}\{\cos(s+k)\pi - 1\} - \frac{1}{s-k}\{\cos(s-k)\pi - 1\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+k} + \frac{1}{s-k}\right] - \frac{1}{s+k}\{\cos s\pi.\cos k\pi - \sin s\pi.\sin ks\}$$

$$-\frac{1}{s-k}\{\cos\pi.\cos k\pi+\sin s\pi.\sin k\pi\}$$

$$=\frac{1}{2}\left[\frac{2s}{s^2-k^2}-\cos s\pi.\cos k\pi\left\{\frac{1}{s+k}+\frac{1}{s-k}\right\}\right]$$

$$=\frac{1}{2}\left[\frac{2s}{s^2-k^2}-\cos s\pi\ .\cos k\pi.\frac{2s}{s^2-k^2}\right]$$

$$=\frac{s}{s^2-k^2}[1-\cos s\pi\ .\cos k\pi]$$

$$F_s(s)=\frac{s}{s^2-k^2}\left\{1+(-1)^{s+1}\cos k\pi\right\} \text{ where } k\neq s$$

7.2 : *Fourier Transforms*

Let $f(x)$ be a piecewise continuous function and have piecewise continuous first derivative in any finite interval and $f(x)$ be absolutely integrable. Then the function F defined by, $-\infty < x < \infty$

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \qquad ...(1)$$

is called, the Fourier transform of $f(x)$ and we denote it by

$$F(s)=F\left[f(x)\right] \qquad ...(2)$$

Inverse Fourier Transform :

Let $f(x)$ be a function satisfying Dirichlets conditions in every finite interval $(-l, l)$

Let $F(s)$ denotes the Fourier transform of a continuous function $f(x)$. Then at every point of continuity of $f(x)$, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \qquad ...(3)$$

is called the Inverse Fourier transform of $F(s)$, We denote it by

$$f(x)=F^{-1}\left[F(s)\right] \qquad ..(4)$$

Note : In some texts the Fourier transform of $f(x)$, is defined by

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx$$

In this situation the inverse Fourier transform of $f(x)$ becomes.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}ds$$

Consequently the results proved in the subsequent theorems may differ in their sign. It is therefore advise to take care of this fact. The coefficient

$\frac{1}{\sqrt{2\pi}}$ may also be different.

PROPERTIES OF FOURIER TRANSFORM :

Uniqueness of Fourier Transform :

Theorem –1 : If f(x) and g(x) are piecewise continuous and absolutely integrable functions having piecewise continuous first derivative in any finite interval and which have the same Fourier transform then they may differ at only a finite no of points of discontinuity in any finite interval.

Proof : Since $f(x)$ and $g(x)$ have the same fourier transform we have $F[f] = F[g] = F(s)$ say.

or $$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ixs}ds=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{ixs}ds \qquad ...(1)$$

Further, by fourier integral Theorem,

$$\frac{f(x^+)+f(x^-)}{2}=\frac{g(x^+)+g(x^-)}{2}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{ixs}ds \qquad ...(2)$$

form this it is clear that at points where $f(x)$ and $g(x)$ are continuous $f(x) = g(x)$ this implies that the Fourier transform and its inverse is unique if the function is continuous and that two functions differing at their finite points of discontinuity may have the same fourier transform.

Theorem –2 : Fourier transform is continuous.

Proof : Let $f(x)$ be the function whose Fourier transform $F(s) = F[f(x)]$ exists; then by definition of Fourier transform we have.

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{ixs}ds \qquad ...(1)$$

and $$F(s+h)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+h)x}ds \qquad ...(2)$$

From this it is clear that at points where $f(x)$ and $g(x)$ are continuous $f(x) = g(x)$ implies that Fourier transform and its inverse are unique if the function is continuous and that two functions differing at their finite points of discontinuity may have same fourier transforms.

The Fourier integral theorem ensures the existence of the transforms given by (1) and (2)

Theorem –3 : The Fourier transform is linear; that is if $f_1(x)$ and $f_2(x)$ function whose Fourier transforms exist and c_1 and c_2 are scalars, then,

$$F\left[c_1 f_1(x)+c_2 f_2(x)\right]=c_1F[f_1(x)]+c_2F[f_2(x)]$$

Proof: By definition of Fourier transform,

$$F\left[c_1f_1+c_2f_2\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[c_1f_1(x)+c_2f_2(x)e^{ixs}ds\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c_1f_1(x)e^{ixs}+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c_2f_2(x)\, e^{isx}dx$$

$$=\frac{c_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)\, e^{-ixs}ds+\frac{c_2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}dx$$

$= c_1F[f_1]+c_2F[f_2]$ this proves the result.

7.3 : *Fourier Integral Theorem*

Theorem – 4 :Let f(x) be a function which satisfies the Dirchlet's conditions in every interval (–l, l), then

$$f(x) = \frac{1}{2\pi}\int_{\infty}^{\infty}\int_{\infty}^{\infty} f(t)ds\, dx\, e^{i(x-t)s}$$

Proof : We know that the complex Fourier series of $f(x)$ in $(-l, l)$ given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{l}} \qquad \text{...(1)}$$

where $c_n = \dfrac{1}{2\pi}\displaystyle\int_{-l}^{l} f(t) e^{\frac{-in\pi t}{l}} dt$

Substituting the value of c_n in (1), we get

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2l}\int_{-l}^{l} f(t) e^{\frac{in\pi(x-t)}{l}} dt$$

Put $\delta s = \dfrac{\pi}{l}$ so that $\delta s \to 0$ when $l \to \infty$

$$\therefore\quad f(x) = \sum_{n=1}^{\infty} \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{in(x-t)\delta s}{l}} dt \;= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \delta s \int_{-l}^{l} f(x)\, e^{in(d-t)\delta s} dt$$

Taking limit $l \to \infty$ and changing the summation to a definite integral, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) e^{i(t-x)s} dt\; ds$$

7.4 : *Inversion Formula for Fourier Transform*

Theorem – 5 :Let f(x) be a function satisfying Dirchlet's conditions in every function interval (–l, l). Let F(s) denote the Fourier transform of f(x). Then at every point of continuity of f(x), we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s) e^{-ixs} ds$$

Proof : By Fourier integral theorem, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot e^{i(x-t)s} dt\; ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixs}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{-its} dt\right] ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixw}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cdot e^{itw} dt\right] dw \quad \text{(putting } s = -w\text{)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixw} F(w)\, dw = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s) e^{-ixs} ds \quad \text{(By definition of Fourier transform)}$$

7.5 : *Fourier Transforms and its properties*

In previous article we discussed the infinite exponential Fourier transform and its properties. In this article we shall discuss some other infinite transforms namely Fourier sine and Fourier Cosine transforms.

Fourier sine transforms :

Definition :

The Fourier sine transform of $f(x)$, $0<x<\infty$ is defined by $F_s(s)$ where

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = F_s[f(x)]$$, Whenever it exists.

(ii) Fourier cosine transform :

The Fourier cosine transform of $0<x<\infty$ is defined by $F_c(s)$ where

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = F_c[f(x)]$$, Whenever it exists.

Properties of sine and cosine transforms:

Fourier sine and cosine transforms have many properties similar to those of exponential Fourier transforms discussed in previous chapter for example:

(i) F_s and F_c respectively, the Fourier sine and consine transform operators are linear.

Theorem – 6 : Fourier transform, Fourier sine transform and Fourier cosine transform are linear

(i) $F[a\, f(x) + b\, g(x)] = a\, F[f(x)] + b\, F[g(x)]$

(ii) $F_s[a\, f(x) + b\, g(x)] = a\, F_s[f(x)] + b\, F_s[g(x)]$

(iii) $F_c[a\, f(x) + b\, g(x)] = a\, F_c[f(x)] + b\, F_c[g(x)]$

where a and b are real numbers.

Proof : (i) $$F[a\, f(x) + b\, g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a\, f(x) + b\, g(x)] e^{isx} dx$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx = a\, F(x) + b\, F[g(x)]$$

(ii) $$F_s[a\, f(x) + b\, g(x)] = \frac{2}{\sqrt{\pi}} \int_0^\infty [a\, f(x) + b\, g(x)] \sin\, sx\, dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty af(x) \sin sx\, dx + \frac{b}{\sqrt{2\pi}} \int_0^\infty g(x) \sin\, sx\, dx$$

$$= \frac{2a}{\sqrt{\pi}} \int_0^\infty f(x)\, e^{isx} dx + \frac{2b}{\sqrt{\pi}} \int_0^\infty g(x) \sin\, sx\, dx$$

$$= aF_s[f(x)] + bF_s[g(x)]$$

Similarly (iii) can also be proved

Change of Scale Property

For any non zero real a,

$$\mathrm{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Proof: Suppose, a > 0

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2x}} \int_{-\infty}^{\infty} f(t)\ e^{is(t+a)x} \frac{dt}{a} \qquad (\text{put } t = ax)$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{is(t/a)} dt \quad = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Similarly $a < 0$

$$F[f(ax)] = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ .\ e^{\frac{ist}{a}} \frac{dt}{a} \quad = -\frac{1}{a} F\left(\frac{s}{a}\right)$$

Theorem – 7 : Let F_s and F_c respectively the fourier sine and and cosine transforms. Then prove the followings.

(i) $F_s[f(ax)] = a^{-1} F_s[s/a], (a > 0)$

(ii) $F_c[f(ax)] = a^{-1} F_c[s/a], (a > 0)$

(iii) $F[\cos ax\, f(x)] = \frac{1}{2}[F(s+a) + F(s-a)]$

(iv) $F[f(x) \sin ax] = \frac{1}{2i}[F(s+a) - F(s-a)]$

(v) $F_s[\cos(ax) f(x)] = \frac{1}{2}\{F_s(s+a) + F_s(s-a)\}$

$F_s[\sin(ax) f(x)] = \frac{1}{2}\{F_c(s-a) - F_c(s+a)\}$

$F_c[\sin(ax) f(x)] = \frac{1}{2}\{F_s(s+a) + F_s(a-s)\}$

$F_c[\cos(ax) f(x)] = \frac{1}{2}\{F_c(s+a) + F_c(s-a)\}$

(vi) $F_s[f(x)] = F_s(|s|)$ sgn x where sgn s

$$= \begin{cases} +1, & s > 0 \\ -1, & s < 0 \end{cases}$$

Proof: (i) By definition of sine transform, we can write

$$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \sin(xs) dx$$

Putting $t = ax$ so that $dt = a\ dx,\ dx = \frac{dt}{a}$

we get $F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{a} f(t) \sin\left(\frac{s}{a} t\right) dt = \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin\left(\frac{s}{a} t\right) dt$

$$= \frac{1}{a} F_s\left(\frac{s}{a}\right), (a > 0)$$

(ii) Similarly, one can prove that

$$F_c\left[f\left(ax\right)\right]=\frac{1}{a}F_c\left(\frac{s}{a}\right),\left(a>0\right)$$

(iii) $$F\left[f(x)\cos ax\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)\cos ax\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)dx$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(s+a)x}f(x)dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(s-a)x}f(x)dx\right]$$

$$=\frac{1}{2}\left[F(s+a)+F(s-a)\right]$$

(iv) $$F\left[f(x)\sin ax\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)\sin ax\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)\left(\frac{e^{iax}-e^{-iax}}{2i}\right)dx$$

$$=\frac{1}{2i}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(s+a)x}f(x)dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(s-a)x}dx\right]$$

$$=\frac{1}{2i}\left[F(s+a)-F(s-a)\right]$$

(v) $$F_s\left[\cos\left(ax\right)f\left(x\right)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos\left(ax\right)f\left(x\right)\sin\left(sx\right)dx$$

$$=\frac{2}{\sqrt{\pi}}\cdot\frac{1}{2}\int_{-\infty}^{\infty}\left[2\sin sx.\cos ax\right]f(x)\,dx$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty}f\left(x\right)\left[\sin\left\{\left(s+a\right)x\right\}+\sin\left\{\left(s-a\right)x\right\}\right]dx$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty}f\left(x\right)\sin\left\{\left(s+a\right)x\right\}dx+\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty}f\left(x\right)\sin\left\{\left(s-a\right)x\right\}dx$$

$$=\frac{1}{2}\left[F_s(s+a)+F_s(s-a)\right]$$

Similarly other can be proved.

(vi) We know that

$$F_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin sx\,dx=F_s(s)\quad(s>0)\qquad ...(1)$$

When s < 0

$$F_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin sx\,dx\quad(s<0)$$

Putting $s=-\eta$, where $\eta>0$, we get

$$F_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\sin\{(-\eta)x\}\,dx$$

$$=-\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\sin(\eta x)dx=-F_s(\eta)=-F(-s) \quad \text{...(2)}$$

From (1) and (2) we have in general

$$F_s\left[f(x)\right]=F_s(|\,s\,|)\operatorname{sgn} s$$

where $\operatorname{sgn} s=\begin{cases}1, & \text{when } s>0\\ -1, & \text{when } s<0\end{cases}$

Theorem – 8 :

$$F\left[f(x-a)\right]=e^{ias}F(s)$$

Proof : $$F\left[f(x-a)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-a)\,e^{ist}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\,e^{is(a+t)}dx \quad [\text{put } x-a=t]$$

$$=\frac{e^{ias}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{isx}dx=e^{ias}F(s)$$

Theorem – 9 :

$$F\left[e^{iax}f(x)\right]=F(s+a)$$

Proof : $$F\left[e^{iax}f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\;e^{(s+a)x}dx=F(s+a)$$

Theorem – 10 :

(i) $F\left[x^n f(x)\right]=(-i)^n\dfrac{d^n}{ds^n}\left[F(s)\right]$ (ii) $F_s\left[x\,f(x)\right]=\dfrac{-d}{ds}\left[F_c(s)\right]$

(iii) $F_c\left[x\,f(x)\right]=\dfrac{d}{ds}\left[F_s(s)\right]$

Proof : (i) We have $F(s)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}f(x)e^{isx}dx$

Differentiating both sides n times w.r.t. 's' we get,

$$\frac{d^n}{ds^n}\left(F(s)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)(ix)^n\;e^{isx}dx$$

$$=\frac{i^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{isx}x^n dx=i^nF\left[x^n f(x)\right]$$

Hence $F\left[x^n\,f(x)\right]=\dfrac{1}{i^n}\dfrac{d^n}{ds^n}\left[F(s)\right]=(-i)^n\dfrac{d^n}{ds^n}\left(F(s)\right)$

(ii) We have $F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$

Differentiating w.r.t. 's' on both sides

$$\frac{d}{ds}\left[F_c(s)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\frac{d}{ds}(\cos sx)\, dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)(-x\sin sx)\, dx$$

$$=-\sqrt{\frac{2}{\pi}}\int_0^{\infty} x\, f(x)(\sin sx)\, dx=-F_s\left[x f(x)\right]$$

$$\therefore\ F_s\left[x\, f(x)\right]=-\frac{d}{ds}\left[F_c(s)\right]$$ Proof of (iii) is similar.

Theorem – 11 : $F\left[f'(x)\right]=-is\,F(s)$ if $f(x)\to 0$ as $x\to\pm\infty$

Proof : $$F\left[f'(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f'(x)dx\ =\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\frac{d}{dx}\left[f(x)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left\{\left[e^{isx} f(x)\right]-\text{is}\ \int_{-\infty}^{\infty} f(x)e^{isx}dx\right\}$$

$$=\text{is}\ F(s)\left(\text{Since } f(x)\to 0, \text{ as } x\to\pm\infty\right)$$

Note : $F\left[f^{n}(x)\right]=(-is)^{n}F(s)\, if\ f, f', f'',\ f(x-1)\to 0\ \ as\ \ x\to\pm\infty$

Theorem – 12 : **Fourier sine and cosine transforms of derivatives:**

Show that

(i) $F_s\left[f'(x)\right]=-sF_c[f(x)]$ if $f(x)\to 0$ as $x\to\infty$

(ii) $F_c\left[f'(x)\right]=-\sqrt{(2/\pi)}f(0)+sF_s[f(x)]$ if $f(x)\to 0$ as $x\to\infty$

(iii) $F_c\left[f''(x)\right]=-\sqrt{(2/\pi)}f'(0)-s^2F_c[f(x)]$ if $f(x)$ and $f'(x)\to 0$ as $x\to\infty$

(iv) $F_s\left[f''(x)\right]=\sqrt{(2/\pi)}\ sf(0)-s^2F_s[f(x)]$ if $f(x)$ and $f'(x)\to 0$ as $x\to\infty$

Proof: (i) By definition we can write

$$F_s\left[f'(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\sin(xs)\, dx$$

Integrating by parts, the R.H.S. of (1) becomes

$$=\sqrt{\frac{2}{\pi}}\left\{\left[f(x)\sin sx\, dx\right]_0^{\infty}-\int_0^{\infty} f(x).s\cos sx\, dx\right\}$$

$$=-s\frac{2}{\sqrt{\pi}}\int_0^{\infty} f(x)\cos sx\, dx=-s\,F_c(s)$$

$$F_s[f'(x)]=-\int_0^{\infty} f(x)\cos(xs)\, dx=-sF_c(s)$$

(ii) We have

$$F_c[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos(xs)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[\cos(xs)f(x)\right]_{x=0}^{\infty} + s\int_0^\infty f(x)\sin(xs)\,dx\right\} \quad = -\sqrt{\frac{2}{\pi}}f(0) + sF_s[f(x)]$$

(iii) From (3) we can write

$$F_c\left[f''(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[f'(x)\cos sx\,dx\right]_0^\infty - \int_0^\infty f'(x)(-s)\sin sx\,dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\lim_{x\to\infty}\left(f'(x)\cos sx\right) - \lim_{x\to 0}\left(f'(x)\cos sx\right)\right\} + \sqrt{\frac{2}{\pi}}\,.\,s\int_0^\infty f'(x).\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(-f'(0)\right) + \sqrt{\frac{2}{\pi}}.s\left\{\left[f(x)\sin sx\right]_0^\infty - \int_0^\infty f(x).s\cos sx\,dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}f'(0) - s^2\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}f'(0) - s^2F_c(s)$$

(iv) Try yourself.

Theorem – 13 : **Fourier sine and cosine transforms of some elementary / simple functions:**

Show that

(i) $F_s[e^{-ax}] = \sqrt{(2/\pi)}\ \dfrac{s}{(a^2+s^2)},\ (s>0)$

(ii) $F_s[e^{-ax}] = \sqrt{\left(\dfrac{2}{\pi}\right)}\dfrac{a}{a^2+s^2},\ (s>0)$

(iii) $F_s[e^{-ax}] = \sqrt{\left(\dfrac{2}{\pi}\right)}\ \cot^{-1}\left(\dfrac{a}{s}\right),\ (s>0)$

(iv) $F_s[x^{-1}] = \sqrt{(\pi/2)}$

(v) $F_s[xe^{-at}] = \sqrt{\left(\dfrac{2}{\pi}\right)}\ \dfrac{a^2-s^2}{(a^2+s^2)^2},\ (a>0)$

(vi) $F_s[xe^{-ax}] = \sqrt{\left(\dfrac{2}{\pi}\right)}\dfrac{2as}{(a^2+s^2)^2},\ (s>0)$

Proof : To prove (i) and (ii) consider the integrals

$$I = \int_0^\infty e^{-ax}\sin(xs)\,dx,\quad J = \int_0^\infty e^{-ax}\cos(xs)\,dx\ \ (s>0) \qquad ...(1)$$

Involved in the calculation of sine and cosine transforms. Integrating by parts, we get

$$I = \left[-a^{-1}e^{-ax}\sin xs\right]_{s=0}^{\infty} + \frac{s}{a}\int_0^{\infty} e^{-ax}\cos xs\, dx \qquad ...(2)$$

By (1), this may be written as

$$I = \frac{s}{a}J \qquad ...(3)$$

Similarly integrating by parts the integral J in (1), we get

$$J = \frac{1}{a} - \frac{s}{a}I \qquad ...(4)$$

Solving (3) and (4), we get i. e.

$$I = \frac{s}{a}\left(\frac{1}{a} - \frac{s}{a}I\right) = \frac{s}{a^2} - \frac{s^2}{a^2}I$$

$$\Rightarrow \left(I + \frac{s^2}{a^2}\right)I = \frac{s}{a^2}$$

$$I = \frac{s}{a^2+s^2} \quad \text{and} \quad J = \frac{a}{a^2+s^2} \qquad ...(5)$$

Multiplying by the scalar factor $\sqrt{\frac{2}{\pi}}$ in I and J we get

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\ I = \sqrt{\frac{2}{\pi}}\ \frac{s}{(a^2+s^2)} \qquad ...(6)$$

and $$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\ J = \sqrt{\frac{2}{\pi}}\ \frac{a}{(a^2+s^2)} \qquad ...(7)$$

This proves (i) and (ii)

(iii) From (6) above we have

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin(xs)\, dx = \sqrt{\frac{2}{\pi}}\ \frac{s}{a^2+s^2}$$

Integrating this with respect to ‘a’ from 0 to ‘ ∞ ’ we get

$$\int_0^{\infty}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin(xs)dx\right]da = \sqrt{\frac{2}{\pi}}\ s\int_0^{\infty}\frac{da}{a^2+s^2}$$

$$= \sqrt{\frac{2}{\pi}}.s.\frac{1}{s}\left[\tan^{-1}\frac{a}{s}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2} - \tan^{-1}\frac{a}{s}\right)$$

$$= \sqrt{\frac{2}{\pi}}\cot^{-1}\left(\frac{a}{s}\right),\ (s>0)$$

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} x^{-1}e^{-ax}\sin(xs)dx = \sqrt{\frac{2}{\pi}}\cot^{-1}\left(\frac{a}{s}\right) \qquad ...(8)$$

From this we deduce that

$$F_s[x^{-1}e^{-ax}] = \sqrt{\frac{2}{\pi}}\cot^{-1}\left(\frac{a}{s}\right)\ (s > 0) \qquad ...(9)$$

This proves (iii)

(iv) From (9) we can write

$$\lim_{a\to 0} F_s[x^{-1}\,e^{-ax}] = \lim_{a\to 0}\ \sqrt{\frac{2}{\pi}}\cot^{-1}\left(\frac{a}{s}\right) \qquad ...(10)$$

Thus

$$F_s[x^{-1}] = \sqrt{\frac{2}{\pi}}.\frac{\pi}{2} = \sqrt{\frac{2}{\pi}} \qquad ...(11)$$

This proves (iv)

(v) If we differentiate (7) with respect to a, we get

$$\frac{d}{da}\left[\sqrt{\frac{2}{\pi}}\ \int_0^\infty e^{-ax}\cos xs\ dx\right] = \frac{d}{da}\left[\sqrt{\frac{2}{\pi}}\ \frac{a}{(a^2+s^2)}\right]$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty (-x)e^{-ax}\cos xs\ dx = \sqrt{\frac{2}{\pi}}\left(\frac{a^2+s^2-2a^2}{(a^2+s^2)^2}\right) = \sqrt{\frac{2}{\pi}}\ \frac{s^2-a^2}{(a^2+s^2)^2}$$

$$F_c[xe^{-ax}] = \sqrt{\frac{2}{\pi}}\ \frac{(a^2-s^2)}{(a^2+s^2)},\ a > 0$$

From (6), differentiate w.r.t. a, we get

$$F_s[xe^{-ax}] = \sqrt{\frac{2}{\pi}}\ \frac{2as}{(a^2+s^2)},\ a > 0$$

Theorem – 14 : **Show that**

(i) $F[u(x)e^{-kx}\cos(ax)] = \dfrac{1}{\sqrt{2\pi}}\ \dfrac{k+iz}{(k+iz)^2+a^2}$

(ii) $F[xu(x)\ \sin\ (ax) = \dfrac{2iaz}{\sqrt{2\pi}(a^2-z^2)^2}$

Proof:

(i) Let $f(x) = u(x)\ e^{-kx}\cos(ax)$ ...(17)

Then $F[f(x)] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^\infty e^{-kx}\cos\ (ax)e^{-izx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(k+iz)x}\left[\frac{e^{iax}+e^{-iax}}{2}\right]dx$$

$$= \frac{1}{2\sqrt{2\pi}}\left[\frac{e^{-(k+iz-ia)x}}{-k+1(a-z)} + \frac{e^{-(k+iz+ia)x}}{-k+1(-a-z)}\right]_0^\infty$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\frac{1}{k - 1(a - z)} + \frac{1}{k + 1(a + z)} \right]$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\frac{2(k + iz)}{(k + iz)^2 + a^2} \right] = \frac{1}{\sqrt{2\pi}} \frac{k + iz}{(k + iz)^2 + a^2} \qquad ...(18)$$

(ii) Let $f(x) = xu(x) \sin(ax)$ ...(19)

Then if we take $g(x) = u(x) \sin(ax)$ ...(20)

Then $F[g] = \frac{1}{\sqrt{2\pi}} \frac{a}{a^2 - z^2}$ ...(21)

Since we know that $\frac{d}{dz} F[g] = (-i) F[xg]$ ...(22)

Therefore $\frac{d}{dz} F[g] = (-i) F[f]$ ...(23)

Hence, by (21) and (23)

$$F[f] = \frac{1}{-i} \frac{d}{dz} \left[\frac{1}{\sqrt{2\pi}} \frac{a}{a^2 - z^2} \right] = \frac{2iza}{\sqrt{2\pi}(a - z^2)} \qquad ...(24)$$

7.6 : *Fourier Transforms of Even and Odd functions*

(i) Let $f(x)$ be an even function of x i.e., $f(-x) = f(x)$, whose Fourier transform is $F(s)$.

Then $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ixs}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} f(x) e^{ixs} dx + \int_{0}^{\infty} f(x) e^{ixs} dx \right] = \frac{1}{\sqrt{2\pi}} \left[-\int_{0}^{-\infty} f(x) e^{ixs} dx + \int_{0}^{\infty} f(x) e^{ixs} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} f(-x) e^{-ixs} dx + \int_0^{\infty} f(x) e^{ixs} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} f(x) e^{-ixs} dx + \int_0^{\infty} f(x) e^{ixs} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} f(x) (e^{ixs} + e^{-ixs})\, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cos(xs)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(sx)\, dx = F_c\, [f(x)] \qquad ...(1)$$

This shows that Fourier transform of an even function is equal to its cosine transform.

(ii) Let $f(x)$ be an odd function of x i.e. $f(-x) = -f(x)$, whose Fourier transform is F(s), then by definition

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ixs} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} f(x)\, e^{ixs} dx + \int_{0}^{\infty} f(x)\, e^{ixs} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{0}^{\infty} f(x)\, e^{-ixs} dx + \int_{0}^{\infty} f(x)\, e^{ixs} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} f(x)\, (e^{ixs} - e^{-ixs}) dx \right] = -\frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \sin xs\, dx$$

$$= -i\, \sqrt{2/\pi} \int_{0}^{\infty} f(x)\ \sin xs\ dx = -F_s[f(x)] \qquad ...(2)$$

This relates the Fourier transform of an odd function to its sine transforms.

7.7 : *Inversion formula for Fourier sine and cosine transforms*

Let $F_s(s)$ denote the Fourier sine transform of $f(x)$.

Then $f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin sx\, ds$

If we define a function

$$g(x) = \begin{cases} f(x)\ , & x \geq 0 \\ f(-x), & x < 0 \end{cases} \qquad ...(1)$$

where $f(x)$ is defined for positive values of the real variable x, then Fourier transform of $g(x)$ is given by

$$F[g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx} dx = \frac{1}{\sqrt{2x}} \left[\int_{-\infty}^{0} g(x) e^{isx} dx + \int_{0}^{\infty} g(x) e^{-ixs} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{-\infty}^{0} f(-x) e^{-isx} dx + \int_{0}^{\infty} f(x) e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} f(x) \left\{ e^{ixs} + e^{-ixs} \right\} dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_{0}^{\infty} f(x) \cos (xs) dx \right] = F_c[g(x)] = F_c[f(x)] = F_c(s) \qquad ...(2)$$

$$[\because\ g(x) = f(x) \text{for all } x \geq 0]$$

This indicates that the Fourier cosine transform of a function $f(x)$ defined for positive real variable x i.e. equal to the Fourier transform of a function $g(x)$ which is an extension of $f(x)$ given by (1) and is an even function. Thus by Fourier integral theorem, the inverse Fourier transform of $F_c(s)$ give $g(x)$ i.e.,

$$F\left[g(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\ e^{-isx} ds \qquad ...(3)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} F_c(s) e^{-isx} ds + \int_{0}^{\infty} F_c(s) e^{-isx} ds \right]$$

Since $F_c(-s) = F_c(s)$, i.e., $F_c(s)$ is an even function of s, we get

$$g(x) = \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} F_c(s)\left\{\frac{e^{isx}+e^{-isx}}{2}\right\}ds\right.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\ \cos\ sx\ ds \qquad ...(4)$$

From (1) and (4), for $s \geq 0$ we see that

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos\ sx\ ds$$

i.e. $\quad f(x) = F_c[F_c(s)]$ ...(5)

From (2) and (5) we may state that

Definition:

Inverse cosine transform. If $F_c(s)$ denotes the cosine transform of a continuous function $f(x)$ defined for $x \geq 0$ then its inverse transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\ \cos\ sx\,ds \qquad ...(6)$$

From above it is evident that $F_c^{-1} = F_c$, *i.e.,* $F_cF_c = I$ ...(8)

where I denotes the identity operator.

If we define an odd function $g(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$ ...(9)

Then the Fourier transforms of g(x) is

$$F[g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\ e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} g(x)\ e^{isx}\,dx + \int_0^{\infty} g(x)e^{ixs}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\int_{-\infty}^{0} f(-x)\ e^{-ixs}\,dx + \int_0^{\infty} f(x)e^{ixs}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\int_{-\infty}^{0} f(x)\ e^{-ixs}\,dx + \int_0^{\infty} f(x)e^{ixs}\,dx\right]$$

$$= \frac{-1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x)(e^{ixs} - e^{-ixs})\ dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x)\ \sin\ xs\ dx$$

$$= -F_s[g(x)] = -F_s\left[f(x)\right] - iF_s(s) \qquad ...(10)$$

By Fourier inversion formulae

$$g(x) = -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_s(s)\, e^{-ixs}\, ds$$

$$= -i\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} F_s(s)\, e^{-ixs}\, ds + \int_{0}^{\infty} F_s(s)\, e^{ixs}\, ds\right] \qquad ...(11)$$

From (10) when $F_s(-s) = -F_s(s)$, i.e., here $F_s(s)$ is an odd function of *s*, therefore we get

$$g(x) = -i\frac{2i}{\sqrt{2\pi}}\left[\int_{0}^{\infty} F_s(s)\left\{e^{ixs} - e^{-ixs}\right\} ds = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} F_s(s)\sin xs\, ds \qquad ...(12)$$

Hence, for $x \geq 0$, $f(x) = \dfrac{\sqrt{2}}{\pi}\displaystyle\int_{0}^{\infty} F_s(s)\sin xs\, ds = F_s[F_s(s)]$ ...(13)

Now we may define the inverse of sine transform

Definition (inverse sine transform):

It *f*(*x*) is a continuous function defined for $x \geq 0$ and $F_s(s)$ is its sine transform then

$$f(x) = \frac{\sqrt{2}}{\pi}\int_{0}^{\infty} F_s(s)\ \sin xs\, ds \qquad ...(14)$$

Gives the inversion formula for the sine transform. From (14) it is evident that

$F_s^{-1} = F_s$ *i.e.* $F_s^{-1}\, F_s = I$

Theorem -1: The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transforms.
(i.e.) F[f (x) * g(x)] = F (s) · G (s) = F [f (x)] ·F\g(x)].

Proof : $F[f * g] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} (f * g)(x) e^{isx} dx$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) g(x-t) dt\right) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} g(x-t) e^{isx} dx\right) dt$$

(by changing the order of integration

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} g(x-t) e^{isx} dx\right) dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) F[g(x-t)] dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{ist} G(s) dt \qquad \text{(Theorem)}$$

$$= G(s)\cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

= G(s)· F(s) = F (s)·G(s)

Corollary. F^{-1} [F(s) ·G(s)] = (f*g) (x) = F^{-1} [F(s)] * F^{-1}[G(s)].

Theorem - 2: (Parseval's identity). Let F(s) be the Fourier transform of f(x).

$$\textbf{\textit{Then}} \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Proof : For any two functions $f(x)$ and $g(x)$ we have by Theorem. F(f * g) = F(s) G(s).

Hence f * g = F^{-1}[F(s) G(s)].

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s) e^{-isx} ds$$

Putting $x = 0$, we get $\int_{-\infty}^{\infty} f(t) g(-t) dt = \int_{-\infty}^{\infty} F(s) G(s) ds$

Now taking $g(t) = \overline{f(-t)}$, we get

$$\int_{-\infty}^{\infty} f(t)\overline{f(t)} dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)} ds \text{ (using Theorem)}$$

Hence $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$.

Theorem - 3: (Parseval's identity for Fourier sine and cosine transforms). If F_s[f (x)] = F_s(s) and F_c[g(x)] = F_c(s) then

$$\textbf{\textit{(i)}} \int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_S(s)|^2 ds \quad \textbf{\textit{and}} \qquad \textbf{\textit{(ii)}} \int_0^{\infty} |g(x)|^2 dx = \int_0^{\infty} |F_c(s)|^2 ds$$

Proof : (i) and (ii) follow from Parseval's identity for Fourier transform.

Theorem - 4: If F_c(s) and G_c(s) are the Fourier cosine transforms and F_s(s) and G_s(s) are the Fourier sine transforms of f(x) and g(x) respectively, then

$$\int_0^{\infty} f(x) g(x) dx = \int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} F_s(s) G_s(s) ds$$

Proof : $\int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} F_c(s) \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \cos sx \, dx \right] ds$

$$= \int_0^{\infty} g(x) \left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx \; ds \right] dx$$

(by changing the order of integration)

$$= \int_0^\infty g(x) f(x) dx$$

(by inversion formula for Fourier cosine transform)

$$= \int_0^\infty f(x) g(x) dx$$

Similarly $\int_0^\infty F_s(s) G_s(s) ds = \int_0^\infty f(x) g(x) dx$

Evaluation of Fourier Transform :

The following results will be often used in evaluation of Fourier transforms.

(1) $\int_0^\infty e^{-ax} \cos bx \, dx = \dfrac{a}{a^2+b^2}$ (2) $\int_0^\infty e^{-ax} \sin bx \, dx = \dfrac{b}{a^2+b^2}$

(3) $\int_0^\infty \dfrac{\sin ax}{x} dx = \dfrac{\pi}{2}$ if $a > 0$. (4) $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$

Illustrative Examples

Example – 1 : Find the Fourier transform of

$$\boldsymbol{f(x) = \begin{cases} x & if\ |x| \le a \\ 0 & if\ |x| > a \end{cases}}$$

Sol [n] ***:*** The given function can be written as

$$f(x) = \begin{cases} x & if\ -a \le x \le a \\ 0 & otherwise. \end{cases}$$

$$\therefore F(s) = F|f(x)| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x e^{isx} dx = \frac{1}{\sqrt{2\pi}} \left[x\left(\frac{e^{isx}}{is}\right) - \left(\frac{e^{isx}}{i^2 s^2}\right) \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{ae^{isa}}{is} + \frac{e^{isa}}{s^2} \right) - \left(\frac{-ae^{-isa}}{is} + \frac{e^{-isa}}{s^2} \right) \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{a\left(e^{isa} + e^{-isa}\right)}{is} + \frac{\left(e^{isa} - e^{-isa}\right)}{s^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2a \cos sa}{is} + \frac{2i \sin sa}{s^2} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{-i2a \cos sa}{s} + \frac{2i \sin sa}{s^2} \right].$$

$$\therefore F(s) = \frac{i}{s^2} \sqrt{\frac{2}{\pi}} [\sin sa - as \cos sa]$$

Example – 2 : Find the Fourier transform of

$$f(x)=\begin{cases} 1 & \text{if } |x|<a \\ 0 & \text{if } |x|\geq a \end{cases}$$

where 'a' is a positive real number. Hence deduce that

(i) $\displaystyle\int_0^{\infty}\frac{\sin t}{t}\,dt=\frac{\pi}{2}$ ***and (ii)*** $\displaystyle\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$

Sol^n : The given fucntion can be written as

$$f(x)=\begin{cases} 1 & if\ -a<x<a \\ 0 & otherwise. \end{cases}$$

$$\therefore F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{isx}dx=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{e^{ias}-e^{-ias}}{is}\right)=\frac{1}{\sqrt{2\pi}}\left(\frac{2i\sin as}{is}\right).$$

$$\therefore F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right) \qquad \text{... (1)}$$

(i) Now, by Fourier inversion formula we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)(\cos sx-i\sin sx)\,ds \text{ [using (1)]}$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds-\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\sin sx\,ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds\ \left(\because\left(\frac{\sin as}{s}\right)\sin sx \text{ is an odd function in } s\right)$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds\ \left(\because\left(\frac{\sin as}{s}\right)\cos sx \text{ is an even function in } s\right).$$

$$\therefore\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds=\frac{\pi}{2}f(x).$$

Putting $x = 0$, we get $\displaystyle\int_0^{\infty}\left(\frac{\sin as}{s}\right)ds=\frac{\pi}{2}.$

Now, put $t = as$. Then we have $s = \frac{t}{a}$ and $ds = \frac{dt}{a}$. Further $t \to 0$ as $s \to 0$ and $t \to \infty$ as $s \to \infty$.

Hence $\int_0^{\infty} \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$.

(ii) Using Parseval's identity $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$ we get

$$\int_{-a}^{a} 1 dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin as}{s}\right)^2 \text{ds.}$$

$$\text{(i.e.) } 2a = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$\therefore \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = a\pi$$

Putting t = as we get $\int_{-\infty}^{\infty} \left(\frac{\sin t}{t/a}\right)^2 \left(\frac{dt}{a}\right) = a\pi$

$$\therefore 2\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi \left(\therefore \left(\frac{\sin t}{t}\right)^2 \text{is an even function} \right).$$

$$\therefore \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Example – 3 : Find the Fourier transform of f(x) = xe^{-x} where 0 ≤ x < ∞.

Sol^n : F (s) = F[$f(x)$]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(xe^{-x}\right) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} xe^{-(1-is)x} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{xe^{-(1-is)x}}{-(1-is)} - \frac{e^{-(1-is)x}}{(1-is)^2} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1-is)^2} \right] (|e^{-x} e^{isx}| = |e^{-x}| \to 0 \text{ as } x \to \infty)$$

$\therefore e^{-x} e^{isx} \to 0$ as $x \to \infty$)

$$= \frac{1}{\sqrt{2\pi}} \frac{(1+is)^2}{(1+s^2)^2} .$$

Example – 4 : Find the Fourier sine transform of f(x) defined by

$$f(x)=\begin{cases} 1 \text{ if } 0 \le x < 1 \\ 0 \text{ if } x > 1 \end{cases}.$$

Solⁿ : $F_s(s)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\ \sin sx\, dx = \sqrt{\dfrac{2}{\pi}}\int_0^{1} \sin sx\ dx = \sqrt{\dfrac{2}{\pi}}\left[\dfrac{-\cos sx}{s}\right]_0^1$

$$=\sqrt{\frac{2}{\pi}}\left[\left(\frac{-\cos s}{s}\right)-\left(-\frac{1}{s}\right)\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1}{s}-\frac{\cos s}{s}\right].$$

Example – 5 : Find the Fourier cosine transform of $2e^{-3x}+3e^{-2x}$.

Solⁿ : Let $f(x) = 2e^{-3x} + 3e^{-2x}$.

$F_c[f(x)] = F_c[2e^{-3x} + 3e^{-2x}]$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left(2e^{-3x}+3e^{-2x}\right)\cos sx\ dx$$

$$=\sqrt{\frac{2}{\pi}}\left[2\int_0^{\infty} e^{-3x}\cos sx\, dx + 3\int_0^{\infty} e^{-2x}\cos sx\, dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[2\left(\frac{3}{s^2+9}\right)+3\left(\frac{2}{s^2+4}\right)\right] = \frac{6\sqrt{2}}{\sqrt{\pi}}\left[\frac{1}{s^2+9}+\frac{1}{s^2+4}\right].$$

Example – 6 : Find the Fourier cosine transform of

$$f(x)=\begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{if } x \ge a \end{cases}$$

Solⁿ : $F_c(s)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\ \cos sx\, dx = \sqrt{\dfrac{2}{\pi}}\int_0^{a} \cos x \cos sx\, dx$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{1}{2}\right)\int_0^{a}\ [\cos (s+1)x + \cos (s-1)x]\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1}+\frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1}+\frac{\sin(s-1)a}{s-1}\right]$$

provided $s \neq 1, -1$.

Example – 7 : Find the Fourier sine transform of $\dfrac{1}{x}$.

Solⁿ : $F_s\left(\dfrac{1}{x}\right)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}\frac{\sin sx}{x}\, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta} d\theta \quad \text{(putting } \theta = sx \text{ so that } dx = \frac{d\theta}{s}\text{)}$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} \quad \left(\because \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2} \right) = \sqrt{\frac{\pi}{2}}\,.$$

Example – 8 : Find the Fourier sine transform of $\frac{e^{-ax}}{x}$ $a > 0$.

Sol^n : Let $f(x) = \frac{e^{-ax}}{x}$.

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{e^{-ax}}{x} \right) \sin sx \, dx = F(s) \text{ (say).} \qquad \text{............... (1)}$$

Differentiating w.r.t.s we get $\frac{d}{ds}[F(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{ds}\left(\frac{e^{-ax}}{x} \sin sx \right) dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2 + a^2} \right)$$

Integrating w.r.t.s we get $F(s) = \sqrt{\frac{2}{\pi}} \int \left(\frac{a}{s^2 + a^2} \right) ds$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a} \right) + c \qquad \text{............... (2)}$$

From (1) we note that F(s) = 0 when
s = 0, Hence c = 0

$$\therefore F(s) = F_s[F(x)] = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a} \right)$$

Example – 9 : Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$ and hence find $F_c\left[\frac{e^{-ax} - e^{-bx}}{x} \right]$.

Sol^n : $F_c\left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx = F(s)$ (say).

Differentiating both sides w.r.t.s, we get

$$\frac{dF(s)}{ds} = \frac{d}{ds}\left(\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx \right)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d}{ds}\left(\frac{e^{-ax}}{x} \cos sx \right) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} (-x \sin sx) dx = -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx.$$

$$\therefore \frac{dF(s)}{ds} = -\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right].$$

$$\therefore F(s) = -\sqrt{\frac{2}{\pi}}\int \frac{s}{s^2+a^2} = -\sqrt{\frac{2}{\pi}}\left[\frac{1}{2}log\left(s^2+a^2\right)\right].$$

$$\therefore F_c\left[\frac{e^{-ax}}{x}\right] = \frac{-1}{\sqrt{2\pi}}log\left(s^2+a^2\right).$$

Simillarly $F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}}log\left(s^2+b^2\right)$

Now $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$

$$= -\frac{1}{\sqrt{2\pi}}log\left(s^2+a^2\right) + \frac{1}{\sqrt{2\pi}}log\left(s^2+b^2\right) \quad = \frac{1}{\sqrt{2\pi}}log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

Example – 10 : Find the Fourier sine and cosine transforms of xe^{-ax}.

Sol^n : We have $F_s\left[xf(x)\right] = -\frac{d}{ds}F_c\left[f(x)\right]$

Let $f(x) = e^{-ax}$.

$$\therefore F_s\left[xe^{-ax}\right] = -\frac{d}{ds}F_c\left[e^{-ax}\right]$$

$$= -\frac{d}{ds}\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right) \quad = -\sqrt{\frac{2}{\pi}}a\left(\frac{-2s}{\left(s^2+a^2\right)^2}\right) \quad = \frac{2\sqrt{2}as}{\sqrt{\pi}\left(s^2+a^2\right)^2}.$$

We have $F_c\,[x f(x)] = \frac{d}{ds}F_s\left[f(x)\right]$

$$\therefore F_c\left[xe^{-ax}\right] = \frac{d}{ds}F_s\left[e^{-ax}\right] = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s^2+a^2-2s^2}{\left(s^2+a^2\right)^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right)$$

Example – 11 : Find (i) the Fourier cosine transform of $\frac{1}{1+x^2}$ and (ii) Fourier sine transform of $\frac{x}{1+x^2}$.

Sol^n : (i) $F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{cos\,sx}{1+x^2}dx$.

Let $I = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx$ .... (1)

$\therefore \frac{dI}{ds} = \int_0^\infty \frac{-x \sin sx}{1+x^2}\,dx$ .... (2)

$= -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)}\,dx \quad = -\int_0^\infty \frac{\left[\left(1+x^2\right)-1\right]\sin sx}{x(1+x^2)}\,dx$

$= -\int_0^\infty \frac{\sin sx}{x}\,dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$.

$\therefore \frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx$ . ... (3)

Now, $\frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)}\,dx \quad = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I$

$\therefore \frac{d^2 I}{ds^2} - I = 0$. (i.e.) $(D^2 - 1)\,I = 0$ where $D \equiv \frac{d}{ds}$.

$\therefore$ $I = Ae^{-s} + Be^{s}$. .... (4)

$\therefore \frac{dI}{ds} = -Ae^{-s} + Be^{s}$ .... (5)

Putting s = 0 in (1) and (4) we get $A + B = \int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty$

(i.e.) $A + B = \frac{\pi}{2}$ .... (6)

Put $s = 0$ in (2) and (5). We get

$-A + B = -\frac{\pi}{2}$ .... (7)

Solving (6) and (7) we get $A = \frac{\pi}{2}; B = 0$.

Using this in (4) we get $I = \frac{\pi}{2} e^{-s}$. .... (8)

$\therefore F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2} e^{-s}\right] = \sqrt{\frac{\pi}{2}}\, e^{-s}$.

(ii) $F_s\left[\frac{x}{1+x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin sx}{1+x^2}\,dx$

$= -\sqrt{\frac{2}{\pi}}\,\frac{dI}{ds}$ [by (2)]

$$= -\sqrt{\frac{2}{\pi}}\left[-\frac{\pi}{2}e^{-s}\right] \qquad \text{[using (8)]}$$

$$= \sqrt{\frac{\pi}{2}}\ e^{-s}.$$

Example – 12 : Evaluate using $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ ***using Fourier cosine transform.***

Sol^n : We know that $F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right) = F_c(s)$ (say).

Similarly the Fourier cosine transform of $g(x) = e^{-bx}$ is

$$F_c\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+b^2}\right) = G_c(s) \text{ (say).}$$

By Theorem, we have

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(s)G_c(s)ds.$$

$$\therefore \int_0^\infty e^{-ax}e^{-bx} = \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)ds.$$

$$\therefore \frac{2}{\pi}\int_0^\infty \frac{ab}{(s^2+a^2)(s^2+b^2)}ds = \int_0^\infty e^{-(a+b)x}dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = 0 - \left(-\frac{1}{a+b}\right) = \frac{1}{a+b}.$$

$$\therefore \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

$$\text{(i.e.) } \therefore \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Example – 13 : Find f(x) if its Fourier sine transform is $\boldsymbol{\frac{e^{-as}}{s}}$.

Sol^n : Given $F_s\left[f(x)\right] = \frac{e^{-as}}{s}$.

Hence by inversion formula for Fourier sine transform,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\ \sin sx\ ds. \qquad \text{....... (1)}$$

$$\frac{df(x)}{dx} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as} \quad \cos \ sx \ ds = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+x^2}\right].$$

Integrating w.r.t x we get

$$f(x) = \sqrt{\frac{2}{\pi}}\int \frac{a}{a^2+x^2}dx = \sqrt{\frac{2}{\pi}}\, tan^{-1}\left(\frac{x}{a}\right) + A \qquad \text{....... (2)}$$

Where A is a constant to be determined.
By (1), $f(0) = 0$ and by (2), $f(0) = A$. Hence $A = 0$.

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}\, tan^{-1}\left(\frac{x}{a}\right).$$

Example – 14: Find the sine and consine transforms of $x^n\ e^{-ax}$

Sol^n: Let $f(x) = x^n\ e^{-ax}$ and let F_s and F_c respectively be the sine and cosine transforms of $f(x)$ ...(1)

Then $$F_s = \sqrt{\frac{2}{\pi}}\int_0^\infty x^n e^{-ax} \sin\,(xs)\ dx \qquad ...(2)$$

and $$F_c = \sqrt{\frac{2}{\pi}}\int_0^\infty x^n\ e^{-ax} \cos\,(xs)\ dx \qquad ...(3)$$

Non consider the integrals

$$I_1 = \int_0^\infty -e^{-ax} \cos\,(xs)\ dx = \frac{a}{a^2+s^2} \qquad ...(4)$$

$$I_1 = \int_0^\infty -e^{-ax} \cos\,(xs)\ dx = \frac{a}{a^2+s^2} \qquad ...(5)$$

[These integrals have been solved in previous section] Differentiation (4) n times with respect to a we get

$$\frac{d^n I_1}{da^n} = \int_0^\infty (-x)^n e^{-ax} \cos(xs)\ dx = \frac{d^n}{da^n}\left(\frac{a}{a^2+s^2}\right)$$

or $$\int_0^\infty x^n e^{-ax} \cos(xs)\ dx = (-1)^n \frac{d^n}{da^n}\left(\frac{a}{a^2+s^2}\right) \qquad ...(6)$$

Hence from (3) and (6)

$$F_c = \sqrt{\frac{2}{\pi}}(-1)^n \frac{d^n}{da^n}\left(\frac{a}{a^2+s^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{n!\cos\{(n+1)\ \tan^{-1}(s/a)\}}{(a^2+s^2)^{(n+1)/2}} \qquad ...(7)$$

Similarly differentiating (5) in times we get

$$F_c = \sqrt{\frac{2}{\pi}}\ \frac{n!\cos\{(n+1)\ \tan^{-1}(s/a)\}}{(a^2+s^2)^{(n+1)/2}}$$

Example – 15: Find the cosine transform of $\dfrac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}$

Sol^n: Let $f(x)=\dfrac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}$ ...(1)

and let F_c be the cosine transform of $f(x)$, then by definition

$$F_c=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}\cos(xs)\,dx \qquad ...(2)$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}\left(\frac{e^{ixs}+e^{-ixs}}{2}\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{e^{(a+is)x}+e^{-(a+is)x}}{e^{\pi x}+e^{-\pi x}}dx+\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{e^{x(a-is)}+e^{x(a-is)}}{e^{\pi x}+e^{-\pi x}}dx$$

By definite integrals we get

$$F_c=\frac{1}{\sqrt{2\pi}}.\frac{1}{2}\sec\left(\frac{a+is}{2}\right)+\frac{1}{\sqrt{2\pi}}.\frac{1}{2}\sec\left(\frac{a-is}{2}\right)$$

$$=\frac{1}{2\sqrt{2\pi}}\left[\frac{\cos\left(\frac{a+is}{2}\right)+\cos\left(\frac{a-is}{2}\right)}{\cos\left(\frac{a+is}{2}\right)\cos\left(\frac{a-is}{2}\right)}\right]=\frac{4}{2\sqrt{2\pi}}\left[\frac{\cos\left(\frac{a}{2}\right)\cos\left(\frac{is}{2}\right)}{\cos a+\cos(is)}\right]$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{\cos(a/2)\frac{1}{2}(e^{s/2}+e^{-s/2})}{\cos a+\frac{1}{2}(e^{s}+e^{-s})}=\sqrt{\frac{2}{\pi}}\,\frac{\cos(a/2)\frac{1}{2}(e^{s/2}+e^{-s/2})}{e^{s}+2\cos\ a+e^{-s}}$$

Example – 16: Find the cosine transform of $\dfrac{1}{1+x^2}$ ***and then find the sine transform of*** $\dfrac{x}{1+x^2}$ **.**

Sol^n: Let $f(x)=\dfrac{1}{1+x^2}$ ...(1)

Then the cosine transform of $f(x)$ is

$$F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\cos sx}{1+x^2}dx \qquad ...(2)$$

Differentiating with respect of s, we get

$$\frac{dF_c(s)}{ds}=-\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin(sx).x}{1+x^2}dx \qquad ..(3)$$

$$=-\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\left[(1+x^2)-1)\right]\sin sx}{x(1+x^2)}dx$$

$$=-\sqrt{\frac{2}{\pi}}\left[\int_0^{\infty}\frac{\sin sx}{x}dx-\int_0^{\infty}\frac{\sin sx}{x(1+x^2)}dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2} - \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx\right] \qquad ...(4)$$

Differentiating again w.r.t. s we get

$$\frac{d^2\left(F_c(s)\right)}{ds^2} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{x(1+x^2)}\,dx = F_c(s)$$

or
$$\frac{d^2\left(F_c(s)\right)}{ds^2} - F_c(s) = 0 \qquad ...(5)$$

To find C. F it's E is $(D^2 - 1) = 0$
$D = \pm 1$

The solution of (5) is $F_c(s) = Ae^{x} + Be^{-x}$ ...(6)

when $s = 0$, from (2)

$$F_c(s) = \sqrt{\frac{2}{\pi}}\left[\tan^{-1} x\right]_0^\infty = \sqrt{\pi/2} \qquad ...(7)$$

and from (3)

$$\frac{dF_c(s)}{ds} = -\sqrt{\pi/2} \qquad ...(8)$$

Therefore (6) gives

$$A + B = \sqrt{\pi/2} \qquad ...(9)$$

$$A - B = -\sqrt{\pi/2} \qquad ..(10)$$

Hence $A = 0$, and $B = \sqrt{\pi/2}$ ...(11)

Therefore $F_c(s) = \sqrt{\pi/2}\ e^{-s}$ ...(12)

From (3) we see that $\dfrac{dF_c(s)}{ds} = -F_s\left[\dfrac{x}{1+x^2}\right]$

Therefore $F_s\left[\dfrac{x}{1+x^2}\right] = -\dfrac{dF_s}{ds} = \sqrt{\pi/2}\ e^{-s}$

Example – 17: Find f(x) if its cosine transform is $x^n e^{-ax}$.

Sol^n: From definition

$$f(x) = F_c^{-1}[x^n\ e^{-ax}] = F_c[x^n\ e^{-ax}]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty x^n\ e^{-ax}\ \cos(xs)\,dx \qquad ...(1)$$

Since $\displaystyle\int_0^\infty e^{-ax}\ \cos(bx)\,dx = \frac{a}{a^2 + b^2}$ ...(2)

therefore $\displaystyle\int_0^\infty e^{-ax}\ \cos(xs)\,dx = \frac{a}{a^2 + s^2}$ ...(3)

Differentiating n times with respect to a, we get

$$\int_0^\infty (-x)^n e^{-ax}\ \cos(xs)\,dx = \frac{d^n}{da^n}\left[\frac{a}{a^2 + s^2}\right] \qquad ...(4)$$

From (1) and (4)

$$f(x) = \sqrt{\frac{2}{\pi}}\,(-1)^n \frac{d^n}{da^n}\left(\frac{a}{a^2+s^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,(-1)^n\,(-1)^n \frac{n!}{x^{n+1}}\{\cos\,(n+1)\theta\}\,(\sin\theta)^{(n+1)}$$

where $\theta = \tan^{-1}\dfrac{n}{a} = \sqrt{\dfrac{2}{\pi}}\,n!\,\dfrac{\cos\{(n+1)\theta\}}{(s^2+a^2)^{(n+1)/2}}$

Example –18: Find the sine and cosine transforms of e^{-x} and use the inversion formulae to recover the original function, in both the cases.

Sol^n: We know $F_c(s) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_1^{\infty} f(x)\cos\,sx\,dx$ and $F_s(s) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\sin\,sx\,dx$

Let $f(x) = e^{-x}$ ...(1)

Then $F_c[s] = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} e^{-x}\cos(sx)\,dx = \sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{1+s^2}$ ...(2)

and $F_s[s] = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} e^{-x}\sin\,(xs)\,dx = \sqrt{\dfrac{2}{\pi}}\,\dfrac{s}{1+s^2}$ ...(3)

Applying inversion formula on (2), we get

$$f(x) = e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\,\cos\,sx\,ds \qquad ...(4)$$

$$\Rightarrow e^{-x} = \sqrt{\frac{2}{x}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\frac{\cos sx\,ds}{1+s^2} \qquad ...(5)$$

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^{\infty}\frac{\cos\,sx\,ds}{1+s^2}$$

$$\Rightarrow \frac{\pi}{2}e^{-x} = \int_0^{\infty}\frac{\cos\,sx}{1+s^2}ds$$

Again applying inversion formula on (3)

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\,\sin\,sx\,ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}.\frac{s}{s^2+1}\,\sin\,sx\,ds$$

$$e^{-x} = \frac{2}{x}\int_0^{\infty}\frac{s\,\sin\,sx}{s^2+1}\,ds$$

$$\therefore\quad \int_0^{\infty}\frac{s\,\sin sx}{1+s^2}ds = \frac{\pi}{2}e^{-x}$$

Exercise – 7

1. Find the (a) Fourier cosine Integral and (b) Fourier sine integral (representation) of
$$f(x)=\begin{cases} \sin x & if \quad 0\le x\le \pi \\ 0 & if \quad x>\pi \end{cases}$$
2. Find the Fourier transform of
$$f(x)=\begin{cases} \frac{1}{2a} & if \quad |x|\le a \\ 0 & if \quad |x|>a \end{cases}$$
3. Using Fourier Integral show that the given integral represent the indicated functions.

(a) $$\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}dw=\begin{cases} 0 & if \quad x<0 \\ \pi/2 & if \quad x=0 \\ \pi e^{-x} & if \quad x>0 \end{cases}$$

(b) $$\int_0^\infty \frac{\cos xw}{1+w^2}dw=\frac{\pi}{2}e^{-x} if\ x>0$$

(c) $$\int_0^\infty \frac{1-\cos w\pi}{w}\sin xwdw=\begin{cases} \frac{\pi}{2} & if \quad 0<x<\pi \\ 0 & if \quad x>\pi \end{cases}$$

4. Find the Fourier cosine and sine integrals of $f(x)$ = e $^{-kx}$ (x > 0, k > 0)
5. Find the Fourier Integral representation of $f(x)=\begin{cases} x & |x|<1 \\ 0 & |x|>1 \end{cases}$
6. Find the Fourier cosine integral representation of the function given by indicating points.

(a) $$f(x)=\begin{cases} x & if \quad 0<x<a \\ 0 & if \quad x>a \end{cases}$$

(b) $f(x)=e^{-x}\cos x$

(c) $$f(x)=\begin{cases} 1 & if \quad 0<x<1 \\ 0 & if \quad x>1 \end{cases}$$

7. Find the Fourier sine integral representation of the function given by indicating points.

(a) $$f(x)=\begin{cases} 1 & if \quad 0<x<a \\ 0 & if \quad x>a \end{cases}$$

(b) $$f(x)=\begin{cases} \sin x & if \quad 0<x<\pi \\ 0 & if \quad x>\pi \end{cases}$$

(c) $f(x)=e^{-ax}-e^{-bx}$

8. Find the Fourier transform of $f(x)$.

Where $f(x)=\begin{cases} 1 & if \quad |x|<a \\ 0 & if \quad |x|>a \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin ax}{x}dx$

9. Find Fourier cosine and sine transform of $f(x) = \begin{cases} k & if \quad 0 < x < a \\ 0 & if \quad x > a \end{cases}$

10. Find the Fourier transform of $f(x)$

    (a) $f(x) = \begin{cases} x & if \quad |x| \le a \\ 0 & if \quad |x| > a \end{cases}$

    (b) $f(x) = xe^{-x} \quad 0 \le x < \infty$

    (c) $f(x) = \begin{cases} cos\, x & if\ 0 < x < 1 \\ 0 & otherwise \end{cases}$

11. Find Fourier cosine and Fourier sine transform of the given function indicated below. $f(x) = e^{-ax}$ for $x \ge 0$, $a > 0$.

12. Find Fourier sine and cosine transform of $2e^{-5x} + 5e^{-2x}$.

13. Find Fourier cosine transform of $f(x) = \begin{cases} x & if \quad 0 < x < 1 \\ 2 - x & if \quad 1 < x < 2 \\ 0 & if \quad x > 2 \end{cases}$

14. Find $f(x)$ if its Fourier cosine transform is $\frac{1}{1+s^2}$ and Fourier sine transform is $\frac{1}{1+s^2}$

15. Find the Inverse Fourier sine transform of $\frac{1}{s} e^{-as}$.

16. Find $f(x)$ whose Fourier cosine transform is $\frac{sin\, as}{s}$.

17. Solve for $f(x)$ the integral equation $\int_0^\infty f(x)\ \sin xt\ dx = \begin{cases} 1 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$

18. Solve for $f(x)$ the integral equation $\int_0^\infty f(x) cos\, \alpha x dx = e^{-\alpha}$.

Answers

1. **(a)** $f(x) = \frac{2}{\pi} \int_0^\infty \frac{1 + cos\, \alpha \pi}{1 - \alpha^2} cos\, \alpha x\, d\alpha$

(b) $f(x) = \frac{2}{\pi} \int_0^\infty \frac{sin\, \alpha \pi}{(1 - \alpha^2)} sin\, \alpha x\, d\alpha$

2. $\frac{sin(\alpha a)}{2\pi a \alpha}$

4. $\int_0^\infty \frac{cos\, wx}{k^2 + w^2} dw = \frac{\pi}{2k} e^{-kx}, \int_0^\infty \frac{w \sin wx}{k^2 + w^2} dw = \frac{\pi}{2} e^{-kx}$

5. $f(x) = \int_{-\infty}^\infty \frac{sin\, \alpha - \alpha\, cos\, \alpha}{i \pi \alpha^2} e^{i\alpha x} d\alpha$

6. **(a)** $\frac{2}{\pi}\int_0^\infty \left[\frac{a\,sin\,aw}{w}+\frac{cos\,aw-1}{w^2}\right]\cos xw\,dw$

(b) $\frac{2}{\pi}\int_0^\infty \frac{(s^2+2)cos\,sxds}{s^4+4}$ **(c)** $\frac{2}{\pi}\int_0^\infty \frac{sin\,w\cdot cos\,xw}{w}dw$

7. **(a)** $\frac{2}{\pi}\int_0^\infty \frac{1-cos\,aw}{w}sin\,xw\,dw$ **(b)** $\frac{2}{\pi}\int_0^\infty \frac{sin\,xw}{1-w^2}sin\,xw\,dw$

(c) $\frac{2}{\pi}\int_0^\infty \frac{(b^2-a^2)s\cdot sin\,sxds}{(a^2+s^2)(b^2+s^2)}$

8. $\frac{2\,sin\,sa}{s}dx$ for s ≠ 0, for s = 0, F(s) = 2, Integral = $x/2$

9. $\left(F_c(f(x))=\sqrt{\frac{2}{\pi}}k\left(\frac{sin\,aw}{w}\right), F_s(f(x))=\sqrt{\frac{2}{\pi}}k\left(\frac{1-cos\,aw}{w}\right)\right)$

10. **(a)** $\frac{2i}{s^2}\;as\cos sa-\sin sa$ **(b)** $\frac{1}{2\pi}\frac{1+is}{(1+s^2)^2}$

(c) $\frac{1}{4\pi}\left[\frac{sin(s+1)}{(s+1)}+\frac{sin(s-1)}{s-1}+i\left\{\frac{1-cos(s+1)}{s+1}+\frac{1-cos(s-1)}{s-1}\right\}\right]$

11. $\frac{a}{a^2+\alpha^2},\frac{\alpha}{a^2+\alpha^2}$

12. $F_s(f(x))=\frac{2s}{s^2+25}+\frac{5s}{s^2+4}, F_s(f(x))=\frac{10}{s^2+4}+\frac{10}{s^2+25}$

13. $\frac{2\,cos\,s-cos\,2s-1}{s^2}$

14. $f(x)=e^{-x}, f(x)=e^{-x}$

15. $\frac{2}{\pi}tan^{-1}\frac{x}{a}$

16. $f(x)=\begin{cases}1 & if\;\; x<a\\ 0 & if\;\; x>a\end{cases}$

17. $f(x)=\frac{2}{\pi x}[1+cos\,x-2\,cos\,2x]$

18. $\frac{2}{\pi(1+x^2)}$