MODULE – V

[Fourier series, Fourier expansion of functions of any period, Even and odd functions, Half range Expansion, Fourier transform and Fourier Integral.]

Fourier Series

Fourier Transforms

STUDY MATERIAL

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CHAPTER – 6

Fourier Series

6.0 : *Introduction*

In the early eighteenth century D 'Alembert and Euler had given solutions in closed form for the problem of a vibrating string, using two ordinary functions. Where as Daniel Burnoulli (1700-1782) had found a solution in terms of an **infinite series** of trigonometric functions. These series, named after the French physicist Joseph Fourier (1768-1830), represent a very powerful tool in connection with solutions of various problems involving ordinary and partial differential equations. Fourier under took systemetic study of the subject in his memorable monograph "Theories analytic dela chaleur", in 1882 while dealing with the problems of heat conduction along a bar.

These series has a deep influence in the further development of mathematics and mathematical physics. Several phenomenon that are studied in engineering are periodic in nature. They are repeated after an interval of time. For example, the current and voltage in an alternating circuit, conduction of heat in solids, the displacement velocity and acceleration of a piston, electrodynamics and wave propagation in different types of media and many parameters in a vibrating system are all periodic.

In many engineering problems especially in the study of periodic phenomenon in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. In this chapter we discuss the basic concepts relating Fourier series and obtain Fourier series development of several functions.

6.1 *Periodic Functions*

Definition : A function $f : R \to R$ is said to be periodic if there exists a positive number T such that $f(x + T) = f(x)$ for all real numbers x and T is called a period of f. If a periodic function has a smallest positive period T, then T is called the primitive period of f.

Example :

1. The trigonometric functions $\sin x$ and $\cos x$ are periodic functions with primitive period 2π.
2. $\sin 2x$ and $\cos 2x$ are periodic functions with primitive period π.
3. The constant function $f(x) = c$ is a periodic function. In fact every positive real number is a period of f and hence this periodic function has no primitive period.
4. Let $f : R \to R$ be the function defined by

$$f(x)=\begin{cases}0 & if\ \ x\ is\ rational\\ 1 & if\ \ x\ is\ irrational\end{cases}.$$

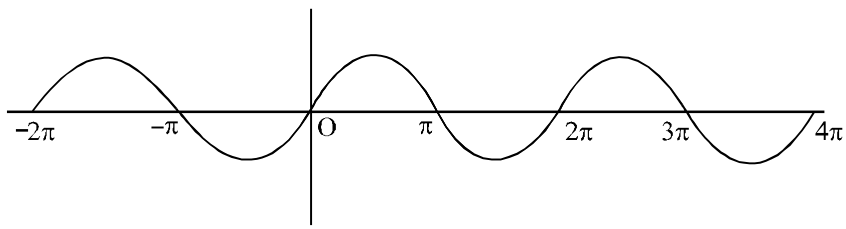
Let T be any rational number. If x is rational, then x + T is also rational and if x is irrational, then x + T is also irrational. Hence

$$f(x+T)=\begin{cases}0 & if\ \ x\ is\ rational\\ 1 & if\ \ x\ is\ irrational\end{cases}=f(x)$$

Hence every rational number is a period of f and f has no primitive period.

Remark. Let f be a periodic function with period T. If the values $f(x)$ are known in an interval of length T, then by periodicity, $f(x)$ can be determined for all x. Hence the graph of a periodic function is obtained by periodic repetition of its graph in any interval of length T.

***Example* – 1.** The graph of the periodic function sin x is given in *(fig.6.1)*



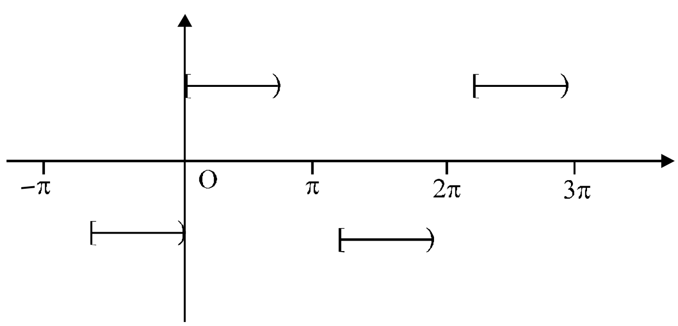
(Fig.6.1)

2. Let f be the periodic function defined by

$$f(x)=\begin{cases}-1 & if\ -\pi\le x<0\\ 1 & if\ 0\le x<\pi\end{cases}$$

and $f(x + 2\pi) = f(x)$.

The graph of f is given in *(fig.6.2)*



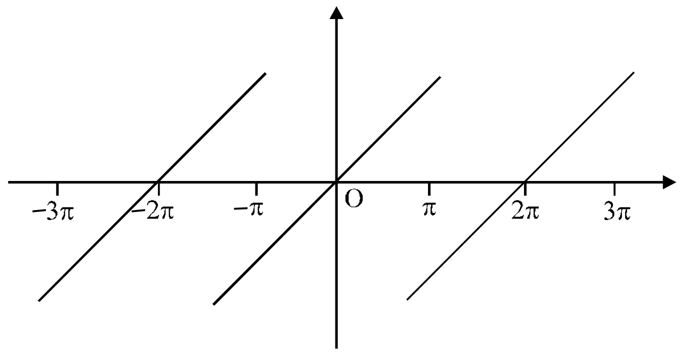
(Fig.6.2)

3. Let f be the periodic function defined by

$$f(x) = x \text{ if} -\pi < x < \pi$$

and $f(x + 2\pi) = f(x)$

The graph of f is given in *(fig.6.3)*



(Fig.6.3)

Theorem – 1. **Let f and g be periodic functions with period T and let a and b be real numbers. Prove that af + bg is also a periodic functions with period T.**

Proof : Since f and g are periodic with period T.

$$f(x+T) = f(x) \quad ....(1)$$

and $g(x + T) = g(x)$ for all x. ....(2)

Now $(af + bg)(x + T) = af(x + T) + bg(x + T)$

$= af(x) + bg(x)$ (by (1) and (2))

$= (af + bg)(x)$

Hence $af + bg$ is periodic with period T.

Theorem – 2. **If T is a period of f, prove that nT where n is any positive integer, is also a period of f.**

Proof : $f(x) = f(x + T) = f(x + 2T) = ... = f(x + (n-1)T) = f(x + nT)$

it follows that nT is a period of f.

Theorem – 3. **Let n be any positive integer. Prove that sin nx is a periodic function with period 2π/n.**

Proof : Since $\sin x$ a periodic function with period 2π, we have

$$\sin(x + 2\pi) = \sin x \text{ for all } x. \quad ....(1)$$

Now let $g(x) = \sin nx$.

Then $g\left(x + \dfrac{2\pi}{n}\right) = \sin\left(n\left(x + \dfrac{2\pi}{n}\right)\right)$

$= \sin(nx + 2\pi) = \sin nx = g(x)$ by (1).

Hence $g(x) = \sin nx$ is a periodic function with period $2\pi/n$.

Theorem – 4. **Let $f(x)$ be a periodic function with period T. Prove that for any positive real number a, $f(ax)$ is a periodic function with period T/a.**

Proof : We have $f(x + T) = f(x)$ for all x.

Let $g(x) = f(ax)$

Now $g\left(x+\frac{T}{a}\right)=f\left(a\left(x+\frac{T}{a}\right)\right)$

$$= f(ax + \mathrm{T}) = f(ax) = g(x).$$

Hence $g(x)$ is a periodic function with period T/a.

Theorem – 5. For a periodic function of period 2π, prove that

(i) $\displaystyle\int_{\alpha}^{\beta} f(\mathrm{x})\,\mathrm{dx} = \int_{\alpha+2\pi}^{\beta+2\pi} f(\mathrm{x})\,\mathrm{dx}$ (ii) $\displaystyle\int_{-\pi}^{\pi} f(\mathrm{x})\,\mathrm{dx} = \int_{\alpha}^{\alpha+2\pi} f(\mathrm{x})\,\mathrm{dx}$

(iii) $\displaystyle\int_{-\pi}^{\pi} f(\mathrm{x})\,\mathrm{dx} = \int_{-\pi}^{\pi} f(\gamma+\mathrm{x})\,\mathrm{dx}$

α, β, γ being any numbers.

Proof : (i) For a periodic function of period 2π, we know that

$f(\mathrm{t}-2\pi) = f(\mathrm{t})$

Hence putting $\mathrm{t} - 2\pi = \mathrm{x}$, $\mathrm{t} = \mathrm{x} + 2\pi$ for all α & β, we get

$$\int_{\alpha}^{\beta} f(\mathrm{x})\mathrm{dx} = \int_{\alpha+2\pi}^{\beta+2\pi} f(\mathrm{t}-2\pi)\mathrm{dt} = \int_{\alpha+2\pi}^{\beta+2\pi} f(\mathrm{t})\,\mathrm{dt} = \int_{\alpha+2\pi}^{\beta+2\pi} f(\mathrm{x})\,\mathrm{dx}$$

(ii) $$\int_{\alpha}^{\alpha+2\pi} f(\mathrm{x})\mathrm{dx} = \int_{\alpha}^{-\pi} f(\mathrm{x})\mathrm{dx} + \int_{-\pi}^{\pi} f(\mathrm{x})\mathrm{dx} + \int_{\pi}^{\alpha+2\pi} f(\mathrm{x})\mathrm{dx} = \int_{\alpha}^{-\pi} f(\mathrm{x})\mathrm{dx} + \int_{-\pi}^{\pi} f(\mathrm{x})\mathrm{dx} + \int_{-\pi}^{\alpha} f(\mathrm{x})\mathrm{dx} \quad \text{(by i)}$$

$$= \int_{\alpha}^{-\pi} f(\mathrm{x})\,\mathrm{dx} + \int_{-\pi}^{\pi} f(\mathrm{x})\,\mathrm{dx} - \int_{\alpha}^{-\pi} f(\mathrm{x})\,\mathrm{dx} = \int_{-\pi}^{\pi} f(\mathrm{x})\mathrm{dx}$$

(iii) Let $\gamma + \mathrm{x} = \mathrm{t}$

when $x = -\pi$, $\mathrm{t} = \gamma - \pi$

$\mathrm{x} = \pi$, $\mathrm{t} = \gamma + \pi$

$$\therefore \int_{-\pi}^{\pi} f(\gamma+x)dx = \int_{\gamma-\pi}^{\gamma+\pi} f(t)dt = \int_{\gamma-\pi}^{-\pi} f(t)dt + \int_{-\pi}^{\pi} f(t)dt + \int_{\pi}^{\gamma+\pi} f(t)dt$$

$$= \int_{\gamma+\pi}^{\pi} f(t)dt + \int_{-\pi}^{\pi} f(t)dt - \int_{\gamma+\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} f(\mathrm{t})\mathrm{dt} = \int_{-\pi}^{\pi} f(\mathrm{x})\mathrm{dx}$$

These results, in fact mean that the integral of a periodic function over any interval whose length is equal to it's period.

Illustrative Examples

Example -1 : Determine the period of each of the following function

(i) sin^2x (ii) $|cos\ x|$ (iii) $x - [x]$

Sol^n:

(i) We have $sin^2 x = \frac{1-cos\,2x}{2} = \frac{1}{2} - \frac{1}{2}cos\,2x = f(x)$

Since cos 2x is a periodic function with period $\frac{2\pi}{2} = \pi$.

So $f(x)$ is periodic with period π.

(ii) We have $f(x) = |cos\,x| = \sqrt{cos^2 x}$ $\left(\because |x| = \sqrt{x^2}\right)$

$= \sqrt{\frac{1+cos\,2x}{2}}$

So $f(x) = |cos\ x|$ is periodic with period $= \frac{2\pi}{2} = \pi$.

(iii) let $f(x) = x - [x]$ be a periodic fucntion with period T.

them $f(x + T) = f(x)$ for all $x \in R$

$\Rightarrow x + T - [x + T] = x - [x]$ for all $x \in R$

$\Rightarrow [x + T] - [x] = T$ for all $x \in R$

$\Rightarrow T = 1, 2, 3, 4$ ....

Since T is the smallest positive real number such that $f(f + T) = f(x)$ for all $x \in R$. therefore $T = 1$.

Example -2: Give an example of a sinusoidal periodic function whose period is

(i) $\frac{c}{z+k}$ (ii) $\frac{\pi}{k}$

Sol^n: Since sin x is a peridic function with period 2π. So sin ax is periodic with period $\frac{2\pi}{a}$.

(i) Here $\frac{2\pi}{a} = \frac{c}{z+k} \Rightarrow a = \frac{2\pi(z+k)}{c}$

So the required function is $sin\left\{\frac{2\pi(z+k)}{c}\right\}x$

(ii) Here $\frac{2\pi}{a} = \frac{\pi}{k} \Rightarrow a = 2k$

Hence the required function is sin $2\,kx$.

Example - 3 : Find, Fundamental Period of the followings

Sin 2x, $Cos\,\pi x$, $Cos^2\pi x$

Sol^n: Period of the function $f(x) = \sin 2x$ is $\frac{2\pi}{2} = \pi$

Period of the function $f(x) = \cos \pi x$ is $\frac{2\pi}{\pi} = 2$

Period of the function $f(x) = \cos 2\pi x$ is $\frac{2\pi}{2\pi} = 1$

Example - 4 : Find the smallest positive period T of the followings

$$Sin\ nx,\ Sin\left(\frac{2\pi x}{k}\right),\ Cos\left(\frac{2\pi nx}{k}\right)$$

Sol^n: Period of the function $f(x) = \sin nx$ is $\frac{2\pi}{n}$

Similarly, period of the function $f(x) = \sin\left(\frac{2\pi x}{k}\right) = k$ is $\frac{2\pi}{\left(\frac{2\pi}{k}\right)} = k$

Period of the fucntion $f(x) = cos\left(\frac{2\pi nx}{k}\right)$ is $\frac{2\pi}{\left(\frac{2\pi n}{k}\right)} = \left(\frac{k}{n}\right)$

Example - 5 : If f(x) and g(x) have period T, show that h(x) = a f(x) + bg(x) where a and b are constants, has the period T. Thus all functions of period T from a vector space.

Sol^n: Since $f(x)$ and $g(x)$ are given to be periodic functions having a period T, so we have

$f(x + T) = f(x)$ for all x

$g(x + T) = g(x)$ for all x

Hence $h(x + T) = a f(x + T) + bg(x + T)$

$= a f(x) + b\, g(x)$

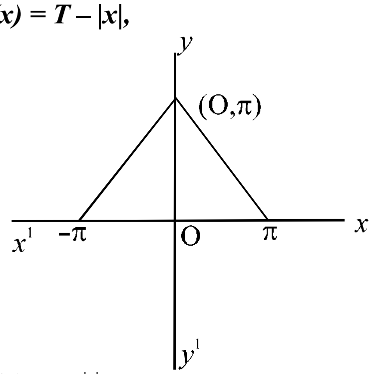
$= h(x)$ for all x

So $h(x)$ a period T.

Draw the graph of the following functions from Ex.-6 to Ex. - 10.

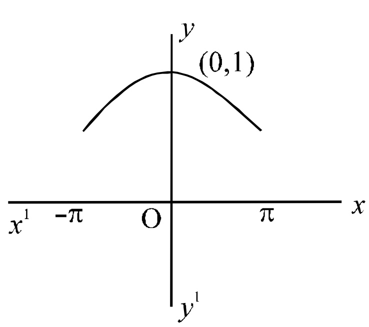
Example - 6 : f(x) = T – |x|,

Sol^n:



Example - 7 : f(x) = e$^{-|x|}$

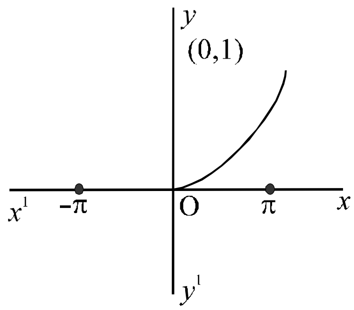
Sol^n:



Example - 8 : $f(x) = \begin{cases} 0 \text{ if } -\pi < x < 0 \\ x^2 \text{ if } 0 < x < \pi \end{cases}$

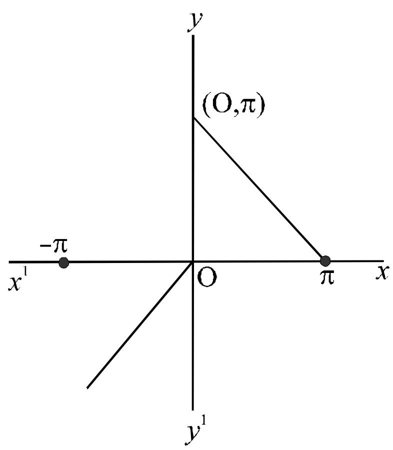
Sol

Sol^n:



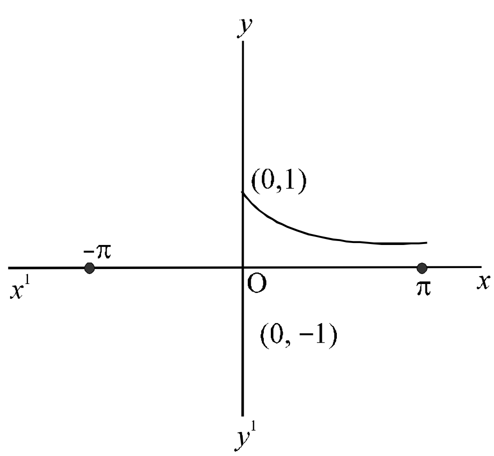
Example - 9 : $f(x) = \begin{cases} x \quad \text{if } -\pi < x < 0 \\ \pi - x \text{ if } 0 < x < \pi \end{cases}$

Sol^n:



Example - 10: $f(x) = \begin{cases} 0 \quad \text{if } -\pi < x < 0 \\ e^{-x} \text{ if } 0 < x < \pi \end{cases}$

Sol^n:



6.2 : Even and odd Functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$. Thus cos x, sec x, x^2, x^4 are even functions. In general, all functions having even powers are even *function*s e.g.

$\sin^n x$, x^n, $\cos^n x$, where n is an even positive integer are even fucntions.

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$. In general all functions having odd powers are odd functions e.g. $\sin^n x$, x^n, where n is an odd integer.

Thus sin x, cosec x, tan x^3 are odd functions

Let the sufficfess'e' and 'o' denote even and odd respectively. If $f(x)$ is an even *function* by def^n $f_e(-x) = f_e(x)$

If $f(x)$ is an odd function by def^n $f_o(-x) = -f_o(x)$.

Any *function*s $f(x)$ can be written in terms of it's even and odd parts such that

$f(x) = f_e(x) + f_o(x)$ ....(1)

Replacing x by – x

$f(-x) = f_e(-x) + f_o(-x) = f_e(x) - f_o(x)$.........(2)

Adding (1) & (2)

$$f(x) + f(-x) = 2f_e(x) \Rightarrow f_e(x) = \frac{f(x) + f(-x)}{2}$$

Substracting (2) from (1)

$$f(x) - f(-x) = 2f_o(x) \Rightarrow f_o(x) = \frac{f(x) - f(-x)}{2}$$

Thus, if $f(x)$ is given we can always find even and odd points. Even and odd functions possess the following properties.

Sum of even functions = even function

sum of odd functions = odd function

even function × even function = even function

odd function × odd function = even function

even function × odd function = odd function.

Thus if $f(x)$ is an odd function $f(x)$ cos nx is also an odd function but $f(x)$ sin nx is an even function.

If an even function is added to an odd function the result is neither even nor odd. The graph of an even function is symmetrical about y - axis while the graph of an odd fucntion is symmetrical about origin.

We also know that

$\int_{-a}^{a} f(x)dx = 0,$ where $f(x)$ is an odd function and

$\int_{-a}^{a} f(x)dx = 2\int_{o}^{a} f(x)dx,$ where $f(x)$ is an even function

Illustration : Determine whether the following functions are even or odd.

(i) $x\left(\dfrac{a^x-1}{a^x+1}\right)$ ***(ii)*** $\log\left(x+\sqrt{x^2+1}\right)$ ***(iii)*** $\sin x-\cos x$

Sol ^{n}***:*** (i) Let $f(x)=x\left(\dfrac{a^x-1}{a^x+1}\right)$

Then $f(-x)=(-x)\left(\dfrac{a^{-x}-1}{a^{-x}+1}\right)=(-x)\left(\dfrac{1/a^x-1}{1/a^x+1}\right)=(-x)\left(\dfrac{1-a^x}{1+a^x}\right)$

$=x\left(\dfrac{a^x-1}{a^x+1}\right)=f(x)$

So $f(x)$ is an even function.

(ii) let $f(x)=\log\left(x+\sqrt{x^2+1}\right)$

Then $f(x)+f(-x)=\log\left(x+\sqrt{x^2+1}\right)+\log\left(-x+\sqrt{x^2+1}\right)$

$=\log(-x^2+x^2+1)=\log 1=0$

$f(x)+f(-x)=0$

$\therefore f(-x)=-f(x)$

So $f(x)$ is an odd function.

(ii) Let $f(x)=\sin x-\cos x$

Then $f(-x)=\sin(-x)-\cos(-x)=-\sin x+\cos x$, clearly $f(-x)$ is neither equal to $f(x)$ nor to $-f(x)$.

So $f(x)$ is neither an even function nor an odd fucntion.

6.3 : Some useful Integrals

The following integrals are useful in determining Fourier series co-efficients, where m & n are integers.

(1) $\displaystyle\int_\alpha^{\alpha+2\pi}\cos nx\,dx=\int_\alpha^{\alpha+2\pi}\sin nx\,dx=0$

(2) $\displaystyle\int_\alpha^{\alpha+2\pi}\cos nx.\cos mx\,dx=\int_\alpha^{\alpha+2\pi}\sin mx\,\sin nx\,dx=0$, $(m\neq n)$

(3) $\displaystyle\int_\alpha^{\alpha+2\pi}\sin nx.\cos nx\,dx=\int_\alpha^{\alpha+2\pi}\cos nx\;\sin nx\,dx=0$, $(n\neq 0)$

(4) $\displaystyle\int_\alpha^{\alpha+2\pi}\cos^2 nx\,dx=\int_\alpha^{\alpha+2\pi}\sin^2 nx\,dx=\pi$, $(n\neq 0)$

(5) $\displaystyle\int_{-\pi}^{\pi}\cos mx.\cos nx\,dx=\begin{cases}0 & m\neq n\\ \pi, & m=n>0\\ 2\pi, & m=n=0\end{cases}$

(6) $\int_{-\pi}^{\pi} \sin mx.\sin nx dx = \begin{cases} 0, & m \neq n, \text{ and } m = n = 0 \\ \pi & \text{for } m = n > 0 \end{cases}$

(7) $\int_{-\pi}^{\pi} \cos mx.\sin nx dx = 0$

Following results will be found useful while attempting problems of Fourier series

(1) $\sin n\pi = \cos\left(n + \frac{1}{2}\right)\pi, = 0, n \in I$

Thus $\sin \pi = \sin 2\pi = \sin 3\pi = \ldots\ldots\ldots\ldots.= 0$

& $\cos \frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \ldots\ldots\ldots\ldots.= 0$

(2) $\cos n\pi = \sin\left(n + \frac{1}{2}\right)\pi = (-1)^n, n \in I$

Thus $\cos \pi = \cos 3\pi = \cos 5\pi = \ldots\ldots\ldots= -1$

$\cos 0 = \cos 2\pi = \cos 4\pi = \ldots\ldots\ldots=1$

$\sin \frac{\pi}{2} = \sin \frac{5\pi}{2} = \sin \frac{9\pi}{2} = \ldots\ldots\ldots\ldots = 1$

$\sin \frac{3\pi}{2} = \sin \frac{7\pi}{2} = \sin \frac{11\pi}{2} = \ldots\ldots\ldots.= -1$

(3) The rule for repeated integration by parts

$\int u\, v dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 = \ldots\ldots$

$u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2}, u''' = \frac{d^3u}{dx^3} \ldots\ldots\ v_1 = \int v dx, v_2 = \int v_1\, dx, v_3 = \int v_2\, dx \ldots\ldots\ldots\ldots\ldots$

(4) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx\right] + c$

(5) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}\left[a \cos bx + b \sin bx\right] + c$

Fourier Series :

Since periodic functions that occur in engineering problems are rather complicated, representation of periodic functions in terms of a simple periodic functions is a matter of great practical importance. We now discuss the problem of representing various functions of period 2π in terms of the simple functions 1, $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, ... $\sin nx$, $\cos nx$, ...

Definition : A series of the form

$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx + \ldots$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

where a_n and b_n are real constants is called a trigonometric series. a_n and b_n are real coefficients of the series.

Since each term of the trigonometric series is a function of period 2π, it follows that if the series conveneges, then the sum is also a function of period 2π.

Let *f(x)* be a periodic function with period 2π, which can be repeated by the following trignometric series

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + ...+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + ...$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ....(1)$$

The series (1) is called Fourier series and it's co-efficients a_0, a_n & b_n (n = 1, 2, 3.......) are called Fourier co-efficient of *f(x)*,

6.4 : *Euler's Formulae*

The Fourier series for the function *f(x)* in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ......(1)$$

where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$$

These values a_0, a_n, & b_n are known as Euler's formulae.

Proof : Let us assume for the time being that the series can be integrated term by term between the limit

α & $\alpha + 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ....(1)$$

To find a_0, Integrating (1) both sides with respect to x between the limits α & $\propto +2\pi$, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \frac{a_0}{2}\int_{\pi}^{\alpha+2\pi} dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx$$

$$= \frac{a_0}{2}[x]_{\alpha}^{\alpha+2\pi} + \sum_{n=1}^{\infty} a_n \left[\frac{\sin nx}{n}\right]_{\alpha}^{\alpha+2\pi} + b_n \sum_{n=1}^{\infty}\left[\frac{-\cos nx}{n}\right]_{\alpha}^{\alpha+2\pi} = \frac{a_0}{2} \times 2\pi + 0 + 0 = a_0\pi$$

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx$$

To find a_n Multiply (1) by cosnx & integrte from x = α to x = α + 2π then we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi}\cos nx\, dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi}\cos nx.\cos nx dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi}\sin nx.\cos nx dx$$

$$= \frac{a_0}{2}\left[\frac{\sin nx}{n}\right]_{\alpha}^{\alpha+2\pi} + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi}\cos^2 nx dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi}\sin nx.\cos nx dx$$

The first and third integral on the right hand side are zero but second integral is equal to π.

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx = a_n \pi$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx$$

To find b_n : Multiplying each side by (1) in sin nx and integrate from x = α to α + 2π, then

$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\, dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi}\sin nx\, dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi}\sin nx \,.\, \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi}\sin nx \,.\, \sin nx\, dx$$

$$= 0 + 0 + b_n \,.\, \pi$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx$$

Cor – 1 : Making α = 0, the integral becomes 0 < x < 2π, and Formulae (1) reduces to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

Cor 2 : Putting α = –π, the interval becomes –π < x < π, the Formulae (1) becomes

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

Working Rule

Let *f (x)* be a periodic function with period 2π. If the given interval is (–π, π) :

Step – 1. Check whether *f* (*x*) is an even function or odd function.

Step – 2. (a) If *f* (*x*) is an even function, then $b_n = 0$ for all *n* and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx, \text{ for all } n \geq 0.$$

(b) If *f* (*x*) is an odd function then $a_n = 0$ for all n ≥ 0 and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx\,.$$

Step – 3. If $f(x)$ is neither an even function nor an odd function in $(-\pi, \pi)$ or if the given interval is not $(-\pi, \pi)$, then calculate the Fourier coefficients by using Euler's formulae.

Illustrative Examples

Example –1 : Find the Fourier expansion of $f(x) = x$ in $-\pi < x < \pi$.

Sol^n: Let $f(x) = x$. Since $f(x)$ is an odd functions, $a_n = 0$ for all $n \geq 0$.

Also, $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{-2}{\pi}\left[\pi\cos n\pi\right] = \frac{-2\cos n\pi}{n}$$

$$= \frac{-2(-1)^n}{\pi} = \frac{2(-1)^{n+1}}{n}$$

$$\therefore x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$$

$$= 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

Example – 2. Find the Fourier series for $f(x) = x^2$ in $-\pi < x \leq \pi$ and deduce that

$$\textbf{(i)}\ \sum_{n=1}^{\infty}\frac{1}{n^2}\ \textbf{and}\ \textbf{(ii)}\ \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\quad \textbf{(iii)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\ \textbf{(iv)}\ \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

***Sol^n :* (i)** Let $f(x) = x^2$; so that its Fourier representation is expressed as

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ ....(1)}$$

where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{+\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}.2\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right] = (-1)^n.\frac{4}{n^2}$$

and $$b_n = \int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx = 0$$

Substituting these values in (1); we get

$$f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{-1}{3^2}\cos 3x...\right]$$

Substituting $x = \pi$; we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - \ldots..\right) \Rightarrow \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..\right)$$

$$\Rightarrow \frac{\pi^2}{6} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..\right) \Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

(ii) $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots..$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots. + \frac{1}{2^2} + \frac{1}{4^2} + \ldots. - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \ldots..$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots\ldots + \ \ldots.. - \frac{1}{2^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..\right)$$

$$= \sum \frac{1}{n^2} - \frac{1}{2^2}\sum \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{1}{4} \times \frac{\pi^2}{6} = \frac{3\pi^2}{24} = \frac{\pi^2}{8}$$

(iii) Now we have to prove $\sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

L.H.S. $\frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$

Now the given series will be

$$x^2 = \frac{\pi^2}{3} + \sum \frac{4(-1)^n}{n^2} \cos\ nx$$

put $x = 0$

$$0 = \frac{\pi^2}{3} + \sum \frac{4(-1)^n}{n^2} \cos\ 0 \qquad \text{or} \qquad \frac{-\pi^2}{3} = \sum \frac{4(-1)^n}{n^2}$$

or $$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \qquad \text{or} \qquad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$$

(iv) $$\sum \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos x}{n^2}, \ a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{4(-1)^n}{n^2},\ b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad .... (1)$$

Multiplying $f(x)$ in (1) and integrate within limit $-\pi$ to π

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx + a_n \sum \int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$\Rightarrow \quad \int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx + a_n \sum \int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$\Rightarrow \quad \int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx + \sum a_n \int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\Rightarrow \quad \int_{-\pi}^{\pi}(x^2)^2 dx = \frac{a_0}{2}\int_{-\pi}^{\pi} x^2 dx + \sum a_n \int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} x^4 dx = \frac{a_0}{2} \times \pi a_0 + a_n \sum a_n \pi$$

$$\Rightarrow \quad \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \pi/2\, a_0^2 + \sum a_n{}^2 \pi$$

$$\Rightarrow \quad \left(\frac{\pi^5}{5} + \frac{\pi^5}{5}\right) = \left[\frac{a_0^2}{2} + \sum a_n{}^2\right]\pi \Rightarrow \frac{2\pi^5}{5} = \pi\left(\frac{4\pi^4}{18} + \sum \frac{16}{n^4}\right)$$

$$\Rightarrow \quad \frac{2\pi^4}{5} = \frac{4\pi^4}{18} + \sum \frac{16}{n^4} \Rightarrow \quad \frac{2\pi^4}{5} - \frac{4\pi^4}{18} = \sum \frac{16}{n^4}$$

$$\Rightarrow \quad \frac{36\pi^4 - 20\pi^4}{90} = 16\sum \frac{1}{n^4} = \frac{16\pi^4}{90} = 16\sum \frac{1}{n^4}$$

$$\Rightarrow \quad \sum \frac{1}{n^4} = \frac{\pi^4}{90}$$

Example – 3 : Find the Fourier series to represents $x - x^2$ from $x = -\pi$ to π

Sol^n : Given, $f(x) = x - x^2$, $x = -\pi$ to $x = \pi$

Let the Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

so, $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}(x - x^2)dx = \dfrac{1}{\pi}\left(\dfrac{x^2}{2} - \dfrac{x^3}{3}\right)_{-\pi}^{\pi} = \left[\dfrac{1}{\pi}\left(\dfrac{\pi^2}{2} - \dfrac{\pi^3}{3}\right) - \left(\dfrac{\pi^2}{2} + \dfrac{\pi^3}{3}\right)\right]$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = -\frac{2\pi^2}{3} \quad ...... (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx \quad = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi} \quad [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\left[(1-2x)\frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} \quad = \frac{1}{n^2\pi}[(1-2\pi)\cos n\pi - (1+2\pi)\cos n\pi]$$

$$= \frac{\cos n\pi}{n^2\pi}[1 - 2\pi - 1 - 2\pi] \; = \frac{-4\pi(-1)^n}{n^2\pi} \; = \frac{-4(-1)^n}{n^2} \quad ......(2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{-\cos nx}{n}\right) - 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi} \quad = -\frac{1}{\pi}\left[\frac{(x - x^2)\cos nx}{n} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi}\left[\left(\frac{(\pi - \pi^2)\cos n\pi}{n} + \frac{2\cos n\pi}{n^3}\right) - \left\{\left(-\pi - \pi^2\right)\frac{\cos n\pi}{2} + \frac{2\cos n\pi}{n^3}\right\}\right]$$

$$= -\frac{1}{\pi}\left[\left((\pi - \pi^2)\frac{\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - (-\pi - \pi^2)\frac{\cos n\pi}{n} - \frac{2\cos n\pi}{n^3}\right)\right]$$

$$= -\frac{\cos n\pi}{\pi n}(\pi - \pi^2 + \pi + \pi^2) = -\frac{-2(-1)^n}{n} \quad ...... (1)$$

$\therefore$ So the required Fourier series becomes

$$-\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left(-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2}.......\right) - 2\left(\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3}\right)$$

$$= -\frac{\pi^2}{3} + 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2}.......\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}.........\right)$$

Example – 4 : Obtain the Fourier series for $f(x) = e^{-x}$, in the interval $0 < x < 2\pi$

Sol**^{n}**: Given $f(x) = e^{-x}$, $0 < x < 2\pi$.

Let the Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{-1}{\pi}\left[e^{-x}\right]_0^{2\pi} = \frac{-1}{\pi}\left[e^{-2\pi} - 1\right] = \frac{1-e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\, dx$$

$$\text{Let } \int_0^{2\pi} e^{-x}\cos nx\, dx = \left[e^{-x}.\frac{\sin nx}{n}\right]_0^{2\pi} - \int_0^{2\pi}(-1)e^{-x}\left(\frac{-\sin nx}{n}\right)dx = \frac{1}{n}\int_0^{2\pi} e^{-x}\sin nx\, dx$$

$$= \frac{1}{n}\left\{\left[e^{-x}\frac{(-\cos nx)}{n}\right]_0^{2\pi} - \int_0^{2\pi}(-1)\,e^{-x}\left(\frac{-\cos nx}{n}\right)dx\right\}$$

$$= \frac{1}{n}\left\{\left[-e^{-x}\frac{\cos nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi} e^{-x}\cos nx\, dx\right\} = \frac{-1}{n^2}\left[e^{-2\pi}\cos 2n\pi - 1\right] - \frac{1}{n^2}\int_0^{2\pi} e^{-x}\cos nx dx$$

$$\Rightarrow \left(1+\frac{1}{n^2}\right)\int_0^{2\pi} e^{-x}\cos nx dx = \frac{1-e^{-2\pi}}{n^2} \Rightarrow \int_0^{2\pi} e^{-x}\cos nx dx = \frac{1-e^{-2\pi}}{n^2}\times\frac{n^2}{n^2+1} = \frac{1-e^{-2\pi}}{n^2+1}$$

$$\Rightarrow \int_0^{2\pi} e^{-x}\cos nx dx = \frac{1-e^{-2\pi}}{n^2+1} \Rightarrow \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx dx = \frac{1-e^{-2\pi}}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}.\sin nx\, dx$$

$$\text{Let } \int_0^{2\pi} e^{-x}.\sin nx\, dx = \left[e^{-x}\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi} e^{-x}(-1).\left(\frac{-\cos nx}{n}\right)dx$$

$$= \left(-e^{-2\pi}\frac{\cos 2n\pi}{n} + \frac{e^0.\cos 0}{n}\right) - \frac{1}{n}\int_0^{2\pi} e^{-x}\cos nx\, dx$$

$$= \frac{1}{n}(1-e^{-2\pi}) - \left[\frac{1}{n}\left\{e^{-x}\frac{\sin nx}{n}\right\}_0^{2\pi} - \int_0^{2\pi} e^{-x}(-1)\frac{\sin nx}{n}\right] \quad (\because \cos 2n\pi = 1)$$

$$= \frac{1}{n}(1-e^{-2\pi}) - \frac{1}{n^2}\int_0^{2\pi} e^{-x}\sin nxdx \quad [\because \sin\, n\pi = 0]$$

$$\Rightarrow \left(1+\frac{1}{n^2}\right)\int_0^{2\pi} e^{-x}\sin nxdx = \frac{1}{n}(1-e^{-2\pi})$$

$$\Rightarrow \int_0^{2\pi} e^{-x}\sin nx dx = \frac{1-e^{-2\pi}}{n} \times \frac{n^2}{(n^2+1)} = \frac{n(1-e^{-2\pi})}{n^2+1}$$

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx dx = \frac{n(1-e^{-2\pi})}{\pi(n^2+1)}$$

so the required Fourier series becomes

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{1-e^{-2\pi}}{\pi(n^2+1)}\cos nx + \sum_{n=1}^{\infty}\frac{n(1-e^{-2\pi})}{\pi(n^2+1)}\sin nx$$

$$= \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{(n^2+1)} + \sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx\right\}$$

Example – 5. Expand f(x) = cos ax as a Fourier series in (–π, π) where a is not an integer.

Sol^n : Since cos ax is an even function, $b_n = 0$ for all n.

Now $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \cos ax\, dx = \dfrac{2}{\pi}\left[\dfrac{\sin ax}{a}\right]_0^{\pi} = \dfrac{2}{\pi}\dfrac{\sin a\pi}{a}$

$$\therefore \frac{a_0}{2} = \frac{\sin a\pi}{a\pi}.$$

Also, $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \cos ax \cos nx\, dx$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\cos(n+a)x + \cos(n-a)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+a)x}{n+a} + \frac{\sin(n-a)x}{n-a}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n+a)\pi}{n+a} + \frac{\sin(n-a)\pi}{n-a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(n-a)\sin(n+a)\pi + (n+a)\sin(n-a)\pi}{n^2-a^2}\right]$$

$$= \frac{n\left[\sin(n+a)\pi + \sin(n-a)\pi\right] - a\left[\sin(n+a)\pi - \sin(n-a)\pi\right]}{n\left(n^2-a^2\right)}$$

$$= -\frac{n(2\sin n\pi \cos a\pi) - a(2\cos n\pi \sin a\pi)}{\pi\left(n^2-a^2\right)} = -\frac{2a\cos n\pi \sin a\pi}{\pi\left(n^2-a^2\right)} = \frac{2a(-1)^{n+1}\sin a\pi}{\pi\left(n^2-a^2\right)}$$

$\therefore$ The Fourier series is given by

$$f(x) = \cos ax = \frac{\sin a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{2a(-1)^{n+1}\sin a\pi}{\pi\left(n^2-a^2\right)}\right]\cos nx.$$

Example – 6. Find the Fourier series corresponding to $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in (0, 2π). Hence prove that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

Sol^n **:** $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}dx=\frac{1}{\pi}\left[\frac{(\pi-x)^3}{-12}\right]_0^{2\pi}=\frac{-1}{12\pi}\left[-\pi^3-\pi^3\right]=\frac{\pi^2}{6}$.

Now $a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$

$$=\frac{-1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}\right]_0^{2\pi}+\frac{2}{4n\pi}\int_0^{2\pi}(\pi-x)\sin nx$$

$$=\frac{1}{2n\pi}\int_0^{2\pi}(\pi-x)d\left(-\frac{\cos nx}{n}\right)=\frac{-1}{2n^2\pi}\left[-\pi(-1)^{2n}-\pi\right]-0$$

$$=\frac{1}{n^2}$$

Also $b_n=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$

$$=\left[-\frac{1}{4\pi}(\pi-x)^2\frac{\cos nx}{n}\right]_0^{2\pi}-\frac{1}{2n\pi}\int_0^{2\pi}(\pi-x)\cos nx$$

$$=\left[\frac{-1}{2n^2\pi}(\pi-x)\sin nx\right]_0^{2\pi}-\frac{1}{2n^2\pi}\int_0^{2\pi}\sin nx\,dx=0$$

$$\therefore\left(\frac{(\pi-x)^2}{2}\right)^2=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2}\right).$$

Put $x=0$ in the above result. $\frac{\pi^2}{4}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2}\right)$

$$\therefore\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{4}-\frac{\pi^2}{12}=\frac{\pi^2}{6}.$$

6.5 : *Conditions For A Fourier Series*

Dirichlet conditions :

So far we have obtained Fourier series for several functions that it is possible to expand most of the functions as Fourier series. It was the German Mathematician peter Gustav Lejeune Dirichlet (1805-1859) who gave the well-known sufficient conditions for uniform convergence of the Fourier series as stated below.

Any function *f(x)* can be expanded as a Fourier series $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \text{sinnx}$ in the interval [α, α + 2π], where a_0, a_n, b_n are costants, provided that

(i) *f(x)* is periodic, single valued & finite

(ii) *f(x)* has only a finite number of finite discontinuities and has no infinite discontinuities.

(iii) *f(x)* has only a finite number of local maxima and minima

These conditions are known as Dirichlet conditions.

In otherwards if *f(x)* is a function defined in a given interval [α, α + 2π] and extended outside this interval by *f* (x + 2π) = *f(x)* having finite number of points of ordinary discontinuity in the interval [α, α + 2π] and having a finite number of local maxima and minima there, then the series.

$$\frac{a_0}{2}+\sum a_n \cos nx+\sum b_n \sin nx$$

with co-efficients a_0, a_n, b_n converges to *f(x)* at every points of continuity and to ½ [*f* (x + 0) + *f* (x – 0)] at a point of discontinuity.

Here *f* (x + 0) and *f* (x – 0) denote the limit on the right and the limit on the left respectively. At a point of discontinuity the sum of the series is equal to the mean of the limits on the right and the left.

Note : Consider the function *f(x)* = sin $\frac{1}{x}$ in the interval (– π, π). It attains it's maximum value when

$$\frac{1}{x} = (2n-1)\,\frac{\pi}{2} \qquad \text{or,} \qquad x = \frac{2}{(2n-1)\pi}$$

Where n is zero or an integer. For large values of n as n → ∞ , the values of x as given by (i) tend to become indefinitely small and are croweded near to the value x = 0. Thus the function $\sin\frac{1}{x}$ has an infinite number of maxima and minima in the neighbourhood at x = 0, Since as such one of the Dirichlet conditions not satisfied. Hence it is not possible to expand *f(x)* = sin $\frac{1}{x}$ in the interval (–π, π) as it contains x = 0 also.

Simillarly, the function *f(x)* = $\frac{1}{x-a}$ has an infinite discontinuity at x = a, and as such it cannot be expanded as Fourier series in the interval which contain the point x = a.

6.6 : *Fourier series for Discontinuous Function*

Let the function *f(x)* defined by *f(x)*

$$f(x) = \begin{cases} f_1(x), \alpha < x < x_0 \\ f_2(x), x_0 < x < \alpha + 2\pi \end{cases}$$

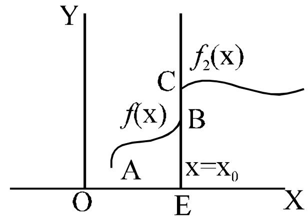
where x_0 is the point of discontinuity in the interval (α. α + 2π)

In such cases also we obtain the Fourier series for *f(x)* in the usual way. The values of a_0, a_n, b_n are evaluated by *(fig. 6.4)*

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)dx + \int_{x_0}^{\alpha+2\pi} f_2(x)dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\cos nxdx + \int_{x_0}^{\alpha+2\pi} f_2(x)\cos nxdx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{x_0} f_1(x)\sin nxdx + \int_{x_0}^{\alpha+2\pi} f_2(x)\sin nxdx\right]$$



(Fig.6.4)

If $x = x_0$, is the point of the finite discontinuity, the sum of the Fourier series

$$= \frac{1}{2}\left[\lim_{h\to 0} f(x_0 - h) + \lim_{h\to 0} f(x_0 + h)\right] = \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right]$$

Note :(1) It may be seen from the graph, that at a point of finite discontinuity at $x = x_0$, there is a finite jumps equal to BC in the value of the function *f(x)* at $x = x_0$.

(2) A given function *f(x)* may be defined by different formula in different regions. Such types of functions are quite common in Fourier series.

(3) At a point of discontinuity the sum of the series is equal to the mean of the limits on the right and left.

Illustrative Examples

Example – 1 : **State give reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.**

(1) cosec x, (Ans - No) *f(x)* = cosec x, $= \frac{1}{\sin x}$ in the interval $(-\pi, \pi)$. It attains maximum value when $x = \frac{1}{n\pi + (-1)^n \frac{\pi}{2}}$ ....(1)

where n is zero or an integer. For large values of n as $n \to 0$, the values of x as given by (1) tend to become indefinitely small and crowded near to the value x = 0. Thus function $\frac{1}{\sin x}$ has an infinite number of maxima and minima in the neighbourhood of x = 0, As such one of the Dirichlet's conditions is not satisfied. It is not possible to expand *f(x)* = cosec x in the interval $(-\pi, \pi)$ as it contains x = 0 also

(ii) *f(x)* $= \sin\frac{1}{x}$, No, because in the interval $(-\pi, \pi)$. It attains it's maximum value when

$$\frac{1}{x} = (2n-1)\frac{\pi}{2} \quad \text{or} \quad x = \frac{2}{(2n-1)\pi} \quad ...... (1)$$

where n is zero or integer

For large values of n as $n \to \infty$, the values of x as given by (i) tend to become indefinitely. Small and are crowded near to the value x = 0. Thus the function sin x has an infinite number of maxima and minima in the neighbourhood of x = 0. As such one of the Dirichlet's condition is not satisfied. Hence it is not possible to expand *f(x)* $= \sin\frac{1}{x}$ in the interval $(-\pi, \pi)$ as it contains x = 0 also.