Example – 2. **Find the Fourier series expansion for *f(x)*, if**

$$f(x) = \begin{cases} -\pi, -\pi < x < 0, \\ x, 0 < x < \pi \end{cases}$$

deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$

Sol^n : Let the Fourier series be,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{Given } f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-\pi)dx + \int_{0}^{\pi} x\,dx\right\} = \frac{1}{\pi}\left\{-\pi[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi)(0+\pi) + \left(\frac{\pi^2}{2} - 0\right)\right\} = \frac{1}{\pi}\left\{(-\pi^2) + \frac{\pi^2}{2}\right\}$$

$$= \frac{1}{\pi}\left(\frac{-\pi^2}{2}\right) = \frac{-\pi}{2} \quad ....(1)$$

$$a_n = \int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\cos nx dx + \int_{0}^{\pi} x\cos nx\,dx\right\} = \frac{1}{\pi}\left\{\left[-\pi\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{0 + 0 + \left[\frac{\cos nx}{n^2}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{1}{n^2\pi}\left((-1)^n - 1\right) = \frac{(-1)^n - 1}{n^2\pi} \quad ....(2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-\pi)\sin nx + \int_{0}^{\pi} x\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[(-\pi)\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \left[x\left(\frac{(-\cos nx)}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi\cos nx}{n}\right]_{-\pi}^{0} - \left[\frac{x\cos nx}{n}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{\frac{\pi}{n}[\cos nx]_{-\pi}^{0} - \frac{1}{n}[x\cos nx]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}\left[1-\cos n\pi\right]-\frac{1}{n}\left[\pi\cos n\pi - 0\right]\right\} = \frac{1}{n}(1-(-1)^n) - \frac{(-1)^n}{n} = \frac{1-2(-1)^n}{n}$$

Hence the required Fourier series is,

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi}\cos nx + \sum_{n=1}^{\infty}\frac{1-2(-1)^n}{n}\sin nx$$

$$= -\frac{\pi}{4} + \left(\frac{-2}{\pi 1^2}\cos x - \frac{2}{\pi 3^2}\cos 3x \ldots\ldots\right) + \left(\frac{3}{1}\sin x - \frac{1}{2}\sin 2x + \sin 3x \ldots.\right)$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots.\right) + \left(3\sin x - \frac{\sin 2x}{2} + \sin 3x \ldots..\right) \ldots\ldots\ldots(1)$$

which is required result

Putting x = 0, in (1) we obtain

$$\Rightarrow \quad f(0) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) \ldots\ldots\ldots..(ii)$$

Now $f(x)$ is dis continuous at x = 0, As a matter of fact

$f(0-0) = -\pi$, & $f(0+0) = 0$

$$f(0) = \tfrac{1}{2}[f(0-0) + f(0+0) = \frac{-\pi}{2}$$

Hence putting this in the eq^n (iii)

$$\Rightarrow \quad \frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\right] \quad \Rightarrow \quad \frac{\pi}{4} - \frac{\pi}{2} = \frac{-2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\Rightarrow \quad -\frac{\pi}{4} = \frac{-2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right] \quad \Rightarrow \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Example – 3. **Find the Fourier series to represent the function *f(x)*, given by**

$$f(x) = \begin{cases} x \text{ for } 0 \le x \le \pi \\ 2\pi - x \text{ for } \pi \le x \le 2\pi \end{cases}$$

Deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\infty = \frac{\pi^2}{8}$

***Sol^n* :** Given, $f(x) = \begin{cases} x, \text{ for } 0 \le x \le \pi \\ 2\pi - x, \text{ for } \pi \le x \le 2\pi \end{cases}$

Let the Fourier series , be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right\} = \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi^2}{2} - 0\right) + \left\{\left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right\}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \left(4\pi^2 - 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right)\right\} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left\{\pi^2\right\} = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\cos nx}{n^2}\right]_0^{\pi} - \left[\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}\left\{\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) - \left(\frac{\cos 2n\pi}{n^2} - \frac{\cos n\pi}{n^2}\right)\right\}$$

$$= \frac{2(\cos n\pi - 1)}{n^2\pi^2} = \frac{2((-1)^n - 1)}{n^2\pi}$$

$$a_1 = \frac{-4}{1^2\pi}, a_2 = 0, a_3 = \frac{-4}{3^2\pi}, a_4 = 0.......$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} + \left[-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(-\pi\frac{\cos n\pi}{n} - 0\right) + \left(0 + \pi\frac{\cos n\pi}{n}\right)\right\} = \frac{1}{\pi}\left\{-\pi\frac{\cos n\pi}{n} + \pi\frac{\cos n\pi}{n}\right\} = 0$$

Hence, the required Fourier series becomes,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2 \pi} \cos nx + 0$$

$$= \frac{\pi}{2} + \left(\frac{-4}{1^2 \pi} \cos x + \frac{(-4)}{3^2 \pi} \cos 3x + \frac{(-4)}{5^2 \pi} \cos 5x + \ldots\ldots \right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots \right)$$

Let putting x = 0, we have

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\ldots \right) \ldots\ldots\text{(ii)}$$

Now, *f(x)* is continuous at $x = \pi$

$f(\pi - 0) = \pi$, & $f(\pi + 0) = \pi$

so $f(\pi) = \frac{1}{2}[f(\pi + 0) + f(\pi - 0)] = \frac{2\pi}{2} = \pi$

Putting this value in the eq^n (ii)

$$\pi = \frac{\pi}{2} + \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \right) \qquad \Rightarrow -\frac{\pi}{2} + \pi = \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \right)$$

$$\Rightarrow \left(\frac{\pi}{2} \right) \times \left(\frac{\pi}{4} \right) = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \qquad \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Hence it's proved.

Example – 4 : If f (x) = $\begin{cases} 0 & \text{for } -\pi < x < 0 \\ \sin x & \text{for } 0 \le x < \pi \end{cases}$

Prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos mx}{4m^2 - 1}$

Hence show that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots\ldots\ldots\ldots \infty = \frac{1}{4}(\pi - 2) \text{ and } \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} + \cdots = \frac{1}{2}$$

Sol^n : If $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ \sin x & \text{for } 0 \le x < \pi \end{cases}$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi}\sin x dx\right] \quad = \frac{1}{\pi}\int_{0}^{\pi}\sin x dx = \frac{1}{\pi}\left[(-\cos x)\right]_0^{\pi} = \frac{1}{\pi}(-\cos\pi + \cos 0) = \frac{2}{\pi}$$

$$a_0 = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\, dx + \int_{0}^{\pi}\sin x.\cos nx\, dx\right] \quad = \frac{1}{2\pi}\int_{0}^{\pi} 2.\sin x.\cos nx\, dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx = \frac{1}{2\pi}\left[\left\{\int_{0}^{\pi}\sin(n+1)x\, dx\right\} - \left\{\int_{0}^{\pi}\sin(n-1)x\, dx\right\}\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-\cos(n+1)x}{(n+1)}\right]_0^{\pi} - \left[\frac{-\cos(n-1)x}{(n-1)}\right]_0^{\pi}\right\} = \frac{-1}{2\pi}\left\{\left[\frac{\cos(n+1)x}{(n+1)}\right]_0^{\pi} - \left[\frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}\right\}$$

$$= \frac{-1}{2\pi}\left[\left(\frac{\cos(n+1)\pi}{n+1} - \frac{1}{n+1}\right) - \left(\frac{\cos(n-1)\pi}{n-1} - \frac{1}{n-1}\right)\right]$$

$$= -\frac{1}{2\pi}\left[\frac{(-1)^{n+1}}{n+1} - \frac{1}{n+1} - \frac{(-1)^{n-1}}{n-1} + \frac{1}{n-1}\right]$$

Hence $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{-2}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$

$$a_n = \frac{-((-1)^n + 1}{\pi(n^2-1)}$$

a_1= is not defined so,

$$a_n = \frac{1}{\pi}\int_{0}^{\pi}\sin x.\cos x dx$$

$$\Rightarrow \quad a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin x.\cos x\ dx = \frac{1}{2\pi}\int_{0}^{\pi}(2\sin x\cos x)\ dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\sin 2x\ dx = \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{4\pi}[\cos 2\pi - \cos 0] = 0$$

$$\Rightarrow \quad a_1 = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi}\sin x.\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\sin x.\sin nx\, dx = \frac{1}{2\pi}\int_{0}^{\pi}(2\sin x.\sin nx)\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}(\cos(n-1)x - \cos(n+1)x)\,dx = \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin x.\sin nx\,dx$$

$$\Rightarrow \quad b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x.\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2\pi}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - (0-0)\right] = \frac{1}{2\pi}\times\pi = \frac{1}{2}$$

So the required Fourier series becomes

$$f(x) = \frac{1}{\pi} + \sum_{n=2}^{\infty}\frac{-((-1)^n+1)}{\pi(n^2-1)}\cos nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1}\cos nx + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{1}{\pi}\left(\frac{2\cos 2x}{3} + \frac{2}{15}\cos 4x + \ldots\ldots\right) + \frac{1}{2}\sin x$$

$$\Rightarrow \quad f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots\ldots\right) + \frac{1}{2}\sin x$$

Putting the value of x = 0

$$\Rightarrow \quad f(0) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \ldots\ldots\right) + 0 \quad \Rightarrow \quad f(0) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\ldots\right) \ldots\ldots\ldots(i)$$

Let us taken $x = \pi/2$, $\sin \pi/2 = 1$

So, $f(0 - \pi/2) = 0$ & $f(0 + \pi/2) = 1$

$$f(0) = \frac{1}{2}\left[f\left(0 - \pi/2\right) + f\left(0 + \pi/2\right)\right] = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\ldots\ldots\right) \quad \Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\ldots\ldots\right)$$

$$\Rightarrow \quad \frac{1}{\pi} - \frac{1}{2} = \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\ldots\ldots\right) \quad \Rightarrow \quad \frac{2-\pi}{2\pi} = \frac{2}{\pi}\left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\ldots\ldots\right)$$

$$\Rightarrow \quad \frac{2-\pi}{4} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\ldots\ldots$$

Putting $x = 0$ in (1) $\sin 0 = 0$

$$0=\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{1}{1\cdot 3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\cdots\right]$$

$$\therefore \frac{2}{\pi}\left[\frac{1}{1\cdot 3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\cdots\right]=\frac{1}{\pi}$$

$$\therefore \frac{1}{1\cdot 3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\cdots=\frac{1}{2}$$

Example – 5. **Find the Fourier series of the following function**

$$f(x)=\begin{cases} x^2, 0\le x\le \pi \\ -x^2, -\pi < x \le 0 . \end{cases}$$

Sol^n: Let, $f(x)$ = $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x).dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x^2\ dx+\int_0^{\pi} x^2 dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[-\frac{x^3}{3}\right]_{-\pi}^{0}+\left[\frac{x^3}{3}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{\left(0-\frac{\pi^3}{3}\right)+\left(\frac{\pi^3}{3}-0\right)\right\}=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x^2.\cos nx\, dx+\int_0^{\pi} x^2 \cos nx\, dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[-x^2.\frac{\sin nx}{n}-(-2x).\left(\frac{-\cos nx}{n^2}\right)+(-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{0}\right.$$

$$\left.+\left[x^2\frac{\sin nx}{n}-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}\right\}$$

$$=\frac{1}{\pi}\left\{\left[-2x\frac{\cos nx}{n^2}\right]_{-\pi}^{0}+\left[2x\frac{\cos nx}{n^2}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{\left(\frac{-2\pi\cos n\pi}{n^2}\right)+\left(\frac{2\pi\cos n\pi}{n^2}\right)\right\}=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2 \sin nx\, dx+\int_0^{\pi} x^2 \sin nx\, dx\right]$$

$$=\frac{1}{\pi}\left\{\left[-x^2\left(\frac{-\cos nx}{n}\right)-(-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{0}\right.$$

$$\left.+\left[x^2\left(\frac{-\cos nx}{n}\right)-2x\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}\right\}$$

$$=\frac{1}{\pi}\left\{\left[x^2\frac{\cos nx}{n}-2\frac{\cos nx}{n^3}\right]_{-\pi}^{0}+\left[\frac{-x^2\cos nx}{n}+2\frac{\cos nx}{n^3}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\left(0-\frac{2}{n^3}\right)-\left(\frac{\pi^2\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}\right)\right]+\left[\left(\frac{-\pi^2\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}\right)-\left(0+\frac{2}{n^3}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[-\frac{2}{n^3}-\frac{\pi^2\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}-\frac{\pi^2\cos n\pi}{n}+\frac{2\cos n\pi}{n^3}-\frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4}{n^3}-\frac{2}{n}\pi^2\cos n\pi-\frac{4}{n^3}\cos n\pi\right] = \frac{1}{\pi}\left[-\frac{4}{n^3}(1+\cos n\pi)-\frac{2\pi^2}{n}\cos n\pi\right]$$

$$= \frac{-2}{\pi}\left[\frac{2}{n^3}(1+\cos n\pi)-\frac{\pi^2}{n}\cos n\pi\right] = -\frac{2}{\pi}\left[\frac{2}{n^3}(1+\cos n\pi)-\frac{\pi^2}{n}\cos n\pi\right]$$

$$= -\frac{2}{\pi}\left[\frac{2}{n^3}(1+(-1)^n)-\frac{\pi^2}{n}(-1)^n\right]$$

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{n^3}(1+(-1)^n)-\frac{\pi^2}{n}(-1)^n\right].\sin nx$$

***Example – 6*. Find the series of sines and cosines of multiples of x which represents *f(x)* in the interval $-\pi < x < \pi$.**

$$f(x)=\begin{cases} 0, \text{ when } -\pi < x < 0 \\ \dfrac{\pi x}{4}, \text{ when } 0 < x \le \pi \end{cases}$$

and hence deduce that $\dfrac{\pi^2}{8} = 1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\ldots$

Sol*[n] *: Let *f(x)* be represented by Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

$$\text{where } a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx \qquad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx dx \qquad \ldots(3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx dx \qquad \ldots(4)$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[0+\int_0^{\pi}\frac{\pi x}{4}dx\right] = \frac{1}{\pi}.\frac{\pi}{4}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{4}.\frac{\pi^2}{2} = \frac{\pi^2}{8} \qquad \ldots..(5)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx + \int_0^{\pi} f(x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[0+\int_0^{\pi}\frac{\pi x}{4}\cos nx\,dx\right] = \frac{1}{4}\int_0^{\pi} x\cos nx\,dx = \frac{1}{4}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{1}{4n^2}(1-\cos n\pi) = -\frac{1}{4n^2}[1-(-1)^n] = \frac{(-1)^n-1}{4n^2} \quad ....(6)$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \int_0^{\pi}\frac{\pi x}{4}\sin nx = -\frac{(-1)^n\pi}{4n} \quad ...(7)$$

Substituting the values of a_0, a_n and b_n; we get

$$f(x) = \frac{\pi^2}{16}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{4n^2}\cos nx+\sum_{n=1}^{\infty}\left\{-\frac{(-1)^n\pi}{4n}\sin nx\right\}$$

$$= \frac{\pi^2}{16}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{4n^2}\cos nx-\frac{(-1)^n\pi}{4n}\sin nx\right]$$

$$= \frac{\pi^2}{16}+\left(-\frac{1}{2}\cos x+\frac{\pi}{4}\sin x\right)-\frac{\pi}{4.2}\sin 2x-\frac{1}{2.3^2}\cos 3x+\frac{\pi}{4.3}\sin 3x \quad .... (8)$$

This is the required representation of *f(x)* substituting x = π in (8)

$$f(\pi) = \frac{\pi^2}{16}+\frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+....\right) \quad ...(9)$$

But $f(\pi) = \frac{1}{2}[f(\pi-0)+f(\pi-0)]$

$$= \frac{1}{2}\left[0+\left(\frac{\pi x}{4}\right)_{x=\pi}\right] = \frac{\pi^2}{8}$$

Hence equation (9) gives

$$\frac{\pi^2}{8} = \frac{\pi^2}{16}+\frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+....\right)$$

$$\frac{\pi^2}{16} = \frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+....\right)$$

Hence $\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+....$

Example – 7*. If** $f(x) = \begin{cases} -c & \text{when } -\pi < x < 0 \\ c & \text{when } 0 \le x < \pi \end{cases}$ **and** $f(x+2\pi) = f(x)$ ***for all x, obtain the Fourier for f(x). Deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots = \frac{\pi}{4}$**.**

***Sol^n* :** Since f(x) is an odd function $a_n = 0$ for all n ≥ 0.

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} c\sin nx\,dx = -\frac{2c}{\pi}[\cos nx]_0^{\pi} = \frac{2c}{n\pi}\left[1-(-1)^n\right]$.

$$\therefore b_n = \begin{cases} \dfrac{4c}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ The Fourier series for $f(x)$ is given by

$$f(x) = \frac{4c}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots\right]$$

Putting $x = \dfrac{\pi}{2}$ in the abvoe result we get

$$f\left(\frac{\pi}{2}\right) = \frac{4c}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\therefore c = \frac{4c}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\therefore 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Example – 8 :* If** $f(x) = \begin{cases} -1 + x, & -\pi < x < 0 \\ 1 + x, & 0 < x < \pi \end{cases}$ ***with period 2π find the Fourier series for f(x).

Sol^n : $f(x)$ is an odd function in $(-\pi, \pi)$ since it is the sum of the odd function $g(x) = x$ and

$$h(x) = \begin{cases} -1 & \text{in } -\pi, x < 0 \\ 1 & \text{in } 0 < x < \pi \end{cases}$$

Hence $a_n = 0$ for all $n \geq 0$.

Now $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}(1 + x)\sin nx \, dx$

$$= \frac{2}{\pi}\left[\frac{-(1 + x)\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

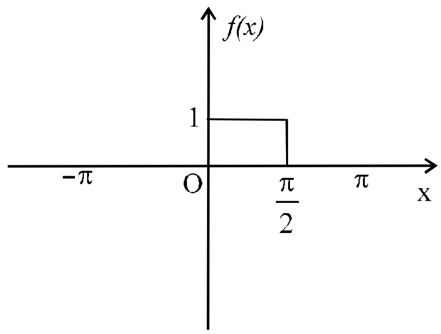
$$= \frac{2}{\pi}\left[\left(\frac{-(1 + \pi)\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - \left(-\frac{1}{n} + 0\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-(1 + \pi)(-1)^n}{n} + \frac{1}{n}\right] = \frac{2}{n\pi}\left[1 - (1 + \pi)(-1)^n\right].$$

$\therefore$ The Fourier series for $f(x)$ is given by

$$f(x) = \frac{2}{n\pi}\sum_{n=1}^{\infty}\left[1 - (1 + \pi)^n\right]\sin nx.$$

Example – 9 : Find the fourier series of the function f(x) which is assumed to have the period 2π. Shown in the given figure.



(Fig. 6.5)

Sol^n : The formula is

$$f(x) = \begin{cases} 0 \ if -\pi < x < 0 \\ 1 \ if 0 < x < \pi/2 \\ 0 \ if \ \pi/2 < x < \pi \end{cases} \quad \text{where} \quad f(x + 2x) = f(x)$$

The fourier expansion of the above periodic function $f(x)$ with period 2π in the interval $-\pi < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \text{ Cos } nx + b_n \text{ Sin } nx) \qquad \text{........(1)}$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Cos\ nx\, dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sin\ nx\, dx$

To Find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi/2} f(x)dx + \int_{\pi/2}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} 1.dx = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$

To Find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Cos\ nx\ dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) Cos\, nx\ dx + \int_{0}^{\pi/2} f(x) Cos\, nx\ dx + \int_{\pi/2}^{\pi} f(x) Cos\, nx\ dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} Cos\ nx\, dx = \frac{1}{n\pi}\left|Sin\, nx\right|_0^{\pi/2} = \frac{1}{n\pi} Sin\left(\frac{n\pi}{2}\right)$$

$$\therefore a_n = \frac{1}{n\pi} Sin\left(\frac{n\pi}{2}\right)$$

$$a_1 = \frac{1}{\pi}, a_2 = 0,\ a_3 = \frac{-1}{3\pi}, a_4 = 0, a_5 = \frac{-1}{5\pi}, a_6 = 0$$

To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,Sin\ nx\ dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) Sin\,nx\ dx + \int_{0}^{\pi/2} f(x) Sin\,nx\ dx + \int_{\pi/2}^{\pi} f(x) Sin\ nx\ dx\right]$$

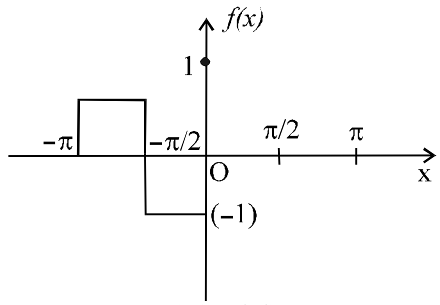
$$= \frac{1}{\pi}\int_{0}^{\pi/2} Sin\ nx\,dx = \frac{1}{n\pi}\left|Cos\,nx\right|_{0}^{\pi/2} = \frac{1}{n\pi}\left(1 - Cos\left(\frac{n\pi}{2}\right)\right)$$

$$b_1 = \frac{1}{\pi}, b_2 = \frac{2}{2\pi}, b_3 = \frac{1}{3\pi}, a_4 = 0, a_5 = \frac{1}{5\pi}, b_6 = \frac{2}{6\pi}, b_7 = \frac{1}{7\pi}$$

Hence the required fourier series of the given function $f(x)$ is

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\left(Cos\,x - \frac{1}{3}Cos3x + \frac{1}{5}Cos\,5x - \ldots\ldots\right) + \frac{1}{\pi}\left(Sin\,x + Sin\,2x + \frac{1}{3}Sin3x + \ldots\ldots\right)$$

Example – 10 : Find the fourier Series of the function f(x), which is assumed to have the period 2π. Shown in the given figure.



(Fig. 6.6)

Sol^n : The formula is in *(fig 6.6)*

$$f(x) = \begin{cases} 1\ if -\pi < x < -\pi/2 \\ -1\ if\ \pi/2 < x < 0\ and\ f(x + 2\pi) = f(x) \\ 0\ if\ 0 < x < \pi \end{cases}$$

The fourier expansion of the above periodic function $f(x)$ with period 2π in the interval $-\pi < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \text{ Cos } nx + b_n \text{ Sin } nx)$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Cos\ nx\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sin\ nx\,dx$

To Find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} f(x)dx + \int_{-\frac{\pi}{2}}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} 1 \cdot dx + \int_{-\pi/2}^{0} (-1)dx\right] = \frac{1}{\pi}\left[|x|_{-\pi}^{-\pi/2} - |x|_{-\pi/2}^{0}\right]$$

$$\therefore a_0 = \frac{1}{\pi}\left[\left(-\frac{\pi}{2}+\pi\right)-\left(0+\frac{\pi}{2}\right)\right] = \frac{1}{\pi}\left[\frac{\pi}{2}-\frac{\pi}{2}\right] = 0$$

To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{(-\pi/2)} \cos nx\, dx + \int_{-\frac{\pi}{2}}^{0} (-) \cos nx\, dx\right] = \frac{1}{n\pi}\left[|\sin nx|_{-\pi}^{\frac{-\pi}{2}} - |\sin nx|_{-\frac{\pi}{2}}^{0}\right]$$

$$= \frac{1}{n\pi}\left[-\sin\left(\frac{n\pi}{2}\right)-\left(0+\sin\left(\frac{n\pi}{2}\right)\right)\right] = \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore a_1 = \frac{-2}{\pi}, a_2 = 0, a_3 = \frac{2}{3\pi},\ a_4 = 0, a_5 = \frac{-2}{5\pi}.................$$

To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \operatorname{Sin}\ nx\ dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} f(x) \operatorname{Sin} nx\ dx + \int_{-\pi/2}^{0} f(x) \operatorname{Sin} nx\ dx + \int_{0}^{\pi} f(x) \operatorname{Sin}\ nx\ dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} \operatorname{Sin} nx\ dx - \int_{-\pi/2}^{0} \operatorname{Sin} nx\, dx\right]$$

$$= \frac{1}{n\pi}\left[|\operatorname{Cos} nx|_{-\pi/2}^{0} - |\operatorname{Cos} nx|_{-\pi}^{-\pi/2}\right]$$

$$= \frac{1}{n\pi}\left[1 - \operatorname{Cos}\left(\frac{n\pi}{2}\right) - \operatorname{Cos}\left(\frac{n\pi}{2}\right) + \operatorname{Cos} n\pi\right] = \frac{1}{n\pi}\left[1+(-1)^n - 2\operatorname{Cos}\left(\frac{n\pi}{2}\right)\right]$$

$$\therefore\ b_1 = 0,\ b_2 = \frac{4}{2\pi} = \frac{2}{\pi},\ b_3 = 0,\ b_4 = 0,\ b_5 = 0, b_6 = \frac{4}{6\pi} = \frac{2}{3\pi}$$

Hence the required fourier series of the given function $f(x)$ is given by

$$f(x) = -\frac{2}{\pi}\left(\operatorname{Cos} x - \frac{1}{3}\operatorname{Cos} 3x + \frac{1}{5}\operatorname{Cos} 5x - ......\right) + \frac{2}{\pi}\left(\operatorname{Sin} 2x + \frac{1}{3}\operatorname{Sin} 6x + \frac{1}{5}\operatorname{Sin} 10x + ...\right)$$

6.7 : *Change of Interval*

So far we have come accross with Fourier series expansions of functions having period 2π. But in many of the problems the function may have arbitary periods (not necessarily 2π). We now obtain Fourier coefficients for functions having period $2l$, where l is any positive number.

The interval $[-l, l]$, l is a real number. If 'f' is bounded, integrable and picewise monotonic in $[-l, l]$, then the sum of the series

$$f(x) = \frac{1}{2}a_0 + \Sigma\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

where $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\frac{n\pi x}{l}dx$, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$, is $\frac{1}{2}[f(x-)+f(x+)]$ for every x between $-l$ and l and is $\frac{1}{2}[f(l-)+f(l+)]$ for $x = \pm l$ and is periodic with period $2l$.

Proof : In many Engineering application we required an expansion of a given function $f(x)$ over an interval of length different from 2π. Some other interval $2l$. Let $f(x)$ be periodic function defined in the interval $(\alpha, \alpha+2l)$. To change the problem to period 2π.

Put $z = \pi x/l$, or $x = lz/\pi$.

So that when $x = \alpha$, $z = \frac{\alpha\pi}{l} = \beta\,(\text{say})$

When $x = \alpha + 2l$, $z = \frac{(\alpha+2l)\pi}{l} = \left(\frac{\alpha\pi}{l} + 2\pi\right) = \beta + 2\pi\ (say)$

Thus the function $f(x)$ is of period $2l$ in $(\alpha, \alpha + 2l)$ is transferred to the function $f(lz/\pi) = F(z)$ say of period 2π in $(\beta, \beta + 2\pi)$. Hence $f(lz/\pi)$ can be expressed as the Fourier series

$$f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \quad ...(2)$$

$$\left.\begin{aligned} Where\ a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{lz}{\pi}\right)dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{lz}{\pi}\right)\cos nz\,dz \\ b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{lz}{\pi}\right)\sin nz\,dz \end{aligned}\right\} \quad ....(3)$$

Making the inverse substitution $z = \pi x/l$, $dz = \frac{\pi}{l}dx$ in (2) & (3), the Fourier expansion of $f(x)$ in the interval $(\alpha, \alpha + 2l)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Where $a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)dx$

$a_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\frac{n\pi x}{l}dx$, $b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\frac{n\pi x}{l}dx$

Put $\alpha = 0, -l$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx, \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx, \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx, \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Illustrative Examples

Example –1 : **If $f(x) = x$ so defined in $-l < x < l$ with period $2l$, find the Fourier expansion of $f(x)$.**

Solⁿ : Since $f(x)$ is an odd function $a_n = 0$ for all $n \geq 0$.

Now $b_n = \dfrac{2}{l}\displaystyle\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{l}\left[\frac{-ln}{n\pi}\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left(\frac{-l^2\cos n\pi}{n\pi}\right) = \frac{-2l(-1)^n}{n\pi} = \frac{2(-1)^{n+1}l}{n\pi}$$

$\therefore$ The Fourier series is $x = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}l}{n}\sin\left(\frac{n\pi x}{l}\right)\right]$.

Example –2 : **Obtain the Fourier series for $f(x)$ defined in $(-1, 1)$ by**

$$f(x) = \begin{cases} c_1 & \text{if } -1 < x < 0 \\ c_2 & \text{if } 0 < x < 1 \end{cases}$$

Solⁿ : $a_0 = \displaystyle\int_{-1}^{1} f(x)dx = \int_{-1}^{0} c_1 dx + \int_0^1 c_2 dx$

$$= c_1[x]_{-1}^0 + c_2[x]_0^1 = c_1 + c_2$$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx$$

$$= \int_{-1}^{0} c_1\cos n\pi x\,dx + \int_0^1 c_2\cos n\pi x dx = \frac{c_1}{n\pi}[\sin n\pi x]_{-1}^0 + \frac{c_2}{n\pi}[\sin n\pi x]_0^1 = 0$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi\,dx$$

$$= \int_{-1}^{0} c_1 \sin n\pi x \, dx + \int_{0}^{1} c_2 \sin n\pi x \, dx \quad = -\frac{c_1}{n\pi}\left[\cos n\pi x\right]_{-1}^{0} - \frac{c_2}{n\pi}\left[\cos n\pi x\right]_{0}^{1}$$

$$= -\frac{c_1}{n\pi}\left[1-(-1)^n\right] - \frac{c_2}{n\pi}\left[(-1)^n - 1\right] \quad = \frac{c_2 - c_1}{n\pi}\left[1-(-1)^n\right].$$

$\therefore$ The required Fourier series is given by

$$f(x) = \frac{c_1 + c_2}{2} + \sum_{n=1}^{\infty}\left(\frac{c_2 - c_1}{n\pi}\right)\left[1-(-1)^n\right]\sin nx \,.$$

***Example –3 :* Expand in a Fourier series the periodic function *f(x)* = e^{-x} in the interval (–*l*, *l*)**

***Sol^n* :** Let $e^{-x} = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\dfrac{n\pi x}{l} + b_n \sin\dfrac{n\pi x}{l}\right)$ ..... (1)

Then $a_0 = \dfrac{1}{l}\int_{-l}^{l} e^{-x}dx = \dfrac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \dfrac{1}{l}\left[e^{l} - e^{-l}\right] = \dfrac{2\sin hl}{l}$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}.\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l} = \frac{2l(-1)^n \sin hl}{l^2 + (n\pi)^2}$$

$$\therefore\ a_1 = -\frac{2l\sin hl}{l^2+\pi^2}, a_2 = \frac{2l\sin hl}{l^2+2^2\pi^2}, a_3 = \frac{-2l\sin hl}{l^2+3^2\pi^2} \text{ etc.}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l} = \frac{2n\pi(-1)^n \sin hl}{l^2 + (n\pi)^2}$$

$$\therefore\ b_1 = -\frac{2\,\pi\sin hl}{l^2+\pi^2}, b_2 = \frac{4\pi\sin hl}{l^2+2^2\pi^2}, b_3 = -\frac{6\pi\sin hl}{l^2+3^2\pi^2} \text{ etc.}$$

Substituting these values in (1), we get

$$e^{-x} = \sin hl\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \ldots\right)\right.$$

$$\left. -2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \ldots\right)\right]$$

Example – 4 : **Expand in a Fourier series the periodic function *f(x)* with period $2l$ which on the interval $[-l, l]$ is given by *f(x)* = |x| .**

Sol^n : Since the function given is even, it follows that

$$b_n = 0, a_0 = \frac{2}{l}\int_0^l x\,dx = l$$

$$a_n = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}dx = \left[\frac{l^2}{n\pi}x\sin\frac{n\pi x}{l} + \frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2l}{n^2\pi^2}(\cos n\pi - 1) = 0 \text{ when n is even}$$

$$= -\frac{4l}{n^2\pi^2} \text{ when n is odd.}$$

Hence, the expansion is of the form

$$|x| = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots \frac{1}{(2l+1)^2}\cos\frac{(2l+1)}{l} + \ldots\right]$$

Example – 5 : **If $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ show that in the interval (0, 2)**

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

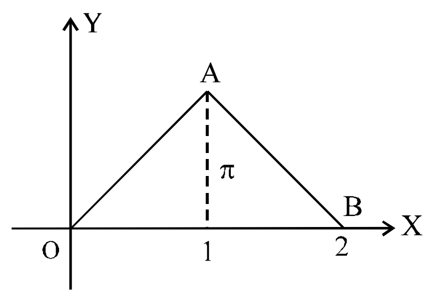


Fig.6.7

Sol^n: Let $f(x) = \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x$

Then $a_0 = \int_0^2 f(x)dx$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \pi\cdot\frac{1}{2} + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx$$

$$a_n = \int_0^1 \pi\, x \cos n\pi x\, dx + \int_1^2 \pi(2-x)\cos n\pi x\, dx$$

$$= \left[\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - \pi(-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\pi\cos n\pi}{n^2\pi^2} - \frac{\pi}{n^2\pi^2}\cos 0\right] + \left[\frac{-\pi\cos 2n\pi}{n^2\pi^2} + \frac{\pi\cos n\pi}{n^2\pi^2}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1)$$

$= 0$ or $\dfrac{-4}{n^2\pi}$ according as n is even or odd.

Similarly it can be shown that $b_n = 0$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + ...\right)$$

The function being continuous, the relation holds for all x.

i.e. f is bounded, integrable the sum of the series must be $f(0+) = 0$, at $x = 0$ & $f(\pi -) = \pi$. at $x = \pi$.

At $x = 0$, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8} \text{ (Proved)}$$

Example – 6 : **Find the Fourier series for the function**

$$f(x) = \begin{cases} 0 & \text{in } -8 < x < 0 \\ 4 & \text{in } 0 < x < 4 \\ 0 & \text{in } 4 < x < 8 \end{cases}$$

Solⁿ: $a_0 = \dfrac{1}{8}\displaystyle\int_{-8}^{8} f(x)dx = \dfrac{1}{8}\left[\int_{-8}^{0} 0\,dx + \int_{0}^{4} 4dx + \int_{4}^{8} 0\,dx\right] = \dfrac{1}{8}[4x]_0^4 = 2$.

$$a_n = \frac{1}{8}\int_{-8}^{8} f(x)cos\left(\frac{n\pi x}{8}\right)dx = \frac{1}{8}\int_{0}^{4} 4\,cos\left(\frac{n\pi x}{8}\right)dx = \frac{1}{2}\left[\frac{8}{n\pi}sin\left(\frac{n\pi x}{8}\right)\right]_0^4 = \frac{4}{4\pi}sin\left(\frac{n\pi}{2}\right).$$

$$b_n = \frac{1}{8}\int_{0}^{4} 4\,sin\left(\frac{n\pi x}{8}\right)dx = \frac{1}{2}\left[\frac{-8}{n\pi}cos\left(\frac{n\pi x}{8}\right)\right]_0^4 = \frac{-4}{n\pi}\left[cos\left(\frac{n\pi}{2}\right) - 1\right]$$

$\therefore$ The Fourier series of $f(x)$ is given by

$$f(x) = 1 + \frac{4}{\pi}\frac{\infty}{2}\frac{1}{n}sin\left(\frac{n\pi}{2}\right)\cdot cos\left(\frac{n\pi x}{8}\right) - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[cos\left(\frac{n\pi}{2}\right) - 1\right]sin\left(\frac{n\pi x}{8}\right).$$

Example – 7 : **A sinusoidal voltage E sin wt is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function**

$$U(t) = \begin{cases} 0, & \text{when } \dfrac{-T}{2} < t < 0 \\ Esinwt, & 0 < t < \dfrac{T}{2} \end{cases} \qquad v(t) = \begin{cases} 0, \text{when } T/2 < t < 0 \\ E\sin wt, \text{when } 0 < t < T/2 \end{cases}$$

and T = $2\pi/w$**, in a Fourier series.**

Solⁿ: Let $U(t) = \dfrac{a_0}{2} + \sum a_n \cos nwt + \sum b_n \sin nwt$ Since $\dfrac{2\pi}{T} = w$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} U(t)dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} 0\,dt + \int_0^{T/2} E\sin wt\,dt\right] = \frac{2}{T}\left[\frac{-E}{w}\cos wt\right]_0^{T/2}$$

$$= \frac{2E}{wT}\left[\cos\frac{wT}{2} - \cos 0\right] = \frac{-2E}{wT}(-2) = \frac{2E}{\pi}$$

Since $wT = 2\pi$, and $\cos \pi = -1$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} U(t)\cos wt\,dt \;\; = \frac{2}{T}\int_0^{T/2} E\sin wt\cos nwt\,dt$$

$$= \frac{E}{T}\int_0^{T/2}[\sin(n+1)wt + \sin(1-n)wt]dt$$

$$= \frac{-E}{T}\left[\frac{\cos(1+n)wt}{(1+n)w} + \frac{\cos(1-n)wt}{(1-n)w}\right]_0^{T/2}, n \neq 1$$

$$= \frac{-E}{T}\left[\frac{\cos(1+n)\frac{wT}{2}}{(1+n)w} + \frac{\cos(1-n)\frac{wT}{2}}{(1-n)w} - \frac{1}{(1+n)w} - \frac{1}{(1-n)w}\right]$$

$$= \frac{-E}{wT}\left[-\frac{1}{1+n} - \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right] = \frac{E}{2\pi}\left[\frac{2}{1+n} + \frac{2}{n-1}\right] = \frac{-2E}{\pi(n^2-1)}$$

When n is even, If n is odd, $(n \neq 1)$, $a_n = 0$ in case $n = 1$, we have

$$a_1 = \frac{2}{T}\int_0^{T/2} E\sin wt.\cos wt\,dt$$

$$= \frac{E}{T}\int_0^{T/2}\sin 2wT\,dt = \frac{E}{T}\left[-\frac{\cos 2wT}{2w}\right]_0^{T/2} = \frac{-E}{2wT}(\cos wt - \cos 0) = \frac{-E}{4\pi}(\cos 2\pi - \cos 0) = 0$$

$$b_1 = \frac{2}{T}\int_0^{T/2} E\sin wt\sin wt\,dt$$

$$= \frac{E}{T}\int_0^{T/2}(1-\cos 2wt)dt = \frac{E}{T}\left[t - \frac{\sin 2wt}{2w}\right]_0^{T/2}$$

$$= \frac{E}{T}\left[\frac{T}{2} - \frac{\sin wT}{2w}\right] = \frac{E}{2} \text{ since } \sin wt = \sin 2\pi = 0$$

If n ≠ 1, then simillar to a_n we have

$$b_n = \frac{2}{T}\int_0^{T/2} E\sin wt.\sin nwt\, dt, \left(\text{since} \frac{2\pi}{T} = w\right)$$

$$= \frac{E}{T}\int_0^{T/2}[\cos(n-1)wt - \cos(n+1)wt]dt = 0$$

Which can be easily evaluated

$$\therefore \quad U(t) = \frac{E}{\pi} + \frac{E}{2}\sin wt - \frac{2E}{\pi}\left(\frac{\cos 2wt}{1.3} + \frac{\cos 4wt}{3.5} + \frac{\cos 6wt}{5.7} + \ldots\right)$$

Example –8. **Find the Fourier series of the function**

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

Sol^n: $f(x) = \frac{1}{2}a_0 + \sum a_n \cos\left(\frac{n\pi x}{2}\right) + \sum b_n \sin\left(\frac{n\pi x}{2}\right)$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\left[\int_{-2}^{-1} 0.dx + \int_{-1}^{1} kdx + \int_{1}^{2} 0.dx\right] = \frac{1}{2}[kx]_{-1}^{1} = \frac{k}{2} \times 2 = k$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_0^2 k\cdot\cos\left(\frac{n\pi x}{2}\right)^{dx} = \int_0^1 k\cdot\cos\frac{n\pi x}{2}dx$$

$$= \left[k\cdot\frac{\sin n\pi x/2}{\frac{n\pi}{2}}\right]_{0a}^{1} = \frac{2k}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} \frac{2k}{n\pi}, \text{ when } n \text{ is odd} \\ 0 \text{ when } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\left[\int_{-2}^{-1} + \int_{-1}^{1} + \int_{1}^{2}\right] f(x)\sin\left(\frac{n\pi x}{2}\right)$$

$$= \int_{-1}^{1} k\sin\frac{n\pi x}{dx} = \frac{k}{2}\cdot 0 = 0$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi x}{2}\right)$$

$$\therefore f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi x}{2} + \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2}\ldots\ldots\ldots\ldots\right]$$