2.8 : Similar Matrices

Let A and B be square matrices of same order. The matrix A is said to be similar to B if there exists an invertible matrix P such that $A = P^{-1} BP$.

or PA = PB.

Multiplying P^{-1}, we have

$PAP^{-1} = (PB) P^{-1} = B (PP^{-1}) = B.I = B$

$B = PAP^{-1}$

A is similar to B if and only if B is similar to A. The matrix P is called similarity matrix.

Theorem – 1 : Similar matrix have the same characteristic equation and hence the same eigen values. Also if X is an eigen vector of corresponding to eigen value λ then $P^{-1}X$ is an eigen vector of B corresponding to the eigen value λ where P is similarity matrix.

Proof : From the above definition (B is similar to A and P is similarity matrix)

$AP = PB$ or $P^{-1} AP = B$

Let λ be the eigen value and X be the corresponding eigen vector of A.

$AX = \lambda X$ ....(1)

Now $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}(\lambda IP) = P^{-1}(A - \lambda I)P$

$$|B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I|\,|P|$$

$$= |A - \lambda I|\,|P^{-1}P| = |A - \lambda I|$$

Similar matrices have the same characteristic polynomials. Multiply both sides by an invertible matrix P^{-1}.

$P^{-1} (AX) = P^{-1}(\lambda X) = \lambda P^{-1}X$.

Let $X = PY$ $\therefore$ $P^{-1}(APY) = \lambda P^{-1}(PY)$

Or $(P^{-1}AP)Y = \lambda (P^{-1}P)Y$ or $BY = \lambda Y$ ....(2)

$\therefore$ B has the same eigen value λ as that of A which shows that eigen values of similar matrices are same.

$\therefore$ Similar matrices have the same characteritic equation and hence the same eigen values.

Now from (2) Y is an eigen vector of B corresponding to λ, the eigen value of B.

$\therefore$ Eigen vector of $B = Y = P^{-1}X$

Theorem – 2 : (Linear independence of eigen vectors) Let $\lambda_1, \lambda_2, ..., \lambda_n$ be distinct eigen values of an $n \times n$ matrix. Then corresponding eigen vectors $x_1, x_2, ... x_k$ form a linearly independent set (proof is left an exercise)

Illustrative Examples

Example – 1 : ***Examine whether A is similar to B where*** $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Solution : We know that A will similar to B if there exista a non singular matrix P such that $A = P^{-1}BP$ or $PA = BP$

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$

$\therefore \quad a + c = a, \quad b + d = b$

$c = c, \quad d = d$

If $a + c = a \Rightarrow c = 0$ and $b + d = b \Rightarrow d = 0$

$\therefore \quad P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ which is a singular matrix

$\therefore$ A, B are not similar matrices.

Example – 2 : ***Verify simillar matrices have equal spectra for A and B = $P^{-1}AP$***

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

Solution: Here $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

Let λ be some unknown scalar for which $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

So the characteristic equation is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(-3-\lambda) - 16 = 0 \Rightarrow -9 - 3\lambda + 3\lambda + \lambda^2 - 16 = 0$

$\Rightarrow \lambda^2 - 25 = 0 \quad \Rightarrow \boxed{\lambda = \pm 5}$

Hence the eigen values of A are $\lambda = 5, -5$

$\therefore$ For $\lambda = 5$, $[A - 5.I]X = 0$

$\Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow -2x_1 + 4x_2 = 0$ .... (1)

and $4x_1 - 8x_2 = 0$ .... (2)

$\therefore$ equation (1) $\Rightarrow -2x_1 = -4x_2$

$\Rightarrow x_1 = 2x_2 \Rightarrow \boxed{\dfrac{x_1}{2} = \dfrac{x_2}{1}}$

$\therefore$ an eigen vector of A corresponding to the value of $\lambda = 5$ is $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Again for $\lambda = -5, [A + 5I]X = 0$

$$\Rightarrow \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow 8x_1 + 4x_2 = 0$ ....(3) and $4x_1 + 2x_2 = 0$ .... (4)

$\therefore$ equation (3) $\Rightarrow 8x_1 = -4x_2 \Rightarrow 2x_1 = -x_2 \Rightarrow \boxed{\frac{x_1}{-1} = \frac{x_2}{2}}$

$\therefore$ an eigen vector of A corresponding to the value of $\lambda = -5$ is $X = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Given $P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

$$\Rightarrow P^{-1} = \frac{Adj\ P}{|P|} = \frac{\begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}}{(-2)} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

$$\therefore B = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} & -1 \\ \frac{25}{2} & 0 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$$

Let λ be some unknown scalar for which $BY = \lambda Y$

$\Rightarrow (B - \lambda I)Y = 0$

So the characteristic equation of B is $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -25-\lambda & 12 \\ -50 & 25-\lambda \end{vmatrix} = 0 \Rightarrow (-25-\lambda)(25-\lambda) + 600 = 0$$

$\Rightarrow -625 + \lambda^2 + 600 = 0 \Rightarrow \lambda^2 - 25 = 0$

$\Rightarrow \lambda^2 = 25 \Rightarrow \boxed{\lambda = \pm 5}$

Hence the eigen values of B are $\lambda = 5, -5$ which are same as the eigenvalues of A. So they are called as similar matrices.

$\therefore$ For $\lambda = 5, [B - 5I]Y = 0 \Rightarrow \begin{bmatrix} -30 & 12 \\ -50 & 20 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$

$\Rightarrow -30y_1 + 12y_2 = 0$ ..... (5) and $-50y_1 + 20y_2 = 0$ ..... (6)

$\therefore$ equation (5) $\Rightarrow -30y_1 = -12y_2 \Rightarrow 5y_1 = 2y_2 \Rightarrow \boxed{\frac{y_1}{2} = \frac{y_2}{5}}$

$\therefore$ an eigen vector of B corresponding to the value of $\lambda = 5$ is $Y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$\therefore P^{-1}X = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = Y$$

Hence $Y = P^{-1}X = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is an eigen vector of B corresponding to the same eigen value is $\lambda = -5$.

Again for $\lambda = -5, [B + 5I]Y = 0 \Rightarrow \begin{bmatrix} -20 & 12 \\ -50 & 30 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$

$\Rightarrow -20y_1 + 12y_2 = 0$ ...... (7) and $-50y_1 + 30y_2 = 0$ .....(8)

$\therefore$ equation (7) $\Rightarrow -20y_1 = -12y_2 \Rightarrow 5y_1 = 3y_2 \Rightarrow \boxed{\frac{y_1}{3} = \frac{y_2}{5}}$

$\therefore$ an eigen vector of B corresponding to the value of $\lambda = -5$ is $Y = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\therefore P^{-1}X = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = Y$$

Hence $Y = P^{-1}X = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is an eigen vector of B corresponding to the same eigen value i.e. $\lambda = -5$.

Example – 3 : ***Verify similar matrices have equal spectra for A and B = A⁻¹AP***

$$\mathbf{A} = \begin{bmatrix} \mathbf{7} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{2} \end{bmatrix}, \mathbf{P} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution: Here $A = \begin{bmatrix} 7 & 0 & 3 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}, P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let λ be some unknown scalar for which $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 0 & 3 \\ 2 & 1-\lambda & 1 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (7-\lambda)(1-\lambda)(2-\lambda)-3.2(1-\lambda)=0$

$\Rightarrow (1-\lambda)\{(7-\lambda)(2-\lambda)-6\}=0 \Rightarrow (1-\lambda)\{14-9\lambda+\lambda^2-6\}=0$

$\Rightarrow (1-\lambda)(\lambda^2-9\lambda+8)=0 \Rightarrow (1-\lambda)(\lambda^2-\lambda-8\lambda+8)=0$

$\Rightarrow (1-\lambda)\{\lambda(\lambda-1)-8(\lambda-1)\}=0 \Rightarrow (1-\lambda)(\lambda-1)(\lambda-8)=0$

$\Rightarrow \lambda = 8, 1$ (twice)

Hence the eigen values of A are λ = 8, 1 (twice)

$\therefore$ For $\lambda = 8$, $[A-8.I]X=0$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 3 \\ 2 & -7 & 1 \\ 2 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$\Rightarrow -x_1+0.x_2+3.x_3=0$ .... (1) $2x_1-7x_2+x_3=0$ .... (2)

and $2x_1+0.x_2-6x_3=0$ ...... (3)

$\therefore$ equation (1) $\Rightarrow \boxed{x_1=3x_3} \Rightarrow \boxed{\frac{x_1}{3}=\frac{x_3}{1}}$

equation (2) $\Rightarrow 6x_3-7x_2+x_3=0$

$\Rightarrow 7x_3=6x_2 \Rightarrow \boxed{\frac{x_2}{1}=\frac{x_3}{1}}$

Hence $\boxed{\frac{x_1}{3}=\frac{x_2}{1}=\frac{x_3}{1}}$

$\therefore$ an eigen vector of A corresponding to the value of $\lambda=8$ is $X=\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Again for $\lambda=1$, $[A-1.I]X=0$

$$\Rightarrow \begin{bmatrix} 6 & 0 & 3 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$\Rightarrow 6x_1+0.x_2+3x_3=0$ ..... (4) and $2x_1+0.x_2+1.x_3=0$ ...... (5)

$\therefore$ equation (4) $\Rightarrow 6x_1=-3x_3$

$\Rightarrow \boxed{\frac{x_1}{-1}=\frac{x_3}{2}}$ taking $x_2=0$

We get an eigen vector of A corresponding to eigen value $\lambda=1$ as $X=\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

But given $P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

We know cofactor matrix of $P=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}$

$\Rightarrow Adj\ P=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}^T=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}$ and $|P|=\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix}=-1\neq 0$

Hence $P^{-1}=\dfrac{Adj\ P}{|P|}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

We know that $B=P^{-1}AP$

$$=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}7&0&3\\2&1&1\\2&0&2\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&1&1\\7&0&3\\2&0&2\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&1\\0&7&3\\0&2&2\end{bmatrix}$$

Let λ be some unknown scalar for which $BY=\lambda Y\Rightarrow(B-\lambda I)Y=0$

So the characteristic equation of B is $|B-\lambda I|Y=0$

$$\Rightarrow\begin{vmatrix}1-\lambda&2&1\\0&7-\lambda&3\\0&2&2-\lambda\end{vmatrix}=0$$

$\Rightarrow(1-\lambda)\{(7-\lambda)(2-\lambda)-6\}=0\Rightarrow(1-\lambda)(14-9\lambda+\lambda^2-6)=0$

$\Rightarrow(1-\lambda)(\lambda^2-9\lambda+8)=0\Rightarrow(1-\lambda)(\lambda-1)(\lambda-8)=0$

$\Rightarrow\lambda=8,1$ (twice)

Hence the eigen values of B are $\lambda=8,1$ (twice) which are same as the eigen values of A. So they are called as similar matrices.

$\therefore$ For $\lambda=8$, $[B-8.I]Y=0$

$$\Rightarrow\begin{bmatrix}-7&2&1\\0&-1&3\\0&2&-6\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=0$$

$\Rightarrow -7y_1 + 2y_2 + y_3 = 0$ ..... (6) $0.y_1 - 1.y_2 + 3.y_3 = 0$ ....(7) and $0.y_1 + 2y_2 - 6y_3 = 0$ ... (8)

$\therefore$ equation (7) $\Rightarrow y_2 = 3y_3 \Rightarrow \boxed{\dfrac{y_2}{3} = \dfrac{y_3}{1}}$

equation (6) $\Rightarrow -7y_1 + 6y_3 + y_3 = 0$

$\Rightarrow -7y_1 = -7y_3 \Rightarrow \boxed{\dfrac{y_1}{1} = \dfrac{y_3}{1}}$

Hence $\boxed{\dfrac{y_1}{1} = \dfrac{y_2}{3} = \dfrac{y_3}{1}}$

$\therefore$ an eigen vector of B corresponding to the value of $\lambda = 8$ is $Y = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$$\therefore P^{-1}X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = Y$$

Hence $Y = P^{-1}X = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ is an eigen vector of B corresponding to the same eigen value i.e. $\lambda = 8$

Again for $\lambda = 1, [B - 1.1]Y = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 1 \\ 0 & 6 & 3 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$\Rightarrow 0.y_1 + 2y_2 + 1.y_3 = 0$ ...... (9) and $0.y_1 + 6.y_2 + 3.y_3 = 0$ ..... (10)

$\therefore$ equation (9) $\Rightarrow 2y_2 = -y_3$

$\Rightarrow \boxed{\dfrac{y_2}{y_1} = \dfrac{y_3}{2}}$ Taking $y_1 = 0$, we get an eigen vector of B corresponding to eigen value $\lambda = 1$ as $Y = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

$$\text{Again } P^{-1}X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = Y \Rightarrow X = PY$$

Example – 4 : Similar matrices have equal spectra verify this for A and B = P^{-1} AP

$$\mathbf{A} = \begin{bmatrix} 10 & -3 & 5 \\ 0 & 1 & 0 \\ -15 & 9 & -10 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

Solution : Here $A = \begin{bmatrix} 10 & -3 & 5 \\ 0 & 1 & 0 \\ -15 & 9 & -10 \end{bmatrix}, \ P = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

Let λ be some unknown scalar for which $AX = \lambda X$

$$\Rightarrow (A - \lambda I)X = 0.$$

So the characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 10-\lambda & -3 & 5 \\ 0 & 1-\lambda & 0 \\ -15 & 9 & -10-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (10-\lambda)(1-\lambda)(-10-\lambda) + 15.5(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)\{(10-\lambda)(-10-\lambda) + 75\} = 0 \quad \Rightarrow (1-\lambda)\{-100 + \lambda^2 + 75\} = 0$$

$$\Rightarrow (1-\lambda)\{(\lambda^2 - 25\} = 0 \quad \Rightarrow \lambda = 1 \text{ or } \lambda^2 = 25 \quad \Rightarrow \lambda = 1,5,-5$$

Hence the eigen values of A are $\Rightarrow \lambda = 1,5,-5$

$\therefore$ For $\lambda = 1$, [A –1.I] X = 0

$$\Rightarrow \begin{vmatrix} 9 & -3 & 5 \\ 0 & 0 & 0 \\ -15 & 9 & -11 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 9x_1 - 3x_2 + 5x_3 = 0$ .... (1)

$\Rightarrow 0.x_1 - 0.x_2 + 0.x_3 = 0$ ..... (2)

and $-15.x_1 + 2x_2 - 11x_3 = 0$ .... (3)

On solving equation (1) and equation (3) by cross multiplication, we get

$$\frac{x_1}{-12} = \frac{x_2}{24} = \frac{x_3}{36} \Rightarrow \boxed{\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{3}}$$

$\therefore$ An eigen vector of A corresponding to the value of $\lambda = 1$ is $X = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

Again for $\lambda = 5$, [A +5.I] X = 0

$$\Rightarrow \begin{vmatrix} 5 & -3 & 5 \\ 0 & -4 & 0 \\ -15 & 9 & -15 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 5x_1 - 3x_2 + 5x_3 = 0$ .... (4)

$0.x_1 - 4.x_2 + 0.x_3 = 0$ .... (5) and $-15.x_2 + 9.x_3 - 15.x_3 = 0$ ....(6)

$\therefore$ Equation (5) $\Rightarrow \boxed{x_2 = 0}$

Equation (4) $\Rightarrow 5x_1 = -5x_3 \Rightarrow \boxed{\frac{x_1}{-1} = \frac{x_3}{1}}$

$\therefore$ An eigen vector of A corresponding to the value of $\lambda = 5$ is $X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Similarly for $\lambda = -5$, $[A + 5.I]X = 0$

$$\Rightarrow \begin{vmatrix} 15 & -3 & 5 \\ 0 & 6 & 0 \\ -15 & 9 & -5 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 15x_1 - 3x_2 + 5x_3 = 0$ .... (7)

$0.x_1 + 6.x_2 + 0.x_3 = 0$ .... (8) and $-15.x_2 + 9.x_2 - 15.x_3 = 0$ ....(9)

$\therefore$ Equation (8) $\Rightarrow \boxed{x_2 = 0}$

Equation (7) $\Rightarrow 15x_1 = -5x_3 \Rightarrow 3x_1 = x_3 \Rightarrow \boxed{\frac{x_1}{-1} = \frac{x_3}{3}}$

An eigen vector of A corresponding to the value of $\lambda = -5$ is $X = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

But given $P = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$, and $|P| = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix} = 1 \neq 0$

We know that cofactor matrix of $P = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix} \Rightarrow adj\ P = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}$

$$\therefore P^{-1} = \frac{adjP}{|P|} = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

We know $B = P^{-1}AP = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 10 & -3 & 5 \\ 0 & 1 & 0 \\ -15 & 9 & -10 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 95 & -42 & 55 \\ 0 & 1 & 0 \\ -60 & 27 & -35 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 355 & -42 & 56 \\ 0 & 1 & 0 \\ -225 & 27 & -355 \end{bmatrix}$$

Let l be some unknown scalar function which $By = \lambda y \Rightarrow (B - \lambda I)\, y = 0$

So the characteristic equation of B is $|B - \lambda I| = 0$

$$= \begin{bmatrix} 355-\lambda & -42 & 560 \\ 0 & 1-\lambda & 0 \\ -225 & 27 & -355-\lambda \end{bmatrix}$$

$\Rightarrow (355 - \lambda)\,(1-\lambda)\,(-355 - \lambda) + 225\,(1-\lambda)\,560 = 0$

$\Rightarrow (1-\lambda)\,\{(355-\lambda)\,(-355-\lambda) + 126000\} = 0$

$\Rightarrow (1-\lambda)\,\{-126025 + \lambda^2 + 126000\} = 0$

$\Rightarrow (1 - \lambda)\,(\lambda^2 - 25) = 0$

$\Rightarrow (1-\lambda) = 0$ or $(\lambda^2 - 25) = 0$

$\Rightarrow \lambda = 1, 5, -5$

Hence the eigen values of B are $\lambda = 1, 5\ -5$ which are same as the eigen values of A. So they are called as similar matrices.

$\therefore$ For $\lambda = 1$, $[B - 1 \cdot I]\,\lambda = 0$

$$\Rightarrow \begin{bmatrix} 354 & -42 & 560 \\ 0 & 0 & 0 \\ -225 & 27 & -356 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $354y_1 - 42y_2 + 560y_3 = 0$ ....(10)

$0 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3 = 0$ ....(11)

$-225y_1 + 27y_2 - 356y_3 = 0$ ....(12)

Solving (10) & (12), we get $= \dfrac{y_1}{-168} = \dfrac{y_2}{24} = \dfrac{y_3}{108}$

$$\Rightarrow \boxed{\frac{y_1}{-14} = \frac{y_2}{2} = \frac{y_3}{9}}$$

$\therefore$ an eigen vector of B corresponding to the value of $\lambda = 1$, is $y = \begin{bmatrix} -14 \\ 2 \\ 9 \end{bmatrix}$

$$\therefore\ P^{-1}X = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -14 \\ 2 \\ 9 \end{bmatrix} = Y$$

Hence $Y = P^{-1}X = \begin{bmatrix} -14 \\ 2 \\ 9 \end{bmatrix}$ is an eigen vector of B corresponding to the same eigen value is $\lambda = 1$.

Again for $\lambda = 5$, $[B - 5\,I]Y = 0 \Rightarrow \begin{bmatrix} 350 & -42 & 560 \\ 0 & -4 & 0 \\ -225 & 27 & -360 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$

$\Rightarrow 350\,y_1 - 42\,y_2 + 560y_3 = 0$ ....(13)

$0\cdot y_1 - 4\cdot y_2 + 0\cdot y_3 = 0$ ....(14)

and $-225y_1 + 27y_2 - 360\,y_3 = 0$ ......(15)

$\therefore$ Equation (14) $y_2 = 0$

Equation (13) $\Rightarrow 350y_1 = -560\,y_3 \Rightarrow 5y_1 = -8y_3 \Rightarrow \boxed{\frac{y_1}{-8} = \frac{y_3}{5}}$

$\therefore$ An eigen vector of B corresponding to the value of $\lambda = 5$ is $y = \begin{bmatrix} -8 \\ 0 \\ 5 \end{bmatrix}$

$$\therefore\ P^{-1}X = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 5 \end{bmatrix} = y$$

Hence $Y = P^{-1}X = \begin{bmatrix} -8 \\ 0 \\ 5 \end{bmatrix}$ is an eigen vector of B corresponding to the same eigen value i.e. $\lambda = 5$.

Similarly $\lambda = -5$, $[B + 5\cdot I]\,Y = 0 \Rightarrow \begin{bmatrix} 360 & -42 & 560 \\ 0 & 6 & 0 \\ -225 & 27 & -350 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$

$\Rightarrow$ $360y_1 - 42y_2 + 560y_3 = 0$ ....(16)

$0\cdot y_1 + 6\cdot y_2 + 0\cdot y_3 = 0$ ....(17) and $-225y_1 + 27y_2 - 350y_3 = 0$ ....(18)

$\therefore$ Equation (17) $\Rightarrow y_2 = 0$

$\therefore$ Equation (16) $\Rightarrow 360y_1 = -560y_3 \Rightarrow 9y_1 = -14y_3 \Rightarrow \boxed{\frac{y_1}{-14} = \frac{y_3}{9}}$

$\therefore$ An eigen value of B corresponding to the value of $\lambda = -5$ is $y = \begin{bmatrix} -14 \\ 0 \\ 9 \end{bmatrix}$

$$\therefore P^{-1}X = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -14 \\ 0 \\ 9 \end{bmatrix} = y$$

Hence $Y = P^{-1}X = \begin{bmatrix} -14 \\ 0 \\ 9 \end{bmatrix}$ is an eigen vector of B corresponding to the same eigen value i.e., $\lambda = -5$

Example – 5 : ***Examine which of the following matrices are similar to diagonal matrices.***

$$\textbf{(i)} \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix} \qquad \textbf{(ii)} \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution : (i) Characteristic equation of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is $|A - \lambda I| = 0$

i.e, $\begin{vmatrix} \lambda - 8 & 6 & -2 \\ 6 & \lambda - 7 & 4 \\ -2 & 4 & \lambda - 3 \end{vmatrix} = 0$ i.e, $\lambda^3 - 18\lambda^2 + 45\lambda = 0$; $\lambda = 0, 3, 15$

Characteristic vectors corresponding to $\lambda = 0$ is given by $(\lambda I - A)X = 0$. Put $\lambda = 0$

$$\sim \begin{bmatrix} -8 & 6 & -2 \\ 6 & -7 & 4 \\ -2 & 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad (R_1 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} -2 & 4 & -3 \\ 6 & -7 & 4 \\ -8 & 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\sim \begin{bmatrix} -2 & 4 & -3 \\ 0 & 5 & -5 \\ 0 & -10 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad (R_2 \leftrightarrow R_2 + 3R_1,\ R_3 \to R_3 - 4R_1)$$

$$\sim \begin{bmatrix} -2 & 4 & -3 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad (R_3 \to R_3 + 2R_2)$$

or $-2x + 4y - 3z = 0$

$5y - 5z = 0$

$y = z, x = \dfrac{z}{2}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{z}{2} \\ z \\ z \end{bmatrix} = \frac{z}{2} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow X = \frac{z}{2} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

We may take single L.I. solution $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Similarly for $\lambda=3$;
$$\begin{bmatrix} -5 & 6 & -2 \\ 6 & -4 & +4 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\sim \begin{bmatrix} -2 & 4 & 0 \\ 6 & -4 & +4 \\ -5 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad (R_1 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 3 & -2 & 2 \\ -5 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad (R_1 \to -\frac{1}{2}R_1, R_2 \to \frac{1}{2}R_2)$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad (R_2 \to R_2 - 3R_1, R_3 \to R_3 + 5R_1)$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad (R_3 \to R_3 + R_2)$$

i.e., $x - 2y = 0$ or $x = 2y$

$4y + 2z = 0$ or $z = -2y$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ -2y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 3$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$
$$\begin{bmatrix} 7 & 6 & -2 \\ 6 & 8 & 4 \\ -2 & 4 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

or
$$\begin{bmatrix} -1 & 2 & 6 \\ 3 & 4 & 2 \\ 7 & 6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \left(R_1 \leftrightarrow R_3,\ R_3\left(\frac{1}{2}\right), R_2\left(\frac{1}{2}\right)\right)$$

$$\begin{bmatrix} -1 & 2 & 6 \\ 0 & 10 & 20 \\ 0 & 20 & 40 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad (R_1 \to R_2 + 3R_1,\ R_3 \longrightarrow R_3 + 7R_1)$$

$$\begin{bmatrix} -1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad (R_3 \to R_3 - 2R_2,\ R_2 \to \frac{1}{10}R_2)$$

$\therefore\ -x + 2y + 6z = 0,\ y + 2z = 0$

$$\therefore\ y = -2z, \qquad x = 2z \quad \therefore\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ Set of L.I. characteristic vectors is

$$P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now, $$P^{-1} = -\frac{1}{27}\begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \qquad \left(\because P^{-1} = \frac{\text{Adj}P}{|P|}\right)$$

$$P^{-1}AP = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

= diag (0,3,15) i.e, diagonal matrix formed by eigen values. Hence A is similar to diagonal matrix.

(ii) $$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution : Characteristic roots of A are $|A - \lambda I| = 0$

$$\begin{vmatrix} \lambda - 2 & -3 & -4 \\ 0 & 2 - \lambda & 1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = 0 \quad \text{or} \quad (\lambda - 2)^2(\lambda - 1) = 0$$

$\lambda = 1,2,2$

Eigen vector corresponding to $\lambda = 1$ is $$\begin{bmatrix} -1 & -3 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-x-3y-4z=0 \quad \therefore \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -7z \\ z \\ z \end{bmatrix}=z\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$

$-y+z=0 \quad \therefore \quad y=z \quad x=-3y-4z=-7z$

$\therefore$ Single eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = 2$, $\begin{vmatrix} 0 & -3 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$

$-3y-4z=0,\ z=0$

$\therefore\ y=0$

$\therefore\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}=x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ Corresponding to $\lambda = 2$ we get only one vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

As there are only two L.I. eigen vectors corresponding to three eigen values.

$\therefore$ There does not exist any non singular matrix P.

Hence A is not similar to diagonal matrix.

Example – 6 : ***Show that the rank of every matrix similar to A is the same as that of A.***

Solution : Let B be similar to A. Then there exists a non singular matrix P such that

$B = P^{-1}AP$

Now, rank of B = rank of $(P^{-1}AP)$ = rank A

$\therefore$ We know that rank of a matrix does not change on multiplication by a non singular matrix.

Rank of B = rank of A

Example – 7 : ***Show that two similar matrices have the same characteristics roots.***

Solution : Let A and B are two square matrices of the same order and there exists a non-singular matrix of the same order such that AP = PB or $B = P^{-1}AP$.

Let λ be an eigen value of A $\therefore$ $|A-\lambda I| = 0$ ....(1)

Now $|B-\lambda I| = |P^{-1}AP - \lambda P^{-1}P| = |P^{-1}(A-\lambda I)P|$ $\quad [\because I = P^{-1}P \ \&\ B = P^{-1}AP]$

$= |P^{-1}|\,|A-\lambda I|\,|P| = |P^{-1}|\,|P|\,|A-\lambda I|$

$= |P^{-1}P|\,|A-\lambda I| = |I|\,|A-\lambda I| = |A-\lambda I| = 0$

$\Rightarrow |B-\lambda I| = 0$

$\Rightarrow \lambda$ is also an eigen value of B.

Hence A and B have same eigen value λ.

2.9 : Principal Axes Theorem

A quadratic form Q= X^TAX can be reduced to principal axes form Q = Y^TAY, where $\lambda_1, \lambda_2, ...\lambda_n$ are (not necessarily destinct) eigen values of the real symmetric matrix A and X is an orthogonal matrix with the corresponding eigen vectors $x_1, x_2, ..., x_n$ respectively as column vectors of the real symmetric matrix A where $Y = X^TX$.

Proof : without restriction we can assure that A is real symmetric. Then A has an orthonormal basis of n eigen vectors. Hence the matrix X with these vectors as column vectors is orthogonal. So that $X^{-1} = X^T$. Since $D = X^{-1}AX$

$\Rightarrow A = XDX^{-1} = XDX^T$

So, $Q = X^TAX = X^T(XDX^T)X$.

$= (X^TX)^T D (X^{-1}X) = Y^TDY$ and which is the required result.

Transformation of Quadratic forms to principal Axes (conic sections) :

Example – 1 : Find out what type of conic section (or pair of straight lines) is repreented by the given quadratic forms. Transform it to principal axes. Express $x^T = (x_1, x_2)$ in terms of new co-ordinate vector $y^T = (y_1, y_2)$.

(a) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

(b) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$

Solution: (a) We know the quadratic form of $7x_1^2 + 6x_1x_2 + 7x_2^2$ is

$$7x_1^2 + 6x_1x_2 + 7x_2^2 = [x_1, x_2]\begin{bmatrix} 7 & 6 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T Ax$$

Where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 7 & 6 \\ 0 & 7 \end{bmatrix}$, $A^T = \begin{bmatrix} 7 & 0 \\ 6 & 7 \end{bmatrix}$

We can find a symmetric matrix C such that $x^TAx = x^TCx$

Where $C = c_{ij}$ and $c_{ij} = c_{ji} = \dfrac{a_{ij} + a_{ji}}{2}$

So $x^T Ax = x^T Cx$ where $C = \dfrac{A + A^{-1}}{2} = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$

Let λ be some unknown scalar for which $CX = \lambda X \Rightarrow (C - \lambda I)X = 0$

We know the C.E. of C is $|C - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} = 0 \Rightarrow (7-\lambda)^2 - 9 = 0$$

$$\Rightarrow (7-\lambda)^2 = 9 \Rightarrow (7-\lambda) = \pm 3 \Rightarrow \lambda = 7 \pm 3$$

$\Rightarrow \lambda = 10, 4$ (eigen values of C)

$\therefore$ For $\lambda = 10$, $[C - 10I]X = 0$

$$\Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -3x_1 + 3x_2 = 0 \quad .....(1) \qquad \text{and } 3x_1 - 3x_2 = 0 \text{ .... } (2)$$

$\therefore$ equation (1) $\Rightarrow -3x_1 = -3x_2$

$$\Rightarrow \boxed{\frac{x_1}{1} = \frac{x_2}{1}}$$

$\therefore$ An eigen vector of C corresponding to the eigen value $\lambda = 10$ is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Again for $\lambda = 4$ $[C - 4 \cdot I]x = 0$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow 3x_1 + 3x_2 = 0 \Rightarrow x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2 \quad \Rightarrow \boxed{\frac{x_1}{1} = \frac{x_2}{-1}}$$

$\therefore$ An eigen vector of C corresponding to the eigen value $\lambda = 4$ is $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$\therefore \quad X = [X_1 \ X_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \text{orthogonal matrix i.e, } X^T = X^{-1}$$

Here the diagonal matrix $D = X^{-1}CX = \begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix}$

$$\therefore \ X^T CX = Y^T DY = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 4y_1^2 + 10y_2^2$$

Again $y = X^T x = X^{-1}x (\because X^T = X^{-1}) \ \Rightarrow \boxed{x = Xy}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow x_1 = \frac{y_1 + y_2}{\sqrt{2}} \text{ and } \Rightarrow x_2 = \frac{y_1 - y_2}{\sqrt{2}}$$

$\therefore$ The given equation is $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

$$\Rightarrow x^T Cx = 200 \Rightarrow y^T Dy = 200 \quad \Rightarrow 4y_1^2 + 10y_2^2 = 200 \Rightarrow \frac{y_1^2}{50} + \frac{y_2^2}{20} = 1$$

where $x_1 = \dfrac{y_1 + y_2}{\sqrt{2}}$ and $x_2 = \dfrac{y_1 - y_2}{\sqrt{2}}$

Hence the given equation represents an ellipse.

(b) $\mathbf{9x_1^2 - 6x_1x_2 + x_2^2 = 40}$ ***[B.P.U.T. – 2011, 2013, 2016***

Sol^n ***:*** We know the quadratic form of $9x_1^2 - 6x_1x_2 + x_2^2$ is

$$9x_1^2 - 6x_1x_2 + x_2^2 = [x_1x_2]\begin{bmatrix} 9 & -6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T Ax$$

Where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $A = \begin{bmatrix} 9 & -6 \\ 0 & 1 \end{bmatrix}$

Here we have to find a symmetric matrix C such that $x^T Ax = x^T Cx$ where $C = c_{ij}$ and $c_{ij} = c_{ji}$ $= \dfrac{a_{ij} + a_{ji}}{2}$

So $x^T Ax = x^T Cx$ where $C = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$

We know the C.E. of C is $|C - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 9-\lambda & -3 \\ -3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (9-\lambda)(1-\lambda) - 9 = 0$$

$$\Rightarrow 9 - 10\lambda + \lambda^2 - 9 = 0 \quad \Rightarrow \lambda^2 - 10\lambda = 0 \Rightarrow \lambda(\lambda - 10) = 0$$

$\Rightarrow \lambda = 0, 10$. (eigen values of C)

$\therefore$ For $\lambda = 0$, $[C - 0.I]X = 0$

$$\Rightarrow \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow 9x_1 - 3x_2 = 0$ .... (1) and $-3x_1 + x_2 = 0$ .... (2)

$\therefore$ Equation (1) $\Rightarrow 9x_1 = 3x_2$

$$\Rightarrow 3x_1 = x_2 \Rightarrow \boxed{\frac{x_1}{1} = \frac{x_2}{3}}$$

$\therefore$ An eigen vector of C corresponding to the eigen value $\lambda = 0$ is $X_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_1 = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$

Again for $\lambda = 10$, $[C - 10.I]X = 0$

$$\Rightarrow \begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -x_1 - 3x_2 = 0 \quad ..... (3) \qquad \text{and } -3x_1 - 9x_2 = 0 \quad .... (4)$$

$\therefore$ Equation (3) $\Rightarrow -x_1 = 3x_2 \Rightarrow \boxed{\frac{x_1}{3} = \frac{x_2}{-1}}$

$\therefore$ An eigen vector of C corresponding to the eigen value $\lambda = 10$ is $X_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_2 = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{bmatrix}$

$\therefore \; X = [X_1 \; X_2] = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix}$ = Orthogonal matirx i.e, $X^T = X^{-1}$

Here the diagonal matrix $D = X^{-1}CX = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}$

$\therefore \; x^T Cx = y^T Dy = [y_1 y_2]\begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 10y_2^2$

Again $y = X^T x = X^{-1}x\left(\because X^T = X^{-1}\right) \Rightarrow x = Xy$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow x_1 = \frac{y_1 + 3y_2}{\sqrt{10}} \text{ and } \Rightarrow x_2 = \frac{3y_1 - y_2}{\sqrt{10}}$$

$\therefore$ The given equation is $9x_1^2 - 6x_1x_2 + x_2^2 = 40$

$$\Rightarrow x^T cx = 40 \Rightarrow y^T Dy = 40 \Rightarrow 10y_2^2 = 40$$

$$\Rightarrow y_2 = \pm 2 \qquad \text{where } x_1 = \frac{(y_1 + 3y_2)}{\sqrt{10}} \text{ and } x_2 = \frac{(3y_1 - y_2)}{\sqrt{10}}$$

Hence the given equation represents the straight lines $y_2 = \pm 2$.

Example – 2 : ***Find out what type of conic section (or pair of straight lines) is repreented by the given quadratic forms. Transform it to principal axes.***

$$\mathbf{-11x_1^2+84x_1x_2+24x_2^2=156}$$

Solution: We know the quadratic form of $-11x_1^2+84x_1x_2+24x_2^2$

$$-11x_1^2+84x_1x_2+24x_2^2=\begin{bmatrix}x_1 & x_1\end{bmatrix}\begin{bmatrix}-11 & 84\\ 0 & 24\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=x^TAx$$

Where $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ and $A=\begin{bmatrix}-11 & 84\\ 0 & 24\end{bmatrix}$ $\quad A^T=\begin{bmatrix}-11 & 0\\ 84 & 24\end{bmatrix}$

We can find a symmetric matrix c such that $x^TAx = x^TCx$

where $C=c_{ij}$ and $c_{ij}=c_{ji}=\dfrac{a_{ij}+a_{ji}}{2}$

So $x^TAx = x^TCx$ where $C=\begin{bmatrix}-11 & 42\\ 42 & 24\end{bmatrix}$ $\quad C=\dfrac{A+A^{\mathrm{T}}}{2}$

Let λ be some unknown scalar for which $CX=\lambda X$

So the C.E. of C is $|C-\lambda|=0$

$$\Rightarrow\begin{vmatrix}-11-\lambda & 42\\ 42 & 24-\lambda\end{vmatrix}=0$$

$$\Rightarrow(-11-\lambda)(24-\lambda)-1764=0\Rightarrow-264-13\lambda+\lambda^3-1764=0$$

$$\Rightarrow\lambda^2-52\lambda+39\lambda-2028=0\Rightarrow\lambda(\lambda-52)+39(\lambda-52)=0$$

$$\Rightarrow(\lambda-52)(\lambda+39)=0\Rightarrow\lambda=52,-39$$

Hence the eigen values of C are $\lambda=52,-39$

$\therefore$ For $\lambda=52,[C-52I]X=0$

$$\Rightarrow\begin{bmatrix}-63 & 42\\ 42 & -28\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$\Rightarrow-63x_1++42x_2=0$ ...... (1) and $42x_1-28x_2=0$ ...... (2)

$\therefore$ equation (1) $\Rightarrow-63x_1=-42x_2\Rightarrow3x_1=2x_2$

$$\boxed{\frac{x_1}{2}=\frac{x_2}{3}}$$

$\therefore$ an eigen vector of C corresponding to the eigen value $\lambda=52$ is $X_1=\begin{bmatrix}2\\ 3\end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_1=\begin{bmatrix}\dfrac{2}{\sqrt{13}}\\ \dfrac{3}{\sqrt{13}}\end{bmatrix}$

Again for $\lambda=-39$ $[C+39I]X=0$

$$\Rightarrow \begin{bmatrix}28 & 42\\ 42 & 63\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$\Rightarrow 28x_1+42x_2=0$ ...... (3) and $42x_1+63x_2=0$ .....(4)

$\therefore$ equation (3) $\Rightarrow 28x_1=-42x_2 \Rightarrow 2x_1=-3x_2$

$$\Rightarrow \boxed{\frac{x_1}{-3}=\frac{x_2}{2}}$$

$\therefore$ an eigen vector of C corresponding to the eigen value $\lambda=-39$ is $X=\begin{bmatrix}-3\\ 2\end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_2=\begin{bmatrix}\frac{-3}{\sqrt{13}}\\ \frac{2}{\sqrt{13}}\end{bmatrix}$

$$\therefore X=[X_1X_2]=\begin{bmatrix}\frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}}\\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}}\end{bmatrix}=\text{orthogonal matrix i.e. } x^T=x^{-1}$$

Here the diagonal matrix $D=\begin{bmatrix}52 & 0\\ 0 & -39\end{bmatrix}=X^{-1}CX$

$$\therefore x^TCx=y^TDy=[y_1\ \ y_2]\begin{bmatrix}52 & 0\\ 0 & -39\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=52y_1^2-39y_2^2$$

Again $y=X^Tx=X^{-1}x\left(\because X^T=X^{-1}\right)$

$\Rightarrow x=Xy$

$$\Rightarrow \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}}\\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}\Rightarrow \boxed{\begin{matrix}x_1=\dfrac{2y_1-3y_2}{\sqrt{13}}\\ x_2=\dfrac{2y_1+2y_2}{\sqrt{13}}\end{matrix}}$$

$\therefore$ The given equation is $-11x_1^2+84x_1x_2+24x_2^2=156$

$\Rightarrow x^TCx=156 \Rightarrow y^TDy=156$

$\Rightarrow 52y_1^2-39y_2^2=156$

$\Rightarrow \boxed{\frac{y_1^2}{3}-\frac{y_2^2}{4}=1}$ where $x_1=\frac{2y_1-3y_2}{\sqrt{13}}$ and $x_2=\frac{3y_1+2y_2}{\sqrt{13}}$

Hence the given equation represents a hyperbola.

Example – 3 : ***Find out what type of conic section (or pair of straight lines) is repreented by the given quadratic forms. Transform it to principal axes.***

$$\mathbf{32x_1^2 - 60x_1x_2 + 7x_2^2 = -52}$$

Solution : We know the quadratic form of $32x_1^2 - 60x_1x_2 + 7x_2^2$ is

$$32x_1^2 - 60x_1x_2 + 7x_2^2 = [x_1x_2]\begin{bmatrix} 32 & -60 \\ 0 & 7 \end{bmatrix} = x^T Ax,$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 32 & -60 \\ 0 & 7 \end{bmatrix}$, $A^T = \begin{bmatrix} 32 & 0 \\ -60 & 7 \end{bmatrix}$

Here we have to find a symmetric matrix C such that $x^T Ax = x^T Cx$ where $C = c_{ij}$ and $c_{ij} = c_{ji} = \dfrac{a_{ij} + a_{ji}}{2}$

So $x^T Ax = x^T Cx$ where $C = \begin{bmatrix} 32 & -30 \\ -30 & 7 \end{bmatrix} = \dfrac{A + A^T}{2}$

We know the C.E. of C is $|C - \lambda| = 0$

$$\Rightarrow \begin{vmatrix} 32-\lambda & -30 \\ -30 & 7-\lambda \end{vmatrix} = 0$$

$\Rightarrow (32-\lambda)(7-\lambda) - 900 = 0 \Rightarrow 224 - 39\lambda + \lambda^2 - 900 = 0$

$\Rightarrow \lambda^2 - 39\lambda - 676 = 0 \Rightarrow \lambda^2 - 52\lambda + 13\lambda - 676 = 0$

$\Rightarrow \lambda(\lambda - 52) + 13(\lambda - 52) = 0 \Rightarrow (\lambda - 52)(\lambda + 13) = 0$

$\Rightarrow \lambda = 52, -13$ (eigen values of C)

$\therefore$ For $\lambda = 52$, $[C - 52.I]\, X = 0$

$$\Rightarrow \begin{bmatrix} -20 & -30 \\ -30 & -45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow -20x_1 - 30x_2 = 0$ ..... (1) and $-30x_1 - 45x_2 = 0$ .... (2)

$\therefore$ equation (1) $\Rightarrow -20x_1 = 30x_2$

$\Rightarrow -2x_1 = 3x_2 \Rightarrow \boxed{\dfrac{x_1}{3} = \dfrac{x_2}{2}}$

$\therefore$ an eigen vector of C corresponding to the eigen value $\lambda = 52$ is $X_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_1 = \begin{bmatrix} \dfrac{3}{\sqrt{13}} \\ \dfrac{-2}{\sqrt{13}} \end{bmatrix}$

Again for $\lambda = -13$, $[C + 13.1]X = 0$

$$\Rightarrow \begin{bmatrix} 45 & -30 \\ -30 & 20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow 45x_1 - 30x_2 = 0$ ..... (3) and $-30x_1 + 20x_2 = 0$ ..... (4)

$\therefore$ equation (3) $\Rightarrow 45x_1 = 30x_2 \Rightarrow 3x_1 = 2x_2$

$$\Rightarrow \boxed{\frac{x_1}{2} = \frac{x_2}{3}}$$

$\therefore$ an eigen vector of C corresponding to the eigen value $\lambda = -13$ is $X_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector is $X_2 = \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix}$

$$\therefore X = [X_1 X_2] = \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix} = \text{orthogonal matrix i.e. } X^T = X^{-1}$$

Here the diagonal matrix of C is D where

$$D = X^{-1}CX = \begin{bmatrix} 52 & 0 \\ 0 & -13 \end{bmatrix}$$

$$\therefore x^T cx = y^T Dy = [y_1 y_2]\begin{bmatrix} 52 & 0 \\ 0 & -13 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 52y_1^2 - 13y_2^2$$

Again $y = X^T x = X^{-1}x \left(\because X^T = X^{-1}\right) \Rightarrow \boxed{x = Xy}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \boxed{x_1 = \frac{(3y_1 + 2y_2)}{\sqrt{13}} \text{ and } x_2 = \frac{(-2y_1 + 3y_2)}{\sqrt{13}}}$$

$\therefore$ The given equation is $32x_1^2 - 60x_1x_2 + 7x_2^2 = -52$

$\Rightarrow x^T cx = -52 \quad \Rightarrow y^T Dy = -52$

$$\Rightarrow 52y_1^2 - 13y_2^2 = -52 \Rightarrow \frac{y_1^2}{(-1)} + \frac{y_2^2}{4} = 1$$

$$\Rightarrow \boxed{\frac{y_2^2}{4} - \frac{y_1^2}{1} = 1} \text{ where } x_1 = \frac{(3y_1 + 2y_2)}{\sqrt{13}} \text{ and } x_2 = \frac{(-2y_1 + 3y_2)}{\sqrt{13}}$$

Hence the given equation represents a hyperbola.

Exercise – 2.2

1. If A and B are non-singular matrices of order n, show that the matrices AB and BA are similar.
2. Prove that Eigen values and Eigen vectors are of similar matrices.
3. Show that the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ are similar to diagonal matrix.
4. Verify for A and B = $P^{-1}AP$. Find eigen vectors y of B. Show that x = Py are eigen vectors of A.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 7 & 0 & 3 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}$ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
5. Prove that every orthogonal set of vectors is lenearly independent.
6. Show that the matrix are not similar to diagonal matrix $\begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{bmatrix}$.
7. Examine whether A is similar to B, where **(Ans –** yes**)**

   (a) $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

9. Show that (a) $\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$ is similar to $\begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$
10. Find out what type of conic section (or pair of straight lines) is repreented by the given quadratic forms. Transform it to principal axes.

    $41x_1^2 - 24x_1x_2 + 34x_2^2 = 156$ **(Ans.** Ellipse)

Objective type Questions with Answers

Short Answertype Questions Carries 2 Marks

1. What is a basis of eigenvectors ? When does it exist ?

Ans. Basis of eigen vectors : If n×n matrix A has 'n' distinct eigen values, then A has a basis of eigenvectors for C^n (or R^n).

Existence of Basis : If the Rank of the matrix is less than the number of row vectors, there will be linearly independent eigen vectors possible and form a basis.

Or

Ans. Let A is the given matrix of order n if the rank of the coefficient matrix in the homogenous equation $(A-\lambda I)x = 0$, there will be n number of L.I eigen vectors corresponds to λ and forming a basis of eigen vector.

2. What do you mean by unitary matrix? What is the value of the determinant of an unitary matrix?

Ans. (i) Square matrix $A = [a_{kj}]$ is called unitary if $\overline{A}^T = A^{-1}$

(ii) Determinant of an unitary matrix $= \pm 1$

3. How can you say a real square matrix is orthogonal ?

Ans. A real square matrix H is called an orthogonal matrix if $H^TH = I = HH^T$. i.e., $H^T = H^{-1}$.

4. Define normalized Eigen vector.

Ans. The normalised eigen vector is the eigen vector whose norm is one. For example the noramalised eigen vector corresponds to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is $\begin{pmatrix} \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ \dfrac{x_2}{\sqrt{x_1^2 + x_2^2}} \end{pmatrix}$

5. Verify the following matrix is symmetric or skew-symmetric : $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 1 \end{bmatrix}$.

Ans. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 1 \end{bmatrix}$

$$\therefore A^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 1 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & -1 \end{bmatrix}$$

$\therefore A^T \neq A \, or \, A^T \neq -A$

$\therefore$ A is neither symmetric nor skew-symmetric.

6. Let A be a skew-symmetric matrix of odd order. Then prove that det A = 0.

Ans. Let A be a skew-symmetric matrix of odd order. We know that A is skew-symmetric if $A = -A^T$.

$\Rightarrow \det A = -\det A^T \; But \det A^T = \det A$

$\therefore \det A = -\det A$

$\Rightarrow \det A + \det A = 0$

$\Rightarrow 2\det A = 0$

$\Rightarrow \det A = 0$

7. If x is an eigen vector of A corresponding to the eigen value λ, prove that x is an eigen vector the g(A) corresponding to the elegen value $g(\lambda)$.

Ans. As λ is an elegan value of the matrix A. Then $\exists$ a non-zero vector X such that $AX = \lambda X$.

$\Rightarrow g(AX) = g(\lambda X)$ where g is a polynomial .

$\Rightarrow g(A)X = g(\lambda)X$

$\therefore X$ is an eigen vector of g(A) corresponding to the eigen value $g(\lambda)$

8. Prove that $\lambda = 0$ is an eigen value of the matrix A iff A is singular.

Ans. We have det $(A - \lambda I) = 0$, where λ is an eigen value of the matrix A.

Putting $\lambda = 0$, we get, det A = 0 $\Rightarrow$ A is singular.

Conversely, let A is singular.

$\Rightarrow \det A = 0 \Rightarrow \det(A - 0.1) = 0$

$\Rightarrow \lambda = 0$ is an eigen value of A.

9. If λ is an eigen value A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}, if A is non-singular.

Ans. If X be the given vector corresponding to the eigen value λ, then we have

$AX = \lambda X$

$\Rightarrow A^{-1}(AX) = A^{-1}\lambda X \Rightarrow (A^{-1}A)X = \lambda(A^{-1}X)$

$\Rightarrow IX = \lambda(A^{-1}X) \Rightarrow X = \lambda(A^{-1}X) \Rightarrow A^{-1}X = \left(\frac{1}{\lambda}\right)X$

$\therefore \frac{1}{\lambda}$ is an elegen value of A^{-1}, provided A is non-singular.

10. Define a unitary matrix and give examples.

Ans. A complex square matrix 'U' is called a unitary matrix if $U^* U = I = UU^*$

In other words $U^* = U^{-1}$

Ex : $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is a unitary matrix.

11. If U is unitary, prove that | det U| = 1.

Ans. Let U be a unitary matrix.

Then $U^* U = 1$

Since $|U^*| = |\overline{U}|$ and $|U^* U| = |U^*||U| \Rightarrow |I| = |U^*||U| \Rightarrow |U| = 1$

12. Show that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is unitary.

Ans. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore A^* = \left(\overline{A}\right)^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A^* A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

$\therefore$ A is unitary.

13. Show that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is unitary.

Ans. Let $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\therefore C^* = \left(\overline{C}\right)^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\therefore CC^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1$ $\therefore C$ is unitary.

Multiple type or dash fill up type questions. carries 2 Marks

1. If λ is an eigen value of an orthogonal matrix then its other eigen value is ———.

Ans. $\frac{1}{\lambda}$

2. Find the characterstic values of the given matrix $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$

Ans. 5, –2

3. If λ is an eigen value of an orthogonal matrix find its eigen value

Ans. $1/\lambda$

4. If λ is an eigen value of A and $f(A)$ is any polynomial in A, then find eigen values of $f(A)$

Ans. $f(\lambda)$

5. If A is triangular the elements on its leading diagonal are its

(a) eigen vector (b) eigen values

(c) orthonormal (d) None

Ans. (b)

6. The product of eigen values of a matrix is equal to ——.

Ans. determinant

7. The quadratic form corresponding to the symmetric matrix $\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$ is

(a) $x^2 - 4xy - 4y^2$ (b) $x^2 + 4xy + 4y^2$

(c) $x^2 + 4xy - 4y^2$ (d) $x^2 + 4xy - y^2$

Ans. (c)

8. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ then eigen values of A^2 are

Ans. 1, 4, 9

9. The eigen values of a triangular matrix are

(a) None (b) column

(c) row (d) the elements of its leading diagonal

Ans. (d)

10. The characteristic of an orthogonal martix A is

(a) $A^{-1}.A = I$ (b) $A.A^{-1} = I$

(c) $A'.A^{-1} = I$ (d) $A.A' = I$

Ans. (d)

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