MODULE – III

[Vector differential calculus : vector and scalar functions and fields, Derivatives, Curves tangents and arc length, gradient, divergence, curl.]

Vector Differentiation

Vector Integration

STUDY MATERIAL

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CHAPTER – 3

Vector Differentiation

3.1 : Introduction

Vector analysis provides a convenient shorthand by means of which mathematical relations between physical quantities may be compactly written and exhibited.

The physical quantities considered in vector analysis are of two kinds, scalars and vectors. A scalar quantity is one which is completely specified when its magnitude, size or number of units according to scale is given. Examples of scalars are mass, temperature, electric charge and quantity of heat. A vector quantity is one whose specification involves in addition to magnitude, a direction. Thus displacement, velocity, force, electric current, temperature gradient are all vectors.

A vector is represented algebraically by a letter in bold type or by a bar over the symbol representing it. Geometrically, it is represented by a directed line segment with an arrowhead on it, the length indicating magnitude, the inclination indicating direction, the arrowhead indicating sense. Vector calculus comprising vector differentiation. Many concepts are highly incorporated in various branches of engineering and technology.

3.2 : Vector Function

Definition :

If to each value of a scalar 't'belonging to some interval [a, b] of real numbers, there corresponds a vector **r**, we say taht **r** is a vector valued function (or vector function) of a scalar variable t and we write $\mathbf{r} = \mathbf{r}(t)$ or $\mathbf{r} = \mathbf{f}(t)$

The law of correspondence is called the function. The statement $\mathbf{r} = \mathbf{f}(t)$ implies that the function f associates to each scalar t belonging to some interval $[a, b]$, a vector **r**. The vector **r** is the value of function for scalar t.

Decomposition of a vector valued function :

Vector Equation :

Let i. j, k constitute a right handed triad of mutually perpendecular non-coplanar unit vectors. We further know that every vector in space can be uniquely expressed as a linear combination of three non-planar vectors.

Therefore, we may write

$$\vec{r} = \vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

Where $f_1(t), f_2(t), f_3(t)$ are scalar function with independent variable t.

$\vec{f}(t)$ is called vector valued function because for any particular value of t the value $\vec{f}(t)$ is a vector, relative to the orthonormal triad $\hat{i}, \hat{j}, \hat{k}$ of vector.

Equation of surface :

$\vec{F}(x, y, z) = 0$ represent a surface parametric equation of surface.

$x = f_1(u, v), y = f_2(u, v), z = f_3(u, v)$

Parametric eq^n of surface where u and v are two independent parameter.

Equation of curve :

Def^n : The locus of a point whose p.v. $\vec{r}$ w.r.t. some fixed point origin function of single independent parameter is determined as space curve.

$$\vec{r} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}, \ \vec{r} = \vec{r}(u), \ \vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Equation of curve :

1. $F_1(x, y, z) = 0 = F_2(x, y, z)$ equation of curve in Cartesian co-ordinate.
2. **Parametric equation of curve :** $x = \phi_1(t), y = \phi_2(t), z = \phi_3(t)$
3. **Vector equation of curve :** $\vec{r} = f\left(\vec{t}\right) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

Illustration :

Consider a particle moving in space so that at each instant t of time particle is at some point whose position vector with reference to a given origin O is **r**. Thus, the movement of the particle associates to each instant t of time a vector **r**. Therefore the vector **r** is a function of the scalar variable t.

Illustration – 1 : **Find equation of straight line passing a point $A\left(\vec{a}\right)$ is parallel to vector $\vec{b}$.**

Solution : Let P be any point on the required line whose p.v. is $\vec{r}$ w.r.t. origin 'O'. ***(fig. 3.1)***

$$\therefore\ AP = \vec{r} - \vec{a}, \ \overrightarrow{AP} = \vec{r} - \vec{a}$$

AP parallel to $\vec{b}$.

$$\vec{r} - \vec{a} = \kappa\vec{b}$$

$\therefore \boxed{\vec{r} = \kappa\vec{b} + \vec{a}}$ where κ is some scalar.

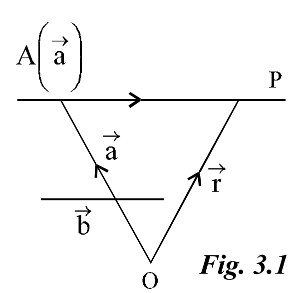


Fig. 3.1

Illustration – 2 : **Find equation of line passing through points whose p.v. are $\vec{a}, \vec{b}$ w.r.t. origin.**

Solution : Let P be any point on the required line whose P.V. is $\vec{r}$ w.r.t. origin O. ***(fig. 3.2)***

$$\overrightarrow{AB} = \vec{b} - \vec{a}\ ,\ \overrightarrow{AP} = \vec{r} - \vec{a}$$

$\overrightarrow{AP}$ is parallel to $\overrightarrow{AB}$ $\therefore \vec{r} - \vec{a} = \kappa\left(\vec{b} - \vec{a}\right)$

$$\therefore \vec{r} = \vec{a} + \kappa\left(\vec{b} - \vec{a}\right)$$

$$\boxed{\vec{r} = (1-\kappa)\vec{a} + \kappa\vec{b}}$$

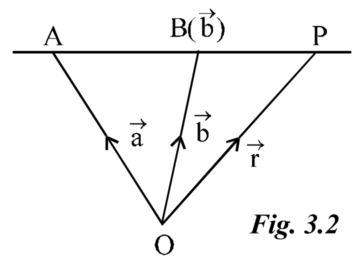


Fig. 3.2

Illustration – 3 : **Find the equation of plane passing through a point whose p.v. is $\vec{a}$ and parallel to two given vector $\vec{b}$ and $\vec{c}$.**

Solution : Let $\vec{P}$ be any point on the required plane whose p.v. is $\vec{r}$ w.r.t origin 'O'. ***(fig. 3.3)***

$$AP = \vec{r} - \vec{a}$$

Required plane is parallel to given two vector $\vec{b}$ & $\vec{c}$ separately.

$\therefore$ $\vec{b}$ & $\vec{c}$ are lie on the required plane $\overrightarrow{AP} = \alpha\vec{b} + \beta\vec{c}$ ·

$$\vec{r} - \vec{a} = \alpha\vec{b} + \beta\vec{c}\ ,\ \boxed{\vec{r} = \vec{a} + \alpha\vec{b} + \beta\vec{c}}$$

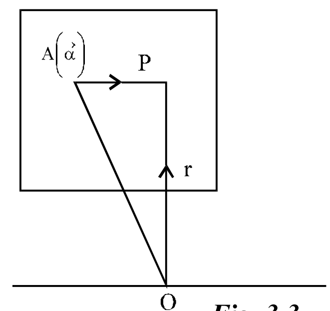


Fig. 3.3

Illustration – 4 : **Find equation of a plane whose normals are parallel to a vector $\vec{n}$.**

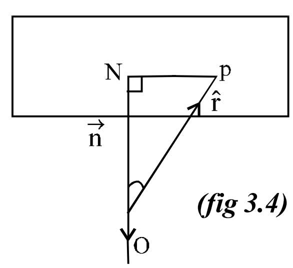
Solution : d = ON = Perpendicular from origin on the required plane. ***(fig. 3.4)***

Normal vector $\vec{n}$ is only ON.

Now ON = Projection of $\overrightarrow{OP} = r \cdot \hat{n}$

The direction of $\overrightarrow{N}$, $\dfrac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = d$

$\Rightarrow$ $\boxed{\vec{r} \cdot \hat{n} = d}$ Required eq^n of plane in normal term.



(fig 3.4)

3.3 : Limit of a vector function

A vector valued function $\boldsymbol{r}$ (t) is said to tend to a limit L when t tends to t_0 if for any given positive number $\in$, however small, there corresponds a positive number δ such that $|f(t) - L| < \in$ whenever $0 < |t - t_0| < \delta$.

We can also write, $\lim\limits_{t \to t_0} \boldsymbol{r}(t) = L$, if and only if $\lim\limits_{t \to t_0} \|\boldsymbol{r}(t) - L\| = 0$

Here the vector $\boldsymbol{r}$(t) tends to L in both length as well as direction as t tends to t_0.

Theorem – 1.

(a) If $\boldsymbol{r}(t) = x(t)\,\vec{\boldsymbol{i}} + y(t)\,\vec{\boldsymbol{j}}$, then $\lim\limits_{t\to t_0}\vec{\boldsymbol{r}}(t) = \lim\limits_{t\to t_0}x(t)\,\vec{\boldsymbol{i}} + \lim\limits_{t\to t_0}y(t)\,\vec{\boldsymbol{j}}$

provided the limits of component functions exists.

(b) If $\boldsymbol{r}(t) = x(t)\,\vec{\boldsymbol{i}} + y(t)\,\vec{\boldsymbol{j}} + z(t)\,\vec{\boldsymbol{k}}$, then $\lim\limits_{t\to t_0}\boldsymbol{r}(t) = \lim\limits_{t\to t_0}x(t)\,\vec{\boldsymbol{i}} + \lim\limits_{t\to t_0}y(t)\,\vec{\boldsymbol{j}} + \lim\limits_{t\to t_0}z(t)\,\vec{\boldsymbol{k}}$,

provided the limits of components functions exists.

***Proof :* (b)** $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$

Let $\lim\limits_{t\to t_0}x(t) = l_1$, $\lim\limits_{t\to t_0}y(t) = l_2$ and $\lim\limits_{t\to t_0}z(t) = l_3$

Let $L = \langle l_1, l_2, l_3\rangle$

$$\therefore \lim_{t\to t_0}\boldsymbol{r}(t) = \left\langle \lim_{t\to t_0}x(t), \lim_{t\to t_0}y(t), \lim_{t\to t_0}z(t)\right\rangle = \langle l_1, l_2, l_3\rangle = L = l_1\boldsymbol{i} + l_2\mathbf{j} + l_3\mathbf{k}$$

$$= \lim_{t\to t_0}x(t)\vec{\boldsymbol{i}} + \lim_{t\to t_0}y(t)\vec{\mathbf{j}} + \lim_{t\to t_0}z(t)\vec{\mathbf{k}}$$

3.4 : Algebra of limits of vector valued functions

Lte $\vec{\boldsymbol{r}}_1(t)$ and $\vec{\boldsymbol{r}}_2(t)$ be vector valued functions in R^2 or R^3. Then the following rules of limits hold.

Theorem – 2

(a) $\lim\limits_{t\to t_0}\left[\boldsymbol{r}_1(t) \pm \boldsymbol{r}_2(t)\right] = \lim\limits_{t\to t_0}\boldsymbol{r}_1(t) \pm \lim\limits_{t\to t_0}\boldsymbol{r}_2(t)$

(b) $\lim\limits_{t\to t_0}\left[\boldsymbol{r}_1(t).\boldsymbol{r}_2(t)\right] = \left(\lim\limits_{t\to t_0}\boldsymbol{r}_1(t)\right).\left(\lim\limits_{t\to t_0}\boldsymbol{r}_2(t)\right)$

(c) $\lim\limits_{t\to t_0}\left[\boldsymbol{r}_1(t) \times \boldsymbol{r}_2(t)\right] = \left(\lim\limits_{t\to t_0}\boldsymbol{r}_1(t)\right) \times \left(\lim\limits_{t\to t_0}\boldsymbol{r}_2(t)\right)$

(d) $\lim\limits_{t\to t_0}\left(c\,\boldsymbol{r}_1(t)\right) = c\lim\limits_{t\to t_0}\boldsymbol{r}_1(t)$, where c is a constant vector.

Proof :

(a) Let $\boldsymbol{r}_1(t) = x_1(t)\,\boldsymbol{i} + y_1(t)\,\mathbf{j} + z_1(t)\,\mathbf{k}$ and $\boldsymbol{r}_2(t) = x_2(t)\,\boldsymbol{i} + y_2(t)\,\mathbf{j} + z_2(t)\,\mathbf{k}$

$$\lim_{t\to t_0}\left[\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)\right]$$

$$= \lim_{t\to t_0}\left[(x_1(t) + x_2(t))\boldsymbol{i} + (y_1(t) + y_2(t))\mathbf{j} + (z_1(t) + z_2(t))\mathbf{k}\right]$$

$$= \left[\lim_{t\to t_0}\{x_1(t) + x_2(t)\}\right]\boldsymbol{i} + \left[\lim_{t\to t_0}\{y_1(t) + y_2(t)\}\right]\mathbf{j} + \left[\lim_{t\to t_0}\{(z_1(t) + z_2(t)\}\right]\mathbf{k}$$

$$= \left[\lim_{t\to t_0}x_1(x)\boldsymbol{i} + \lim_{t\to t_0}y_1(t)\mathbf{j} + \lim_{t\to t_0}z_1(t)\mathbf{k}\right] + \left[\lim_{t\to t_0}x_2(t)\boldsymbol{i} + \lim_{t\to t_0}y_2(t)\mathbf{j} + \lim_{t\to t_0}z_2(t)\mathbf{k}\right]$$

$$= \lim_{t\to t_0}\boldsymbol{r}_1(t) + \lim_{t\to t_0}\boldsymbol{r}_2(t)$$

Similarly,

Proof of $\lim\limits_{t\to t_0}\left[\boldsymbol{r}_1(t) - \boldsymbol{r}_2(t)\right] = \lim\limits_{t\to t_0}\boldsymbol{r}_1(t) - \lim\limits_{t\to t_0}\boldsymbol{r}_2(t)$.

(b), (c), (d) are left as an exercises.

Example – 1: ***Find the following limits***

(a) $\lim\limits_{t\to\infty}\left(\dfrac{t^3+1}{4t^3+2}\boldsymbol{i}+\dfrac{1}{t}\mathbf{j}\right)$ (b) $\lim\limits_{t\to1}\left(\dfrac{2}{t^2}\boldsymbol{i}+\dfrac{\ln t}{t^2-1}\mathbf{j}+\cos 3t\,\mathbf{k}\right)$

Solution :

(a) $$\lim_{t\to\infty}\left(\frac{t+1}{4t^3+2}\boldsymbol{i}+\frac{1}{t}\mathbf{j}\right)=\left(\lim_{t\to\infty}\frac{t^3+1}{4t^3+2}\right)\boldsymbol{i}+\left(\lim_{t\to\infty}\frac{1}{t}\right)\mathbf{j}=\left(\lim_{t\to\infty}\frac{1+\frac{1}{t^3}}{4+\frac{2}{t^3}}\right)\boldsymbol{i}+0\mathbf{j}=\frac{1}{4}\boldsymbol{i}+0\mathbf{j}$$

Alternatively :

$$\lim_{t\to\infty}\boldsymbol{r}(t)=\lim_{t\to\infty}\left\langle\frac{t^3+1}{4t^3+2},\frac{1}{t}\right\rangle=\left\langle\lim_{t\to\infty}\frac{t^3+1}{4t^3+2},\lim_{t\to\infty}\frac{1}{t}\right\rangle=\left\langle\frac{1}{4},0\right\rangle=\frac{1}{4}\boldsymbol{i}+0\mathbf{j}$$

(b) $$\lim_{t\to1}\left(\frac{2}{t^2}\boldsymbol{i}+\frac{\ln t}{t^2-1}\mathbf{j}+\cos 3t\,\mathbf{k}\right)$$

$$=\left\langle\lim_{t\to1}\frac{2}{t^2},\lim_{t\to1}\frac{\ln t}{t^2-1},\lim_{t\to1}\cos 3t\right\rangle=\left\langle 2,\lim_{t\to1}\frac{1/t}{2t},\cos 3\right\rangle=\left\langle 2,\frac{1}{2},\cos 3\right\rangle$$

3.5: Continuity of vector function

If $\boldsymbol{r}(t)=f_1(t)\boldsymbol{i}+f_2(t)\mathbf{j}+f_3(t)\mathbf{k}$, then $\boldsymbol{r}(t)$ is continuous at t_0 when t_0 is in the domain of the component functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ and

$$\lim_{t\to t_0} f_1(t)=f_1(t_0),\quad \lim_{t\to t_0} f_2(t)=f_2(t_0)\text{ and }\lim_{t\to t_0} f_3(t)=f_3(t_0)$$

A vector valued function $\boldsymbol{r}(t)$ is continuous at $t = t_0$ if

(i) $\boldsymbol{r}(t_0)$ is defined

(ii) $\lim\limits_{t\to t_0}\boldsymbol{r}(t)$ exists, i.e. $\lim\limits_{t\to t_0^+}\boldsymbol{r}(t)=\lim\limits_{t\to t_0^-}\boldsymbol{r}(t)$

Thus, $\boldsymbol{r}(t)$ is continuous at $t = t_0$ if $\lim\limits_{t\to t_0}\boldsymbol{r}(t)=\boldsymbol{r}(t_0)$

Continuity on an interval

Let I =[a, b]. Then the vector valued function $\boldsymbol{r}(t)$ is continuous on I if it is continuous at each point of I.

Example – 2 :

- The vector valued function $\boldsymbol{r}(t)=\langle 2t, 3t, t^2+t\rangle$ is continuous, because

  $\lim\limits_{t\to t_0}\boldsymbol{r}(t)$ is $\langle 2t_0, 3t_0, t_0^2+t_0\rangle$ which is $\boldsymbol{r}(t_0)$

  $\therefore$ $\boldsymbol{r}(t)$ is continuous for all values of t_0 i.e., on the interval $(-\infty, \infty)$

Remark :

A vector valued function $\boldsymbol{r}(t)$ is continuous at $t = t_0$ if and only if the component functions are continuous at $t = t_0$.

Example – 3 : Test for continuity of $\boldsymbol{r}(t) = \begin{cases} t^2\boldsymbol{i} + \dfrac{\sin t}{t}\mathbf{j} + t\,\mathbf{k}, & t \neq 0 \\ \mathbf{j}, & t = 0 \end{cases}$ at $t = 0$

Solution : Let $\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $x(t) = t^2, y(t) = \dfrac{\sin t}{t}$ and $z(t) = t$

$\lim\limits_{t\to 0} x(t) = 0$, $\lim\limits_{t\to 0} y(t) = 1$ and $\lim\limits_{t\to 0} z(t) = 0$

Hence the limit exists

$$\therefore \lim_{t\to 0} \boldsymbol{r}(t) = \left(\lim_{t\to 0} t^2\right)\boldsymbol{i} + \left(\lim_{t\to 0}\frac{\sin t}{t}\right)\mathbf{j} + \left(\lim_{t\to 0} t\right)\mathbf{k} = 0\boldsymbol{i} + 1\mathbf{j} + 0\mathbf{k} = \mathbf{j}$$

and $\boldsymbol{r}(0) = \mathbf{j}$

$$\therefore \lim_{t\to 0} \boldsymbol{r}(t) = \boldsymbol{r}(0)$$

Thus $\boldsymbol{r}(t)$ is continuous at $t = 0$

Example – 4 : Determine whether $\boldsymbol{r}(t)$ is continuous at $t = 0$?

(a) $\boldsymbol{r}(t) = 3\sin t\,\boldsymbol{i} + 3t\,\mathbf{j}$

(b) $\boldsymbol{r}(t) = t^2\boldsymbol{i} + \dfrac{\cos t}{t}\mathbf{j} + t\,\mathbf{k}$

Solution : (a) $\lim\limits_{t\to 0} \boldsymbol{r}(t) = \left(\lim\limits_{t\to 0} 3\sin \mathbf{t}\right)\boldsymbol{i} + \left(\lim\limits_{t\to 0} 3t\right)\mathbf{j} = 0\,\boldsymbol{i} + 0\,\mathbf{j} = \mathbf{0}$

$\boldsymbol{r}(0) = 0\,\boldsymbol{i} + 0\,\mathbf{j} = \mathbf{0}$

$\therefore \lim\limits_{t\to 0} \boldsymbol{r}(t) = \boldsymbol{r}(0)$

$\therefore \boldsymbol{r}(t)$ is continuous at $t = 0$

(b) Since $\lim\limits_{t\to 0}\dfrac{\cos t}{t}$ does not exist, therefore $\boldsymbol{r}(t)$ is not continuous at $t = 0$

3.6 : Differentiation and Integration of Vector function

In this section we shall discuss vector derivatives, tangent vectors, properties of vector derivatives, velocity vector, speed, direction of motion, acceleration vector.

The differentiation of vector-valued function is similar to that of differentiation of real-valued function.

3.7 : Differentiation of Vector Function

Differentiability – We define the derivative of a vector-valued function $f(t)$ at a point $t = a$ by the same type of limit equation as we use for scalar functions. Thus,

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} \qquad ......(1)$$

provided the limit on the right exists. We then expect that f is differentiable at $t = a$ each of its components is differentiable at $t = a$. In this connection we prove the following result :

Definition : Vector derivative

Let $\boldsymbol{r}(t)$ be a vector valued function. The derivative of $\boldsymbol{r}(t)$ with respect to t is another vector-valued function $\boldsymbol{r}'(t)$defined by $\boldsymbol{r}'(t) = \lim\limits_{h\to 0}\dfrac{\boldsymbol{r}(t+h)-\boldsymbol{r}(t)}{h}$, provided the limit exists.

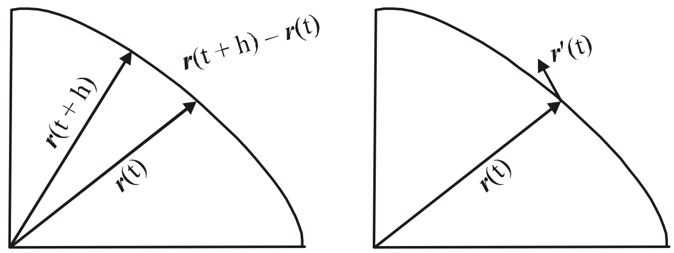


Fig. 3.4 (a) Derivatives of Vectors

Note : We can use symbols $\dfrac{d\boldsymbol{r}}{dt}, \dfrac{d}{dt}[\boldsymbol{r}(t)]$ or $\boldsymbol{r}'(t)$ for the derivative of $\boldsymbol{r}(t)$.

Theorem – 3 : If $\boldsymbol{r}(t)$ be a vector valued function, there $\boldsymbol{r}$ is differentiable at t if and only iff each of the component functions of $\boldsymbol{r}(t)$ are differentiable at t.

Proof : Let $\boldsymbol{r}(t)$ be a vector-valued function in R^3.

Suppose $\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$

$$\therefore \boldsymbol{r}'(t) = \lim_{h\to 0}\frac{r(t+h)-r(t)}{h}$$

$$= \lim_{h\to 0}\frac{[x(t+h)\boldsymbol{i} + y(t+h)\mathbf{j} + z(t+h)\mathbf{k}] - [x(t)\boldsymbol{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]}{h}$$

$$= \left(\lim_{h\to 0}\frac{x(t+h)-x(t)}{h}\right)\boldsymbol{i} + \left(\lim_{h\to 0}\frac{y(t+h)-y(t)}{h}\right)\mathbf{j} + \left(\lim_{h\to 0}\frac{z(t+h)-z(t)}{h}\right)\mathbf{k}$$

$$= x'(t)\,\boldsymbol{i} + y'(t)\,\mathbf{j} + z'(t)\mathbf{k}$$

Theorem – 4 : Every differentiable vector-valued functions are continuous but not the converse is not true.

Proof : Let $\boldsymbol{r}(t)$ be a differentiable vector valued function at t.

$$\lim_{h\to 0}[\boldsymbol{r}(t+h)-\boldsymbol{r}(t)]$$

$$= \lim_{h\to 0}\left(\frac{\boldsymbol{r}(t+h)-r(t)}{h}.h\right) = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h)-\boldsymbol{r}(t)}{h}\lim_{h\to 0}(h) = \boldsymbol{r}'(t).0 = 0$$

$\Rightarrow \boldsymbol{r}(t)$ is continuous at t.

Hence every differentiable vector valued function is continuous at t

The converse is not true. Take $\boldsymbol{r}(t) = |t|\,\boldsymbol{i}$

$$\therefore |\boldsymbol{r}(t)-\boldsymbol{r}(0)| = ||t|\,i - 0\boldsymbol{i}| = |t|$$

$$\therefore \lim_{t\to 0}\boldsymbol{r}(t) = 0 = \boldsymbol{r}(0)$$

Hence $\boldsymbol{r}(t)$ is continuous at $t = 0$

But, $\lim\limits_{t\to 0}\dfrac{\boldsymbol{r}(t)-\boldsymbol{r}(0)}{t-0} = \lim\limits_{t\to 0}\dfrac{|t|\,\boldsymbol{i}}{t}$ does not exist.

$\therefore\ \boldsymbol{r}'(t)$ does not exist

$\Rightarrow \boldsymbol{r}(t)$ is not differentiable at $t = 0$.

Example – 5 : If $\boldsymbol{r}(t) = (a\cos t)\,\boldsymbol{i} + (a \sin t)\,\mathbf{j} + bt\,\mathbf{k}$, then find $\boldsymbol{r}'(t)$.

Solution : We have

$$\boldsymbol{r}'(t) = \frac{d}{dt}(a\cos t)\boldsymbol{i} + \frac{d}{dt}(a\sin t)\mathbf{j} + \frac{d}{dt}(bt)\mathbf{k} = (-a\sin t)\,\boldsymbol{i} + (a\cos t)\,\mathbf{j} + b\mathbf{k}$$

Rules for differentiating vector functions

Theorem – 5 :

If $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$ be vector valued functions, then the following two results holds good in differentiation.

1. $\dfrac{d}{dt}(\boldsymbol{c}) = \mathbf{0}$ where $\boldsymbol{c}$ is a constant vector
2. $(a\boldsymbol{r}_1 + b\boldsymbol{r}_2)' = a\boldsymbol{r}_1' + b\boldsymbol{r}_2'$

   In particular if $a = 1$, $b = 1$, then $(\boldsymbol{r}_1 + \boldsymbol{r}_2)' = \boldsymbol{r}'_1 + \boldsymbol{r}'_2$
3. $(a\boldsymbol{r})' = a\boldsymbol{r}'$
4. $(\boldsymbol{r}_1.\boldsymbol{r}_2)' = \boldsymbol{r}_1'.\boldsymbol{r}_2 + \boldsymbol{r}_1.\boldsymbol{r}_2'$ i.e., $\dfrac{d}{dt}\left[\boldsymbol{r}_1(t).\boldsymbol{r}_2(t)\right] = \boldsymbol{r}_1(t).\left[\dfrac{d}{dt}\boldsymbol{r}_2(t)\right] + \left[\dfrac{d}{dt}\boldsymbol{r}_1(t)\right].\boldsymbol{r}_2(t)$
5. $(\boldsymbol{r}_1\times\boldsymbol{r}_2)' = \boldsymbol{r}_1'\times\boldsymbol{r}_2 + \boldsymbol{r}_1\times\boldsymbol{r}_2'$ i.e., $\dfrac{d}{dt}[\boldsymbol{r}_1(t)\times\boldsymbol{r}_2(t)] = \boldsymbol{r}_1(t)\times\left[\dfrac{d}{dt}\boldsymbol{r}_2(t)\right] + \left[\dfrac{d}{dt}\boldsymbol{r}_1(t)\right]\times\boldsymbol{r}_2(t)$
6. $[\boldsymbol{r}(h(t))]' = h'(t)\boldsymbol{r}'(h(t))$
7. $\dfrac{d}{dt}\left[f(t)\boldsymbol{r}(t)\right] = \left[\dfrac{d}{dt}f(t)\right]\boldsymbol{r}(t) + f(t)\left[\dfrac{d}{dt}\boldsymbol{r}(t)\right]$, where $f(t)$ is a real valued differentiable function

Proof :

(1) Let $\boldsymbol{c} = c_1\boldsymbol{i} + c_2\mathbf{j} + c_3\mathbf{k}$

$$\therefore \frac{d}{dt}(\boldsymbol{c}) = \left[\frac{d}{dt}(c_1)\right]\boldsymbol{i} + \left[\frac{d}{dt}(c_2)\right]\boldsymbol{j} + \left[\frac{d}{dt}(c_3)\right]\boldsymbol{k} = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k} = 0$$

(2) $\dfrac{d}{dt}[\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)]$

$$= \lim_{h\to 0}\frac{[\boldsymbol{r}_1(t+h) + \boldsymbol{r}_2(t+h] - [\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)]}{h}$$

$$=\left[\lim_{h\to 0}\frac{\boldsymbol{r}_1(t+h)-\boldsymbol{r}_1(t)}{h}\right]+\left[\lim_{h\to 0}\frac{\boldsymbol{r}_2(t+h)-\boldsymbol{r}_2(t)}{h}\right]=\frac{d}{dt}\left[\boldsymbol{r}_1(t)\right]+\frac{d}{dt}\left[\boldsymbol{r}_2(t)\right] \text{ i.e.,}$$

$$(\boldsymbol{r}_1+\boldsymbol{r}_2)'=\boldsymbol{r}_1'+\boldsymbol{r}_2'$$

(4) Let $\boldsymbol{f}(t) = \boldsymbol{r}_1(t).\, \boldsymbol{r}_2(t)$

$$\therefore f'(t)=\frac{d}{dt}[\boldsymbol{r}_1(t).\boldsymbol{r}_2(t)]=\lim_{h\to 0}\frac{\boldsymbol{f}(t+h)-\boldsymbol{f}(t)}{h}$$

$$=\lim_{h\to 0}\frac{\boldsymbol{r}_1(t+h).\boldsymbol{r}_2(t+h)-\boldsymbol{r}_1(t).\boldsymbol{r}_2(t)}{h}$$

$$=\lim_{h\to 0}\frac{\boldsymbol{r}_1(t+h).\boldsymbol{r}_2(t+h)-\boldsymbol{r}_1(t+h).\boldsymbol{r}_2(t)+\boldsymbol{r}_1(t+h).\boldsymbol{r}_2(t)-\boldsymbol{r}_1(t).\boldsymbol{r}_2(t)}{h}$$

$$=\lim_{h\to 0}\frac{\boldsymbol{r}_1(t+h).\dfrac{\boldsymbol{r}_2(t+h)-\boldsymbol{r}_2(t)}{h}+\boldsymbol{r}_2(t).\dfrac{\boldsymbol{r}_1(t+h)-\boldsymbol{r}_1(t)}{h}}{h}$$

$$=\boldsymbol{r}_1(t).\frac{d}{dt}\boldsymbol{r}_2(t)+\boldsymbol{r}_2(t).\frac{d}{dt}\boldsymbol{r}_1(t) = \boldsymbol{r}_1\cdot\boldsymbol{r}_2'+\boldsymbol{r}_2\cdot\boldsymbol{r}_1'$$

Remark : Is $\lim_{h\to 0}\boldsymbol{r}_1(t+h)=\boldsymbol{r}_1(t)$?

Orthogonality condition

If $\boldsymbol{r}(t)$ be a vector valued function such that $\boldsymbol{r}(t).\boldsymbol{r}'(t) = 0$, then $\boldsymbol{r}$ and $\boldsymbol{r}'$ are orthogonal.

***Illustration* : Obtain the derivative of $f(t) = (\sin^2 t)\,\hat{i} + (\ln t)\,\hat{j} + \tan^{-1}(3t)\,\hat{k}$**

Solution : The function and its derivative are defined at every positive value of t.

$$f'(t)=(2\sin t\cos t)\hat{i}+\frac{1}{t}\hat{j}+\frac{3}{1+9t^2}\hat{k}.$$

We not give the geometrical interpretation of the derivative at a vector valued function.

Geometrical Representation of Derivative :

Draw the vector $f(t)$ for values of the independent variable t in some interval containing t and $t + \Delta t$ from the same initial point O. Then the locus of head of arrows representating f for different values of t traces out a space curve. ***(fig. 3.5)***

Let $OP = f(t)$ and $OQ = f(t + \Delta t)$, then

$$f(t + \Delta t) - f(t) = OQ - OP = PQ = \Delta f, \text{ (say)},$$

Hence $\dfrac{df}{dt}=\lim_{\Delta t\to 0}\dfrac{\Delta f}{\Delta t}$

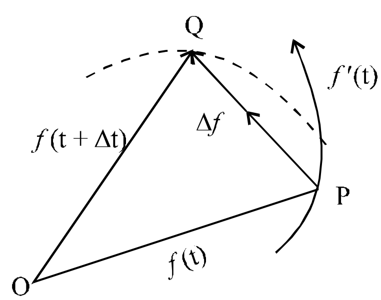


***Fig. 3.5* : Derivative of a vector function**

The direction of $\frac{df}{dt}$ is the limiting direction $\frac{\Delta f}{\Delta t}$ or of $|\Delta f|$. But as Q tends to P, PQ tends to the tangent line at P. Hence the direction of $\frac{df}{dt}$ is along the tangent to the space curve traced out by P. Let s denote the length of the arc of this curve from a fixed point on it upto P. Then the magnitude of $\frac{df}{dt}$ is given by

$$\left|\frac{df}{dt}\right| = \lim_{\Delta t \to 0} \frac{|\Delta f|}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\Delta f|}{\Delta s} \cdot \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

since the ratio $\frac{|\Delta f|}{\Delta t} = \frac{\text{chord PQ}}{\text{arc PQ}} \to 1 \text{ as } \Delta t \to 0$.

Thus derivative of a vector function represents a vector whose direction is tangent to the space curve traced by the vector function and the magnitude is $\frac{ds}{dt}$ where s is the arc length from a fixed point on the curve to the variable point representing the vector function.

You may note here that a vector will change if either its magnitude changes or direction changes or both direction and magnitude changes. In this regard, the following results may be remembered :

(a) The necessary and sufficient condition for f (t) to be constant is $\frac{df}{dt} = 0$.

(b) The necessary and sufficient condition for f (t) to have constant magnitude is $f \cdot \frac{df}{dt} = 0$.

(c) The necessary and sufficient condition for f (t) to have constant/uniform direction is $f \times \frac{df}{dt} = 0$.

The familiar rules of differentiation on real functions yield corresponding rules for differentiating vector functions; for example,

(i) $(Cf)' = Cf'$ (C is a constant).

(ii) $(u \pm v)' = u' \pm v'$

(iii) $(uf)' = \frac{du}{dt} f + u\frac{df}{dt}$ (u is a scalar function of t).

(iv) $(u \cdot v)' = u' \cdot v + u \cdot v'$

(v) $(u \times v)' = u' \times v + u \times v'$

(vi) $[u\ v\ w]' = [u'\ v\ w] + [u\ v'\ w] + [u\ v\ w']$

Vector Calculus :

Let $\vec{A}$, $\vec{B}$, $\vec{C}$ are vector valued function in single parameter variable parameter u.

1. $\frac{d}{du}\left(\vec{A} \pm \vec{B}\right) = \frac{d}{du}\left(\vec{A}\right) + \frac{d}{du}\left(\vec{B}\right)$

2. $\dfrac{d}{du}\left(\alpha \vec{A}\right) = \alpha \cdot \dfrac{d}{du}\left(\vec{A}\right)$

3. $\dfrac{d}{du}\left(\vec{A}\cdot\vec{B}\right) = \dfrac{d}{du}\left(\vec{A}\right)B + \vec{A}\dfrac{d\vec{B}}{du}$

4. $\dfrac{d}{du}\left(\vec{A}\times\vec{B}\right) = \dfrac{d}{du}\left(\vec{A}\right)\times\vec{B} + \vec{A}\times\dfrac{d\vec{B}}{du}$

5. $\dfrac{d}{du}\left[\vec{A}\,\vec{B}\,\vec{C}\right] = \left[\dfrac{d\vec{A}}{du}\cdot\vec{B}\vec{C}\right] + \left[\vec{A}\dfrac{d\vec{B}}{du}\vec{C}\right] + \left[\vec{A}\vec{B}\dfrac{d\vec{C}}{du}\right]$

6. $\dfrac{d}{du}\left[\vec{A}\times(\vec{B}\times\vec{C})\right] = \dfrac{d\vec{A}}{du}\times\left(\vec{B}\times\vec{C}\right) + \vec{A}\times\left(\dfrac{d\vec{B}}{du}\times\vec{C}\right) + \vec{A}\times\left(\vec{B}\times\dfrac{d\vec{C}}{du}\right)$

7. If '$\vec{f}$' is constant $\dfrac{df}{du} = 0$.

   $\vec{f}(u) = \vec{f}_1(u)\,\hat{i} + \vec{f}_2(u)\hat{j} + \vec{f}_3(u)\hat{k}$

8. If $\vec{f}$ is vector constant in magnitude having variable direction. $\vec{f}\times\dfrac{d\vec{f}}{dt} = 0$

   i.e. $\vec{f}$ is parallel to . $\dfrac{d\vec{f}}{dt}$

Applications of Derivatives.

The simplest application of vector calculus is some basic facts about curves in space.

Given a Cartesian coordinate system, we may represent a curve C by a vector function

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Here to each value of the real variable t, there corresponds a point of C having position vector $r(t_0)$. ***(fig. 3.6)***

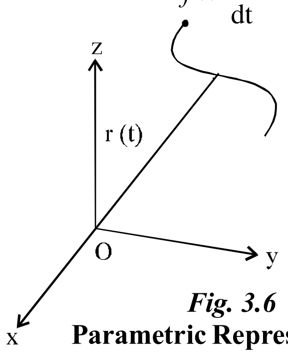
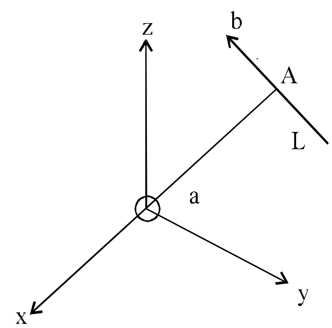


Fig. 3.6
Parametric Representation of a curve

For example, any straight line L can be represented in the form r(t) = a + tb,

Where a and b are constant vectors and line L passes through the point A with position vector r = a and has the direction of b. ***(fig. 3.7)***



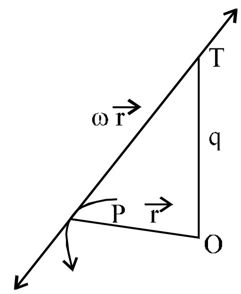
(Fig. 3.7)
(Parametric Representations of Straight line)

The vector function $\vec{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j}$ represents an ellipse in the xy-plane with centre at origin and axes in the directions of x and y axes.

Further, if a curve C is represented by a continuously differentiable vector function $\vec{r}(t)$, where t is any parameter, then the vector.

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

has the direction of the tangent to the curve C at P(t). ***(fig. 3.8)*** Thus, the position vector of a point on the tangent is the sum of the position vector $\vec{r}$ of a point. P on the curve and a vector in the direction of the tangent. Hence the parametric representation of the tangent is where both $\vec{r}$ and $\frac{d\vec{r}}{dt}$ depend on P and the parameter ω is a real variables.



(Fig. 3.8)
Representation of the tangent to a curve

3.8 : *Vector Function of a Single Variable and the Derivative of a Vector*

Let the position vector of a point $P(x, y, z)$ in space be

$$\vec{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$$

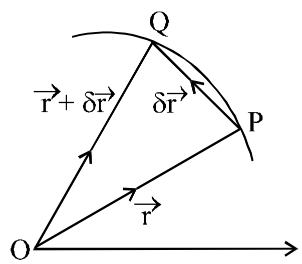
If x, y, z are all functions of a single parameter t, then $\vec{r}$ is said to be a vector function of t which is also referred to as a vector point function usually denoted as

$$\vec{r} = x(t)i + y(t)j + z(t)k$$

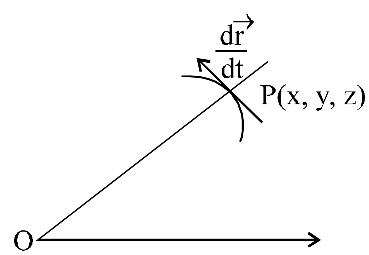
is called as the vector equation of the curve.

On giving a small increment δt to t $\vec{r}$ gets an increament $\delta\vec{r}$
Let P and Q be two neighbouring points on the curve.

Let $\overrightarrow{OP} = \vec{r}$ and $\overrightarrow{OQ} = \vec{r} + \delta\vec{r}$



(Fig. 3.9)



(Fig. 3.10)

$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{\delta r}$ by a basic property.

The derivative of the vector $\vec{r}(t)$ is denoted and is defined as follows.

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \underset{\delta t \to 0}{lt} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t} \quad ....(1)$$

$\lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$ is a vector parallel to the tangent at P to the curve $\vec{r} = \vec{r}\ t$.

Geometrical meaning of the derivative of a vector

As $\delta t \to 0$, Q $\to$ P which means that the chord PQ approaches the tangent to the curves at P. Thus, geometrically we can say that $\frac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at P.

Physical meaning of $\frac{d\vec{r}}{dt}$

Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ represents the positive vector of a point moving along a curve, *x, y, z* will be a function of the time variable *t* and accordingly $\vec{r}$ is a function of the time variable *t*.

$\therefore\ \vec{v} = \frac{d\vec{r}}{dt}$ gives the velocity of the particle at time *t*.

Further $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity $\vec{v}$ and is called the acceleration of the particle at time *t*.

Further $|\vec{v}| = \left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt}$ is the speed of *P* where *s* is the arc length measured from a fixed point on the curve onto the point *P* along the curve.

Remark :

When a particle moves along the curve velocity vector is directed along the tangent.

Derivative of a constant vector :

A vector is said to be constant if both it's magnitude and direction are fixed. If either of these changes the vector is not constant. i.e., The derivative of a constant vector is always zero or (null) vector.

Geometrical interpretation of $\vec{\mathbf{r}}$ and $\frac{d\vec{\mathbf{r}}}{\mathbf{dt}}$.

Prove that the following :

(i) $\vec{v}\frac{d\vec{v}}{dt} = v\frac{dv}{dt}$. Where $|\vec{v}| = v$.

Proof : We know $\frac{d}{dv}(\vec{u}\cdot\vec{v}) = \frac{d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{d\vec{v}}{dt}$.

Taking $\vec{u} = \vec{v}$ in the above $\frac{d}{dv}(\vec{v}\cdot\vec{v}) = \frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt}$

$$\frac{d}{dv}\left(\vec{v}^2\right) = 2\vec{v}\frac{d\vec{v}}{dt} \quad \text{..................(i)}$$

But we know $\vec{v}^2 = v^2 \quad [\therefore |\vec{v}| = v]$

$$\Rightarrow \frac{d}{dv}\left(\vec{v}^2\right) = 2v\frac{dv}{dt} \quad \text{....................(ii)}$$

Now from (i) and (ii)

$$\Rightarrow 2v\frac{dv}{dt} = 2\vec{v}\cdot\frac{d\vec{v}}{dt} \qquad \Rightarrow \boxed{v\cdot\frac{dv}{dt} = \vec{v}\cdot\frac{d\vec{v}}{dt}}$$

(ii) For a vector $\vec{v}$ of a constant length iff $\vec{v}\cdot\frac{d\vec{v}}{dt}=0$

Proof : **Necessary part**

Let $\vec{v}$ is a constant length i.e., $|\vec{v}| = v$ (say) const.

Now we know $\vec{v}\cdot\frac{d\vec{v}}{dt}=v\cdot\frac{dv}{dt}=v\cdot 0=0$

Sufficient Part : Let us suppose that $\vec{v}\cdot\frac{d\vec{v}}{dt}=0$

$\Rightarrow\quad v\cdot\frac{dv}{dt}\Rightarrow\frac{dv}{dt}=0\ [\because v\neq 0]\Rightarrow\ v=\text{constant}$

$\Rightarrow\quad |\vec{v}| = \text{const.}\qquad\Rightarrow\ \vec{v}$ has const. length.

(iii) $\frac{d}{dt}\left(\vec{u}\times\frac{d\vec{u}}{dt}\right)=\vec{u}\times\frac{d^2\vec{u}}{dt^2}$

Proof : L.H.S. $=\frac{d}{dt}\left(\vec{u}\times\frac{d\vec{u}}{dt}\right)$

We know $\frac{d}{dt}(\vec{u}\times\vec{v})=\frac{d\vec{u}}{dt}\times\vec{v}+\vec{u}\times\frac{d\vec{v}}{dt}$

Taking $\vec{v}=\frac{d\vec{u}}{dt}$

$$\frac{d}{dt}\left(\vec{u}\times\frac{d\vec{u}}{dt}\right)=\frac{d\vec{u}}{dt}\times\frac{d\vec{u}}{dt}+\vec{u}\times\frac{d}{dt}\left(\frac{d\vec{u}}{dt}\right)=0+u\times\frac{d^2\vec{u}}{dt^2}=\vec{u}\times\frac{d^2\vec{u}}{dt^2}$$

(iv) If the vector $\vec{u}$ has a fixed direction if and only if $\vec{u}\times\frac{d\vec{u}}{dt}=0$

Proof : **Necessary part –** Given $\vec{u}$ was a fixed direction we have to show $\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}$

Let $\vec{u}=u\cdot\hat{n}$ (where $|\vec{u}|=u$ and $\hat{n}$ is a unit vector.)

Since $\vec{u}$ has fixed direction $\Rightarrow\hat{n}$ has a fixed direction and also it has a constant length.

$\Rightarrow\hat{n}=$ constant vector.

$\Rightarrow\frac{d\hat{n}}{dt}=0$

Now $\vec{u}\times\frac{d\vec{u}}{dt}=u\cdot\hat{n}\times\frac{d}{dt}(u\cdot\hat{n})=u\cdot\hat{n}\times\left[\frac{du}{dt}\cdot\hat{n}+u\cdot\frac{d\hat{n}}{dt}\right]$

$=\vec{u}\cdot\hat{n}\times\frac{d\vec{u}}{dt}\cdot\hat{n}+\vec{u}\cdot 0\ =\vec{u}\cdot\frac{d\vec{u}}{dt}(\hat{n}\times\hat{n})=\vec{0}$

Sufficient part : Given that $\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}$

We have to show that $\vec{u}$ was a fixed direction. Since $\vec{u}\times\frac{d\vec{u}}{dt}=0$

Since $\vec{u}$ and $\frac{d\vec{u}}{dt}$ are parallel vectors. But $\frac{d\vec{u}}{dt}$ is the vector is the tangential to the curve $\vec{u} = \vec{u}\,(t)$ at the point P (say).

So $\frac{d\vec{u}}{dt}$ has a fixed direction which is tangential to the curves at P.

$\Rightarrow \vec{u}$ has a fixed direction. Since it is parallel to $\frac{du}{dt}$.

3.9 : Tangential Component of Acceleration of Particle Moving Curve

Let the particle 'P' moves along a curve. Let $\vec{r}$ be p.v. of the particle at any instant 't'. Let Q be another position on the curve at instant t + Δt whose p.v. is $\vec{r} + \Delta\vec{r}$ w.r.t. origin.

Let i and j be the unit vector along tangential direction and normal direction at P and i + Δi, j + Δj be the unit vector along tangential and normal direction at Q. ***(fig. 3.11)***

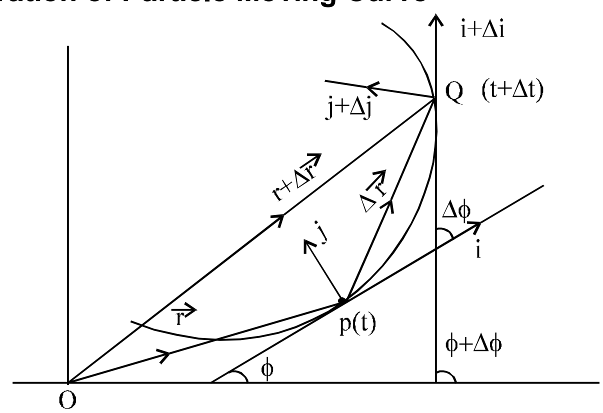


Fig. 3.11

Let tangent at P and Q makes angle ϕ, ϕ + Δ ϕ with X-direction.

Transform unit vectors $\hat{i}, \hat{j}, \hat{i} + \Delta\hat{i}$ and $\hat{j} + \Delta\hat{j}$ for common origin O′. ***(fig.3.12)***

O′AB is an isosceles triangle where $\overrightarrow{O'A} = \hat{i}$,

$\overrightarrow{O'B} = \hat{i} + \Delta\hat{i}$ and $\overrightarrow{AB} = \Delta i$.

and ∠BO′A = Δϕ perpendicular O′C on AB.

$$\angle CO'A = \frac{\Delta\phi}{2},\ \overrightarrow{AC} = \frac{\Delta i}{2}$$

In O′AC right angle triangle,

$$\sin\left(\frac{\Delta\phi}{2}\right) = \frac{AC}{O'A} = \frac{\left|\frac{\Delta i}{2}\right|}{|i|},$$

$$\sin\left(\frac{\Delta\phi}{2}\right) = \left|\frac{\Delta\hat{i}}{2}\right| \quad \therefore |\Delta\hat{i}| = 2\sin\left(\frac{\Delta\phi}{2}\right) (\because |\,i\,| = 1)$$

$$\left|\frac{\Delta\hat{i}}{\Delta\phi}\right| = \frac{\sin\left(\frac{\Delta\phi}{2}\right)}{\frac{\Delta\phi}{2}}, \text{ Taking as } \Delta\phi \to 0.$$

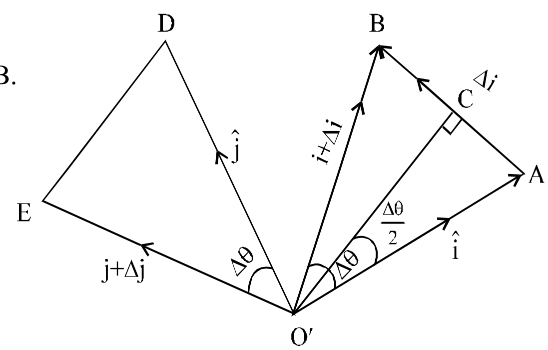


Fig. 3.12

$$\lim_{\Delta\phi\to 0}\left|\frac{\Delta i}{\Delta\phi}\right| = \lim_{\Delta\phi\to 0}\frac{\sin\frac{\Delta\phi}{2}}{\frac{\Delta\phi}{2}}, \quad \left|\frac{d\hat{i}}{d\phi}\right| = 1$$

$\Rightarrow \dfrac{di}{d\phi}$ is an unit vector whose direction is linear direction of Δi. The similar direction of Δi is j.

$$\boxed{\therefore \frac{di}{d\phi} = \hat{j}}$$

Similarly Δ O′DE $\lim\limits_{\Delta\phi\to 0}\left|\dfrac{\Delta i}{\Delta\phi}\right| = \lim\limits_{\Delta\phi\to 0}\dfrac{\sin\frac{\Delta\phi}{2}}{\left(\frac{\Delta\phi}{2}\right)}, \left|\dfrac{d\hat{j}}{d\phi}\right| = 1 \quad \Rightarrow \dfrac{dj}{d\phi}$ is an unit vector whose direction is linear direction of Δj. The similar direction of $\Delta\hat{j}$ is opposite direction of $\hat{i}$.

$$\therefore \boxed{\frac{dj}{d\phi} = -i}$$

Expression for acceleration

The velocity in tangential and normal component is given by $\vec{v} = \dfrac{ds}{dt}\hat{i}$

Differentiate w.r.t. we get

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\cdot\hat{i}\right), \vec{a} = \frac{d}{dt}\left(\frac{ds}{dt}\right)\cdot\hat{i} + \frac{ds}{dt}\left(\frac{d\hat{i}}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\cdot\hat{i} + \frac{ds}{dt}\left(\frac{di}{d\phi}\cdot\frac{d\phi}{ds}\cdot\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\cdot\hat{i} + \left(\frac{ds}{dt}\right)^2\cdot\hat{j}\frac{d\phi}{ds} \quad \left[\because \frac{di}{d\phi} = \hat{j}\right]$$

$$\vec{a} = \frac{d^2s}{dt^2}\cdot\hat{i} + j\left(\frac{ds}{dt}\right)^2\cdot\frac{1}{\rho}$$

Where $\rho = \dfrac{ds}{d\phi}$ = radius of curvature at P.

$$\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{dv}{dt} \quad \boxed{\vec{a} = \frac{dv}{dt}\hat{i} + \frac{v^2}{\rho}\hat{j}}$$

3.10 : Radial and Transverse Component of Velocity and Acceleration

Let a particle P moving along the curve with its position (r, θ) in polar co-ordinate. Where r = length of P from origin O and θ is angle where radius vector $\overrightarrow{OP}$ makes with the direction of x-axis.

As particle moves along the curve. Let Q be another position, whose position is given $(r + \Delta r, \theta + \Delta\theta)$ in polar co-ordinate at instead $t + \Delta t$. Let $\hat{i}$ and $\hat{j}$ be the unit vectors along radial and transverse direction representation at P and $\hat{i} + \Delta\hat{i}$ and $\hat{j} + \Delta\hat{j}$ be the unit vectors along radial and transverse direction at Q. Others unit vectors i, j are functions of θ. ***(fig 3.13)***

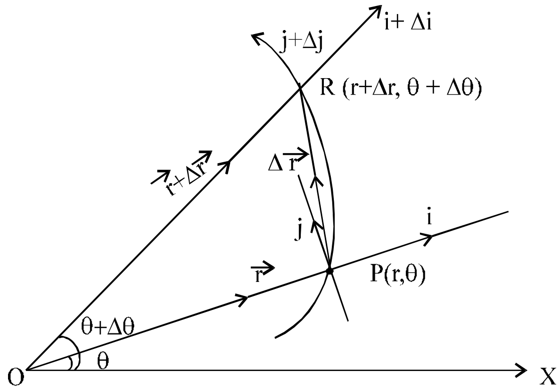


Fig. 3.13

Transfer unit vector $\hat{i}, \hat{j}$ and $\hat{i} + \Delta\hat{i}$, $\hat{j} + \Delta\hat{j}$ to a common origin O′. ***(fig. 3.14)***

O′AB is an isoscale triangle where O′A = i and O′B = $i + \Delta\hat{i}$

$$\therefore \overrightarrow{AB} = \Delta i \text{ and } \angle BO'A = \Delta\theta$$

Draw perpendicular O′C on AB

$$\overrightarrow{AC} = \frac{\Delta i}{2}, \ \angle CO'A = \frac{\Delta\theta}{2}$$

Now right angle Δ CO′A

$$\sin\left(\frac{\Delta\theta}{2}\right) = \frac{\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{O'A}\right|} = \frac{\left|\frac{\Delta i}{2}\right|}{|i|} = \left|\frac{\Delta i}{2}\right|$$

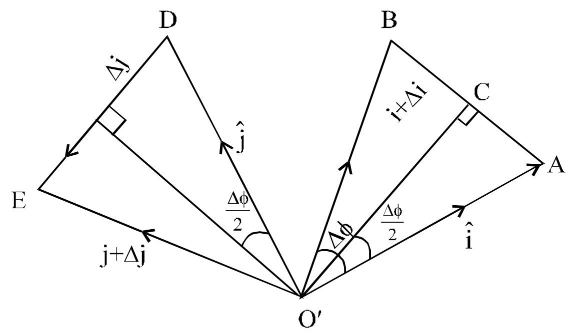


Fig. 3.14

$$\Rightarrow |\Delta i| = 2\sin\left(\frac{\Delta\theta}{2}\right) \quad \Rightarrow \left|\frac{\Delta\hat{i}}{\Delta\theta}\right| = \frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}$$

Taking lim as $\Delta\theta \to 0$.

$$\lim_{\Delta\theta\to 0}\left|\frac{\Delta i}{\Delta\theta}\right| = \lim_{\Delta\phi\to 0}\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} \quad \left|\frac{d\hat{i}}{d\theta}\right| = 1$$

$\Rightarrow \dfrac{di}{d\theta}$ is an unit vector. Whose, direction is similar direction of Δi. The limiting direction of Δi is j.

$$\boxed{\therefore \frac{di}{d\theta} = \hat{j}}$$

Similarly from O′DE $\displaystyle \lim_{\Delta\theta\to 0}\left|\frac{\Delta j}{\Delta\theta}\right| = \lim_{\Delta\theta\to 0}\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} \quad \left|\frac{dj}{d\theta}\right| = 1$

$\Rightarrow \frac{dj}{d\theta}$ is unit vector whose direction is linear direction of Δj. The linear direction of Δj is opposite direction of i. $\boxed{\therefore \frac{dj}{d\theta} = -i}$

Expression for velocity :

OP = $\vec{r}$, Then p.v. of point P i.e., $\overrightarrow{OP} = \vec{r} = r\,\hat{i}$

Velocity at P, $\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r \cdot \hat{i}\right)$

$$v = \frac{d\vec{r}}{dt} \cdot \hat{i} + \vec{r} \cdot \frac{di}{dt} = \frac{dr}{dt} \cdot \hat{i} + \vec{r} \cdot \frac{di}{d\theta} \cdot \frac{d\theta}{dt} \quad \boxed{\vec{v} = \frac{d\vec{r}}{dt} \cdot \hat{i} + \vec{r} \cdot \frac{d\theta}{dt}\hat{j}}$$

Thus, radial and transverse components of velocity are $\Rightarrow \frac{d\vec{r}}{dt}$ and $\vec{r}\frac{d\theta}{dt}$ represented.

Expression for Acceleration :

The velocity is radial and transverse component form is given by.

$$\vec{v} = \frac{d\vec{r}}{dt}\hat{i} + \vec{r}\frac{d\theta}{dt}\hat{j}$$

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{\vec{r}}\,\hat{i} + \vec{r}\,\dot{\theta}\,\hat{j}\left(where\ \frac{d}{dt}(x)\right) = \dot{x}\right)$$

$$= \left(\ddot{\vec{r}}\,\hat{i} + \dot{\vec{r}}\frac{di}{dt}\right) + \left(\dot{\vec{r}}\,\dot{\theta}\,\hat{j} + \vec{r}\,\ddot{\theta}\,\hat{j} + \vec{r}\,\dot{\theta}\frac{dj}{dt}\right) = \left(\ddot{\vec{r}}\,\hat{i} + \dot{\vec{r}}\frac{di}{d\theta} \cdot \frac{d\theta}{dt}\right) + \left(\dot{\vec{r}}\,\dot{\theta}\,\hat{j} + \vec{r}\,\ddot{\theta}\,\hat{j} + \vec{r}\,\dot{\theta}\frac{dj}{d\theta} \cdot \frac{d\theta}{dt}\right)$$

$$= \ddot{\vec{r}}\,\hat{i} + \dot{\vec{r}}\,\dot{\theta}\,\hat{j} + \dot{\vec{r}}\,\dot{\theta}\,\hat{j} + \vec{r}\,\ddot{\theta}\,\hat{j} - \vec{r}\,\dot{\theta}^2\,\hat{i} = \left(\ddot{\vec{r}} - \vec{r}\,\dot{\theta}^2\right)\hat{i} + \left(2\dot{\vec{r}}\,\dot{\theta} + \vec{r}\,\ddot{\theta}\right)\hat{j}$$

$$= \left(\frac{d^2\vec{r}}{dt^2} - \vec{r}\left(\frac{d\theta}{dt}\right)^2\right)\hat{i} + \left(2\frac{d\vec{r}}{dt}\frac{d\theta}{dt} + \vec{r} \cdot \frac{d^2\theta}{dt^2}\right)\hat{j} = \left(\frac{d^2\vec{r}}{dt^2} - \vec{r}\left(\frac{d\theta}{dt}\right)^2\right)\hat{i} + \frac{1}{\vec{r}}\left(2\vec{r}\frac{d\vec{r}}{dt}\frac{d\theta}{dt} + \vec{r}^2\frac{d^2\theta}{dt^2}\right)\hat{j}$$

$$= \left(\frac{d^2\vec{r}}{dt^2} - \vec{r}\left(\frac{d\theta}{dt}\right)^2\right)\hat{i} + \frac{1}{\vec{r}}\left(\frac{d}{dt}\left(\vec{r}^2 \cdot \frac{d\theta}{dt}\right)\right)\hat{j}$$

The radial and transverse component of acceleration are $\frac{d^2\vec{r}}{dt^2} - \vec{r}\left(\frac{d\theta}{dt}\right)^2$ and $\left(2\frac{d\vec{r}}{dt} \cdot \frac{d\theta}{dt} + \vec{r}\frac{d\theta^2}{dt^2}\right)$

or $\frac{1}{\vec{r}}\frac{d}{dt}\left(\vec{r}^2\frac{d\theta}{dt}\right)$

3.11: Arc Length along a space curve

To measure distance along a smooth curve in space is that have a measurable length.

To locate points along these curves by giving their directed distance '*s*' along the curves from same base point. To locate points on co-ordinate axes by giving their directed distance from the origin.

The length of a smooth curve $r(\vec{t}) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$ i.e., 't' increases from

$t = a$ to $t = b$ is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ ....(1)

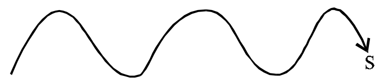


Fig. 3.15

The length of a velocity vector $\frac{d\vec{r}}{dt}$, i.e., the square root of equation (1) is $|v|$ then Arc length formula, $L = \int_a^b |v| dt$ ....(2).

If we choose base point $P(t_0)$ on smooth curve 'C' parameterized by the, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on 'C' and a 'directed distance'.

$$s(t) = \int_{t_0}^{t} |v(\tau)|\, d\tau$$

If $t > t_0$ $S(t)$ is the distance from $P(t_0)$ to $P(t)$. If $t < t_0$ $S(t)$ is negative of the distance. Each value of S determines a point on C and this parametrizes C with respect to S. We call S an arc length parameter for the curve Arc length parameter with base point $P(t_0)$.

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |v(\tau)|\, d\tau .$$

Illustration : A glider is soaring upward along the helix $\vec{r}(t) = (cost)\hat{i} + (sint)\hat{j} + t\hat{k}$. How far does the glider travel along it's path from t = 0 to t = 2π.

Solution : The path segment during this time corresponding to one full turn of the helix. The length of this portion of the curve is $L = \int_a^b |v| dt$.

$$= \int_0^{2\pi} \sqrt{(-sin\, t)^2 + (cos\, t)^2 + (1)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}dt = 2\pi\sqrt{2}$$ units of length.

3.12 : Derivative of a vector in terms of it's components

Let $\vec{r}$ be a vector function of scalar variable t.

Let , $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in directions of x-axis, y-axis, z-axis respectively. x, y, z are components of $\vec{r}$ along these directions respectively. As $\vec{r}$ is a function of t.

$\therefore$ x, y, z are also functions of t.

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i}) + \frac{d}{dt}(y\hat{j}) + \frac{d}{dt}(z\hat{k}) = x\frac{d\hat{i}}{dt} + \frac{dx}{dt}\hat{i} + y\frac{d\hat{j}}{dt} + \frac{dy}{dt}\hat{j} + z\frac{d\hat{k}}{dt} + \frac{dz}{dt}\hat{k}$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \qquad \left[\because \frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0\right]$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

i.e. $\vec{r} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

If $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$

Then $\frac{d\vec{r}}{dt} = f_1'(t)\hat{i} + f_2'(t)\hat{j} + f_3'(t)\hat{k}$

To differentiate a vector, differentiate its components only.

$$\frac{d^2\vec{r}}{dt^2} = f_1''(t)\,\hat{i} + f_2''\,\hat{j} + f_3''(t)\hat{k}$$

$\frac{d}{dt}\{c_1f_1(t) \pm c_2f_2(t)\} = c_1f_1'(t) \pm c_2f_2'(t)$ where c_1 and c_2 are constants.

Property – 1 : $\boldsymbol{\frac{d}{dt}\left(\vec{F}\cdot\vec{G}\right) = \vec{F}\cdot\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\cdot\vec{G}}$

where $\boldsymbol{\vec{F} = \vec{F}(t)}$ ***and*** $\boldsymbol{\vec{G} = \vec{G}(t)}$

Proof. Let $\vec{F} = f_1i + f_2j + f_3k$ and $\vec{G} = g_1i + g_2j + g_3k$ be two vector functions of t.

Now $\vec{F}\cdot\vec{G} = f_1g_1 + f_2g_2 + f_3g_3 = \Sigma f_1g_1$

$$\therefore \quad \frac{d}{dt}\left(\vec{F}\cdot\vec{G}\right) = \frac{d}{dt}\Sigma\, f_1g_1 = \Sigma\frac{d}{dt}(f_1g_1)$$

$$= \Sigma\left(f_1\frac{dg_1}{dt} + \frac{df_1}{dt}g_1\right) = \Sigma f_1i\cdot\Sigma\frac{dg_1}{dt}i + \Sigma\frac{df_1}{dt}i\cdot\Sigma g_1i$$

Thus $\frac{d}{dt}\left(\vec{F}\cdot\vec{G}\right) = \vec{F}\cdot\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\cdot\vec{G}$

Property – 2 : $\boldsymbol{\frac{d}{dt}\left(\vec{F}\times\vec{G}\right) = \vec{F}\times\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\times\vec{G}}$

Proof : Let $\vec{F} = f_1i + f_2j + f_3k$ and $\vec{G} = g_1i + g_2j + g_3k$ be two vector functions of t.

$$\text{Now } \vec{F}\times\vec{G} = \begin{vmatrix} \ddot{i} & \hat{j} & \hat{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} = \Sigma\hat{i}\,(f_2g_3 - f_3g_2)$$

$$\therefore \frac{d}{dt}\left(\vec{F}\times\vec{G}\right) = \frac{d}{dt}\Sigma\hat{i}(f_2g_3 - f_3g_2) = \Sigma\hat{i}\frac{d}{dt}(f_2g_3 - f_3g_2)$$

$$= \Sigma i\left\{\left(f_2\frac{dg_3}{dt} + \frac{df_2}{dt}g_3\right) - \left(f_3\frac{dg_2}{dt} + \frac{df_3}{dt}g_2\right)\right\}$$

$$= \Sigma i\left(f_2\frac{dg_3}{dt} + f_3\frac{df_2}{dt}\right) + \Sigma i\left(\frac{df_2}{dt}g_3 - \frac{df_3}{dt}g_2\right)$$

$$= \begin{vmatrix} i & j & k \\ f_1 & f_2 & f_3 \\ \frac{dg_1}{dt} & \frac{dg_2}{dt} & \frac{dg_3}{dt} \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{df_1}{dt} & \frac{df_2}{dt} & \frac{df_3}{dt} \\ g_1 & g_2 & g_3 \end{vmatrix} = \vec{F} \times \frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt} \times \vec{G}$$

Thus $\frac{d}{dt}\left(\vec{F} \times \vec{G}\right) = \vec{F} \times \frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt} \times \vec{G}$

Derivative of triple products

Now we define the formula for derivative of triple products :

Theorem – 6 : (a) $\frac{\mathrm{d}}{\mathrm{dt}}\left[\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})\right] = \frac{\mathrm{du}}{\mathrm{dt}} \cdot (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u}.\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\boldsymbol{t}} \times \boldsymbol{w}\right) + \boldsymbol{u}.\left(\boldsymbol{v} \times \frac{d\boldsymbol{w}}{dt}\right)$

or $\left[\boldsymbol{u}\ \boldsymbol{v}\ \boldsymbol{w}\right]' = \left[\boldsymbol{u}'\ \boldsymbol{v}\ \boldsymbol{w}\right] + \left[\boldsymbol{u}\ \boldsymbol{v}'\ \boldsymbol{w}\right] + \left[u\ v\ w'\right]$

where $\boldsymbol{u} = \boldsymbol{u}(\mathrm{t})$, $\boldsymbol{v} = \boldsymbol{v}\,(\mathrm{t})$ and $\boldsymbol{w} = \boldsymbol{w}(\mathrm{t})$ be differentiable vector valued functions.

(b) $\frac{\mathrm{d}}{\mathrm{dt}}\left[\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})\right] = \frac{\mathrm{du}}{\mathrm{dt}} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u} \times \left(\frac{\mathrm{dv}}{\mathrm{dt}} \times \boldsymbol{w}\right) + \boldsymbol{u} \times \left(\boldsymbol{v} \times \frac{\mathrm{d}\mathbf{w}}{\mathrm{dt}}\right)$

Proof : (a) $\frac{\mathrm{d}}{\mathrm{dt}}\left[\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})\right] = \frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}}.(\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u}.\frac{\mathrm{d}}{\mathrm{dt}}(\boldsymbol{v} \times \boldsymbol{w})$

$$= \frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}}.(\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u}.\left[\frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}} \times \boldsymbol{w} + \boldsymbol{v} \times \frac{\mathrm{d}\mathbf{w}}{\mathrm{dt}}\right]$$

$$= \frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}}.(\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u}.\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}} \times \boldsymbol{w}\right) + \boldsymbol{u}.\left(\boldsymbol{v} \times \frac{\mathrm{d}\mathbf{w}}{\mathrm{dt}}\right)$$

Proof of (b) is left as an exercise.

3.13 : Vector valued function with constant magnitude

Theorem – 7: The necessary and sufficient condition for the differentiable vector-valued function $\boldsymbol{r}(\mathrm{t})$ in R^2 or R^3 to have constant magnitude $\|\boldsymbol{r}(\mathrm{t})\|$ for all t is $\boldsymbol{r}(\mathrm{t}).\frac{\mathrm{d}}{\mathrm{dt}}(\boldsymbol{r}(\mathrm{t})) = 0$

i.e. $\boldsymbol{r}(\mathrm{t}) . \boldsymbol{r}'(\mathrm{t}) = 0$

Proof : We know that $\boldsymbol{r}(\mathrm{t}) . \boldsymbol{r}(\mathrm{t}) = \|\boldsymbol{r}(\mathrm{t})\|^2$

Differentiating w.r.t. t, we have

$$\boldsymbol{r}(\mathrm{t}).\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{dt}} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{dt}}.\boldsymbol{r}(\mathrm{t}) = 2\|\boldsymbol{r}(\mathrm{t})\|.\frac{\mathrm{d}}{\mathrm{dt}}\|\boldsymbol{r}(\mathrm{t})\|$$

$$\Rightarrow \boldsymbol{r}(\mathrm{t}).\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{dt}} = \|\boldsymbol{r}(\mathrm{t})\|.\frac{\mathrm{d}}{\mathrm{dt}}\|\boldsymbol{r}(\mathrm{t})\| \quad .....(1)$$

Suppose, $\boldsymbol{r}(\mathrm{t}).\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{r}(\mathrm{t}) = 0$

$$\Rightarrow \frac{d}{dt}\| \boldsymbol{r}(t) \| = 0$$

$\Rightarrow \|\boldsymbol{r}(t)\| \infty$ is constant

If $\| \boldsymbol{r}(t) \|$ is constant, then, $\| \boldsymbol{r}(t) \|^2$ is constant

$$\Rightarrow \frac{d}{dt}\| \boldsymbol{r}(t) \|^2 = 0$$

$$\Rightarrow 2\,\boldsymbol{r}(t).\frac{d}{dt}\boldsymbol{r}(t) = 0 \;\Rightarrow \boldsymbol{r}(t).\frac{d}{dt}\boldsymbol{r}(t) = 0$$

Example – 1 : Let $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$. Find $\lim\limits_{t\to 2} \boldsymbol{r}(t)\cdot(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t))$

Solution : We have, $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$.

$$\therefore\ \boldsymbol{r}'(t) = \frac{d}{dt}(t)\,\boldsymbol{i} + \frac{d}{dt}(t^2)\,\mathbf{j} + \frac{d}{dt}(t^3)\,\mathbf{k} = \boldsymbol{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}.$$

$$\boldsymbol{r}''(t) = \frac{d}{dt}(1)\,\boldsymbol{i} + \frac{d}{dt}(2t)\,\mathbf{j} + \frac{d}{dt}(3t^2)\,\mathbf{k} = 0\,\boldsymbol{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}.$$

$$\therefore\ \boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \begin{vmatrix} \boldsymbol{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\,\boldsymbol{i} - 6t\,\mathbf{j} + 2\,\boldsymbol{k}.$$

$$\boldsymbol{r}(t)\cdot(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)) = \langle t, t^2, t^3\rangle \cdot \langle 6t^2, -6t, 2\rangle = 6t^3 + (-6t^3) + 2t^3 = 2t^3$$

$$\therefore\quad \lim_{t\to 2} \boldsymbol{r}(t)\cdot(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)) = \lim_{t\to 2} 2t^3 = 2\times 8 = 16$$

Integration of Vector-valued function

If the vector-valued function $\mathbf{R}(t)$ be the derivative of another vector valued function $\boldsymbol{r}(t)$, then $\boldsymbol{r}(t)$ is an integral of $\mathbf{R}(t)$ with respect to t and we write it as

$\int R(t)\,dt = \boldsymbol{r}(t) + \boldsymbol{c}$, where $\boldsymbol{c}$ is an arbitrary constant vector.

Important results on antiderivative of vector- valued function:

1. $\int 2\boldsymbol{r}(t)\cdot\frac{d\boldsymbol{r}}{dt}\,dt = \boldsymbol{r}(t).\boldsymbol{r}(t) + c = \|\boldsymbol{r}(t)\|^2 + \boldsymbol{c}$

2. $\int [\boldsymbol{r}(t) \times \boldsymbol{r}''(t)]\,dt = \boldsymbol{r}(t) \times \frac{d\boldsymbol{r}}{dt} + \boldsymbol{c}$

3. $\frac{d}{dt}\left[\int \boldsymbol{r}(t)dt\right] = \boldsymbol{r}(t)$ and $\int \boldsymbol{r}'(t)dt = \boldsymbol{r}(t) + \boldsymbol{c}$

4. If $\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then $\int \boldsymbol{r}(t)dt = \left(\int x(t)dt\right)\boldsymbol{i} + \left(\int y(t)dt\right)\mathbf{j} + \int (z(t)dt)\,\mathbf{k}$

Example – 2 : ***Evaluate :***

(a) $\int\left(t^2\boldsymbol{i}-4t\mathbf{j}+\frac{1}{t}\mathbf{k}\right)dt$ (b) $\int\left\langle e^t, e^{-t}, 3t^2\right\rangle dt$

(c) $\int_0^{\pi/2}\langle\cos 2t, \sin 2t\rangle dt$ (d) $\int_1^9\left(t^{1/2}\boldsymbol{i}+t^{-\frac{1}{2}}\mathbf{j}\right)dt$

Solution :

(a) $\int\left(t^2\boldsymbol{i}-4t\mathbf{j}+\frac{1}{t}\mathbf{k}\right)dt$

$$=\left(\int t^2 dt\right)\boldsymbol{i}-\left(\int 4t\,dt\right)\mathbf{j}+\left(\int\frac{1}{t}dt\right)\mathbf{k}=\frac{1}{3}t^3\boldsymbol{i}-2t^2\mathbf{j}+\ln|t|\,\mathbf{k}$$

(b) $\int\left\langle e^t, e^{-t}, 3t^2\right\rangle dt$

$$=\left\langle\int e^t dt, \int e^{-t}dt, \int 3t^2 dt\right\rangle=\left\langle e^t, -e^{-t}, t^3\right\rangle=e^t\,\boldsymbol{i}-e^{-t}\,\mathbf{j}+t^3\,\mathbf{k}$$

(c) $\int_0^{\pi/2}(\cos 2t\,\boldsymbol{i}+\sin 2t\,\mathbf{j})dt$

$$=\frac{\sin 2t}{2}\Bigg]_0^{\frac{\pi}{2}}\boldsymbol{i}+\frac{-\cos 2t}{2}\Bigg]_0^{\frac{\pi}{2}}\mathbf{j}=(0-0)\boldsymbol{i}+\left(\frac{1}{2}+\frac{1}{2}\right)\mathbf{j}=0\,\boldsymbol{i}+\mathbf{j}=\mathbf{j}$$

(d) $\int_1^9\left(t^{1/2}\boldsymbol{i}+t^{-\frac{1}{2}}\mathbf{j}\right)dt=\frac{2}{3}t^{\frac{3}{2}}\Bigg]_1^9\boldsymbol{i}+2t^{\frac{1}{2}}\Bigg]_1^9\mathbf{j}=\frac{2}{3}\left(9^{3/2}-1\right)\boldsymbol{i}+2\left(9^{3/2}-1\right)\mathbf{j}=\frac{2}{3}(26)\boldsymbol{i}+4\mathbf{j}$

$$=\frac{52}{3}\boldsymbol{i}+4\mathbf{j}$$

Definite integral of vector valued function

Let $\boldsymbol{r}$ (t) be continuous (not necessarily differentiable) function on an interval $a \le t \le b$.

The **definite integral** is defined as $\int_a^b \boldsymbol{r}(t)dt=\lim_{\max\Delta t_k\to 0}\sum_{k=1}^{n}\boldsymbol{r}(t_k)\Delta t_k$

$$=\int_a^b x(t)dt\,\boldsymbol{i}+\int_a^b y(t)dt\,\mathbf{j}+\int_a^b z(t)dt\,\mathbf{k}$$

Thus, if $\boldsymbol{r}(t)=x(t)\,\boldsymbol{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$ then, $\int_a^b \boldsymbol{r}(t)dt=\left(\int_a^b x(t)dt\right)\boldsymbol{i}+\left(\int_a^b y(t)dt\right)\mathbf{j}+\left(\int_a^b z(t)dt\right)\mathbf{k}$

Example – 3 : ***Evaluate***

(a) $\int_2^1(e^{2t}\boldsymbol{i}+e^{-t}\mathbf{j}+2t\,\mathbf{k})dt$ (b) $\int_0^2\|t\,\boldsymbol{i}+2t^2\mathbf{j}\|\,dt$

Solution :

(a) $\int_0^1 (e^{2t}\boldsymbol{i} + e^{-t}\mathbf{j} + 2t\,\mathbf{k})dt$

$= \left(\int_0^1 e^{2t}dt\right)\boldsymbol{i} + \left(\int_0^1 e^{-t}dt\right)\mathbf{j} + \left(\int_0^1 2t\,dt\right)\mathbf{k} = \left.\frac{e^{2t}}{2}\right]_0^1 \boldsymbol{i} - e^{-t}\Big]_0^1 \mathbf{j} + t^2\Big]_0^1 \mathbf{k}$

$= \left(\frac{1}{2}e^2 - \frac{1}{2}\right)\boldsymbol{i} - \left(e^{-1} - 1\right)\mathbf{j} + (1-0)\mathbf{k} = \frac{1}{2}(e^2-1)\boldsymbol{i} + \left(1 - \frac{1}{e}\right)\mathbf{j} + \mathbf{k}$

(b) $\| t\,\boldsymbol{i} + 2t^2\mathbf{j} \| = \sqrt{t^2 + 4t^4} = t\sqrt{1+4t^2}$

$\int_0^2 \| t\,\boldsymbol{i} + 2t^2\,\mathbf{j} \|\,dt = \int_0^2 t\sqrt{1+4t^2}\,dt$ [Put $4t^2 = h \ \therefore\ 8t\,dt = dh$]

$= \frac{1}{8}\int_0^{16} \sqrt{1+h}\,dh = \frac{1}{8}.\frac{2}{3}(1+h)^{3/2}\Big]_0^{16} = \frac{1}{12}\left[(17)^{3/2} - 1\right] = \frac{17\sqrt{17}-1}{12}$

Properties of definite integral of vector valued function

If $\boldsymbol{r}(t)$, $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$ be vector-valued function in R^2 or R^3 on the interval $[a, b]$ and k be a scalar, then

(a) $\int_a^b (k\,\boldsymbol{r}(t))dt = k\int_a^b \boldsymbol{r}(t)dt$ (b) $\int_a^b [\boldsymbol{r}_1(t) \pm \boldsymbol{r}_2(t)]dt = \int_a^b \boldsymbol{r}_1(t)dt \pm \int_a^b \boldsymbol{r}_2(t)dt$

(c) $\int_a^b (c_1\boldsymbol{r}_1(t) + c_2\boldsymbol{r}_2(t))dt = c_1\int_a^b \boldsymbol{r}_1(t)dt + c_2\int_a^b \boldsymbol{r}_2(t)\,dt$, where c_1 and c_2 are constants.

Proof : (a) Let $\mathbf{R}(t)$ be differentiable vector valued function such that $\mathbf{R}'(t) = \boldsymbol{r}(t)$

$\therefore\ (kR)'(t) = kr(t)$

$\int_a^b kr(t)dt = (kR)(t)\Big]_a^b = k\mathbf{R}(b) - k\mathbf{R}(a) = k\left[R(b) - R(a)\right] = k\ R(t)\Big]_a^b = k\int_a^b \boldsymbol{r}(t)dt$

Proof of (b) and (*c*) are left as an exercise :

Example – 4 : Find $\mathbf{y}(t)$ given

(a) $\mathbf{y}'(t) = 2t\,\boldsymbol{i} + 3t^2\mathbf{j}, \mathbf{y}(0) = \boldsymbol{i} - 2\mathbf{j}$ (b) $\mathbf{y}''(t) = \boldsymbol{i} + e^t\mathbf{j}, \mathbf{y}(0) = 2\boldsymbol{i}, \mathbf{y}'(0) = \mathbf{j}$

***Solution :* (a)** $\mathbf{y}(t) = \int \mathbf{y}'(t)dt = \int (2t\,\boldsymbol{i} + 3t^2\mathbf{j})dt = t^2\boldsymbol{i} + t^3\mathbf{j} + \boldsymbol{c}$

where $\boldsymbol{c}$ is a vector constant of integration.

To find $\boldsymbol{c}$, put $t = 0$

$\therefore \mathbf{y}(0) = 0\boldsymbol{i} + 0\mathbf{j} + \boldsymbol{c} = \boldsymbol{c}$

But, $\mathbf{y}(0) = \boldsymbol{i} - 2\boldsymbol{i} \Rightarrow \boldsymbol{c} = \boldsymbol{i} - 2\mathbf{j}$

$\therefore \boldsymbol{c} = \boldsymbol{i} - 2\mathbf{j}$

Hence, $\boldsymbol{y}(t) = t^2\,\boldsymbol{i} + t^3\,\mathbf{j} + \boldsymbol{i} - 2\mathbf{j} = (t^2+1)\,\boldsymbol{i} + (t^3-2)\,\mathbf{j}$

(b) Given $\mathbf{y}''(t) = \boldsymbol{i} + e^t\mathbf{j}$

$\therefore \mathbf{y}'(t) = \int \mathbf{y}''(t)\,dt = \int (\boldsymbol{i} + e^t\mathbf{j})\,dt = t\,\boldsymbol{i} + e^t\,\mathbf{j} + \mathbf{b}$, for constant vector $\mathbf{b}$

$\therefore \mathbf{y}'(0) = 0\,\boldsymbol{i} + e^0\mathbf{j} + \mathbf{b} = \mathbf{j} + \mathbf{b}$

But, $\mathbf{y}'(0) = \mathbf{j}$

$\therefore \mathbf{j} + \mathbf{b} = \mathbf{j} \Rightarrow \mathbf{b} = \mathbf{0}$

So, $\mathbf{y}'(t) = t\,\boldsymbol{i} + e^t\mathbf{j}$

$\therefore \mathbf{y}(t) = \int \mathbf{y}'(t)\,dt = \int \left[t\,\boldsymbol{i} + (e^t\mathbf{j}\right]dt = \frac{t^2}{2}\boldsymbol{i} + e^t\mathbf{j} + \boldsymbol{c}$, for constant vector $\boldsymbol{c}$

$\therefore \mathbf{y}(0) = 0\,\boldsymbol{i} + e^0\boldsymbol{j} + \boldsymbol{c} = \boldsymbol{j} + \boldsymbol{c}$

But, $\mathbf{y}(0) = 2\boldsymbol{i}$

$2\,\boldsymbol{i} = \mathbf{j} + \boldsymbol{c} \Rightarrow \boldsymbol{c} = 2\boldsymbol{i} - \mathbf{j}$

Hence,

$$\mathbf{y}(t) = \frac{t^2}{2}\boldsymbol{i} + e^t\mathbf{j} + (2i - j) = \left(\frac{t^2}{2} + 2\right)\boldsymbol{i} + (e^t - 1)\,\mathbf{j}$$

Illustrative Examples

Example – 1 : If $\vec{r} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ ***and*** $\vec{s} = (2t-8)\hat{i} + \hat{j} - t\hat{k}$ ***find***

(a) $\frac{d}{dt}(\vec{r} + \vec{s})$ ***(b)*** $\frac{d}{dt}(\vec{r} \cdot \vec{s})$ ***(c)*** $\frac{d}{dt}(\vec{r} \times \vec{s})$

Solution : Given $\vec{r} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$, $\vec{s} = (2t-8)\hat{i} + \hat{j} - t\hat{k}$

(a) $$\frac{d}{dt}(\vec{r} + \vec{s}) = \frac{d}{dt}\left\{\left(t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\right) + \left((2t-8)\hat{i} + \hat{j} - t\hat{k}\right)\right\}$$

$$= \frac{d}{dt}\left(t^2\hat{i}\right) - \frac{d}{dt}\left(t\hat{j}\right) + \frac{d}{dt}(2t+1)\hat{k} + \frac{d}{dt}(2t-8)\hat{i} + \frac{d}{dt}\left(\hat{j}\right) - \frac{d}{dt}\left(t\hat{k}\right)$$

$$= 2t\hat{i} - \hat{j} + 2\hat{k} + 2\hat{i} + 0 - \hat{k} = (2t+2)\hat{i} - \hat{j} + \hat{k} = 2(t+1)\hat{i} - \hat{j} + \hat{k}$$

(b) $$\frac{d}{dt}(\vec{r} \cdot \vec{s}) = \frac{d(\vec{r})}{dt} \cdot \vec{s} + \vec{r} \cdot \frac{d(\vec{s})}{dt}$$

$$= \frac{d}{dt}\left\{\left(t^2\hat{i} - t\hat{j}\right) + (2t+1)\hat{k}\right\} \cdot \left\{(2t-8)\hat{i} + \hat{j} - t\hat{k}\right\} + \left\{t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\right\}\frac{d}{dt}\left\{(2t-8)\hat{i} + \hat{j} - t\hat{k}\right\}$$

$$= \left(2t\hat{i} - \hat{j} + 2\hat{k}\right)\left\{(2t-8)\hat{i} + \hat{j} - t\hat{k}\right\} + \left(t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\right)\left(2\hat{i} - \hat{k}\right)$$

$$= 4t^2 - 16t - 1 - 2t + 2t^2 - 2t - 1 = 6t^2 - 20t - 2$$

(c) $\frac{d}{dt}(\vec{r}\times\vec{s}) = \frac{d\vec{r}}{dt}\times\vec{s} + r\times\frac{d\vec{s}}{dt}$

$= (2t\hat{i} - \hat{j} + 2\hat{k})\times((2t-8)\hat{i} + \hat{j} - t\hat{k}) + (t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}j)\times(2\hat{i} - \hat{k})$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -1 & 2 \\ (2t-8) & 1 & -t \end{vmatrix} \quad = \hat{i}(t-2) - j(-2t^2 - 4t + 16) + \hat{k}(2t + 2t - 8)$$
$$= (t-2)\hat{i} + \hat{j}(2t^2 + 4t - 16) + \hat{k}(4t-8)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & (2t+1) \\ 2 & 0 & -1 \end{vmatrix} \quad = \hat{i}(t) - \hat{j}(-t^2 - 4t - 2) + \hat{k}(2t)$$
$$= t\hat{i} + \hat{j}(t^2 + 4t + 2) + \hat{k}(2t)$$

$= (t-2)\hat{i} + (2t^2 + 4t - 16)\hat{j} + (4t-8)\hat{k} + t\hat{i} + (4t + t^2 + 2)\hat{j} + 2t\hat{k}$

$= (t - 2 + t)\hat{i} + (3t^2 + 8t - 14)\hat{j} + (4t - 8 + 2t)\hat{k}$

$= (2t-2)\hat{i} + (3t^2 + 8t - 14)\hat{j} + (6t-8)\hat{k}$

Example – 2 : ***A particle moves so that its position vector is given by $\vec{r}$ = cos wt i + sin wt j where w is a constant. Show that***

(a) ***velocity $\vec{v}$ is perpendicular $\vec{r}$***

(b) ***$\vec{r}\times\vec{v}$ is a constant vector***

(c) ***acceleration $\vec{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin.***

Solution : $\vec{r} = \cos wt\, i + \sin wt\, j$ ... (1)

$\vec{v} = \frac{d\vec{r}}{dt} = -w\sin wt\, i + w\cos wt\, j$ ... (2)

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = +w^2\cos wt\, i - w^2\sin wt\, j$ ... (3)

(a) From (1) and (2) we have,

$\vec{v}\cdot\vec{r} = -w\sin wt\cos wt + w\cos wt\sin wt = 0$

$\Rightarrow \vec{v}$ is perpendicular to $\vec{r}$

(b) $\vec{r}\times\vec{v} = \begin{vmatrix} i & j & k \\ \cos wt & \sin wt & 0 \\ -w\sin wt & w\cos wt & 0 \end{vmatrix} = i(0-0) - j(0-0) + kw(\cos^2 wt + \sin^2 wt)$

$\therefore \quad \vec{r}\times\vec{v} = w\,k$, which is a constant vector.

(c) We have from (3)

$$\vec{a} = -w^2(\cos wt\ i + \sin wt\ j) = -w^2\vec{r}$$

$$|\vec{a}| = \left|-w^2\vec{r}\right| = w^2\sqrt{\cos^2 wt + \sin^2 wt} = w^2$$

We know that if $\vec{r} = \overrightarrow{OP}$ where O is the origin and P is any point on the curve. This shows that the acceleration is directed towards the origin.

Also we have $|\vec{a}| = w^2|\vec{r}|$ where w^2 is a constant.

$\Rightarrow$ $|\vec{a}|\omega|\vec{r}|$ and $|\vec{r}|$ is the distance of the point P from O .

Thus the magnitude of the acceleration is proportional to the distance from the origin.

Example – 3 : Evaluate (i) $\dfrac{d^2}{dt^2}\left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\right]$

Solution : First we can find $\dfrac{d}{dt}\left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\right]$

$$= \left[\frac{d\vec{r}}{dt}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^3\vec{r}}{dt^3}\right]$$

$$= \left[\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt}\right]\cdot\frac{d^2\vec{r}}{dt^2} + \vec{r}\cdot\left[\frac{d^2\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r} \times \frac{d\vec{r}}{dt}\right]\cdot\frac{d^3\vec{r}}{dt^3} = 0\cdot\frac{d^2\vec{r}}{dt^2} + \vec{r}\cdot 0 + \left[\vec{r}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^3\vec{r}}{dt^3}\right]$$

$$= \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^3\vec{r}}{dt^3}\right]$$

Again $\dfrac{d^2}{dt^2}\left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\right] = \dfrac{d}{dt}\left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^3\vec{r}}{dt^3}\right] = \left[\dfrac{d\vec{r}}{dt}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^3\vec{r}}{dt^3}\right] + \left[\vec{r}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right] + \left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^4\vec{r}}{dt^4}\right]$

$$= \left[\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt}\right]\cdot\frac{d^3\vec{r}}{dt^3} + \vec{r}\cdot\left[\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}\right] + \vec{r} \times \left[\frac{d\vec{r}}{dt} \times \frac{d^4\vec{r}}{dt^4}\right]$$

$$= 0\cdot\frac{d^3\vec{r}}{dt^3} + \left[\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}\right] + \left[\vec{r}\cdot\frac{d\vec{r}}{dt} \times \frac{d^4\vec{r}}{dt^4}\right] = \left(\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}\right) + \left(\vec{r}\cdot\frac{d\vec{r}}{dt} \times \frac{d^4\vec{r}}{dt^4}\right)$$

$$= \left[\vec{r}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] + \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^4\vec{r}}{dt^4}\right] = \left[\vec{r}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] + \left[\vec{r}\ \dot{\vec{r}}\ \ddddot{\vec{r}}\right]$$

(ii) Find the first and 2nd order derivatives of $\vec{r} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right)$

Solution : We have $\vec{r} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right) = \left(\vec{r}\cdot\dfrac{d^2\vec{r}}{dt^2}\right)\dfrac{d\vec{r}}{dt} - \left(\vec{r}\cdot\dfrac{d\vec{r}}{dt}\right)\cdot\dfrac{d^2\vec{r}}{dt^2}$

$$= \frac{d}{dt}\left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right] = \left[\frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right] + \left[\vec{r} \times \left(\frac{d^2\vec{r}}{dt^2} \times \frac{d^2\vec{r}}{dt^2}\right)\right] + \left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right)\right]$$

$$= \left[\frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right] + \left[\vec{r} \times 0\right] + \left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right)\right]$$

$$= \left[\frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right] + \left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right)\right] = \left[\dot{\vec{r}} \times \left(\dot{\vec{r}} \times \ddot{\vec{r}}\right)\right] + \left[\vec{r} \times \left(\dot{\vec{r}} \times \dddot{\vec{r}}\right)\right]$$

$$\text{Now } \frac{d^2}{dt^2}\left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right] = \frac{d}{dt}\left[\frac{d}{dt}\left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right]\right]$$

$$\frac{d}{dt}\left[\dot{\vec{r}} \times \left(\dot{\vec{r}} \times \ddot{\vec{r}}\right)\right] + \frac{d}{dt}\left[\vec{r} \times \left(\dot{\vec{r}} \times \dddot{\vec{r}}\right)\right]$$

$$= \left[\ddot{\vec{r}} \times \left(\dot{\vec{r}} \times \ddot{\vec{r}}\right)\right] + \left[\dot{\vec{r}} \times \left(\ddot{\vec{r}} \times \ddot{\vec{r}}\right)\right] + \left[\dot{\vec{r}} \times \left(\dot{\vec{r}} \times \dddot{\vec{r}}\right)\right] + \left[\dot{\vec{r}} \times \left(\dot{\vec{r}} \times \dddot{\vec{r}}\right)\right] + \left[\vec{r} \times \left(\ddot{\vec{r}} \times \dddot{\vec{r}}\right)\right] + \left[\vec{r} \times \left(\dot{\vec{r}} \times \ddddot{\vec{r}}\right)\right]$$

$$= \ddot{\vec{r}} \times \left(\dot{\vec{r}} \times \ddot{\vec{r}}\right) + 0 + \dot{\vec{r}} \times \left(\dot{\vec{r}} \times \dddot{\vec{r}}\right) + \dot{\vec{r}} \times \left(\dot{\vec{r}} \times \dddot{\vec{r}}\right) + \vec{r} \times \left(\ddot{\vec{r}} \times \dddot{\vec{r}}\right) + \vec{r} \times \left(\dot{\vec{r}} \times \ddddot{\vec{r}}\right)$$

$$= \ddot{\vec{r}} \times \left(\dot{\vec{r}} \times \ddot{\vec{r}}\right) + 2\dot{\vec{r}} \times \dot{\vec{r}} \times \dddot{\vec{r}} + \vec{r} \times \left(\ddot{\vec{r}} \times \dddot{\vec{r}}\right) + \vec{r} \times \left(\dot{\vec{r}} \times \ddddot{\vec{r}}\right)$$

Example – 4 : ***A particle moves along the curve C: $x = t^3 - 4t$, $y = t^2 + 4t$, $z = 8t^2 - 3t^3$ where t denotes time. Find the components of its acceleration at $t = 2$ along the tangent and normal.***

Solution : $\vec{r} = (t^3 - 4t)i + (t^2 + 4t)\, j + (8t^2 - 3t^3)\, k$

$$\vec{v} = \frac{d\vec{r}}{dt} = (3t^2 - 4)\, i + (2t + 4)\, j + (16\,t - 9t^2)\, k$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6ti + 2j + (16 - 18t)\, k$$

$$\left(\vec{v}\right)_{t=2} = 8i + 8j - 4k = \vec{v} \text{ (say)}$$

$$\left(\vec{a}\right)_{t=2} = 12i + 2j - 20k = \vec{A} \text{ (say)}$$

Since the velocity is along the tangent to the curve the component of acceleration along the tangent is given by

$$\vec{A} \cdot \hat{n} \quad \text{where } \hat{n} = \vec{V}/|\vec{V}|$$

i.e., $(12i+2j-20k)\cdot\dfrac{8i+8j-4k}{\sqrt{64+64+16}}=\dfrac{1}{12}(96+16+80)=\dfrac{192}{12}=16$

Also the component of acceleration along the normal is given by | $\vec{A}$ – (resolved part of $\vec{A}$ along the tangent) |

$$=\left|(12i+2j-20k)-\frac{16}{12}(8i+8j-4k)\right| = \left|(12i+2j-20k)-\frac{16}{3}(2i+2j-k)\right|$$

$$=\left|\frac{1}{3}(4i-26j-44k)\right|=\frac{1}{3}\sqrt{4^2+(-26)^2+(-44)^2} = \frac{\sqrt{2628}}{3}=17$$

$\therefore$ components of acceleration along the tangent and normal are respectively 16 and 17.

Example – 5 : ***A particle moves along the curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$, where 't' denotes time. Find it's velocity and acceleration at time t = 1 also find components of velocity and acceleration at time t = 1, in the direction of $\hat{i}-3\hat{j}+3\hat{k}$.***

Solution. The position vector of the particle at any time 't' is given by

$$\vec{r}=2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$$

The velocity 'v' is given by

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{d}{dt}\left(2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}\right)=4t\hat{i}+(2t-4)\hat{j}+3\hat{k}$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left[4t\hat{i}+(2t-4)\hat{j}+3\hat{k}\right]=4\hat{i}+2\hat{j}$$

$$\left[\frac{d\vec{r}}{dt}\right]_{t=1}=4\hat{i}-2\hat{j}+3\hat{k}=\vec{V}\ ,\quad \left[\frac{d\vec{v}}{dt}\right]_{t=1}=4\hat{i}+2\hat{j}=\vec{A}$$

The unit vector in the direction $i-3j+2k$ is given by

$$\hat{n}=\frac{i-3j+2k}{\sqrt{1+9+4}}=\frac{i-3j+2k}{\sqrt{14}}$$

The required components of velocity and acceleration are respectively $\vec{V}\cdot\hat{n}$ and $\vec{A}\cdot\hat{n}$.

$$\vec{V}\cdot\hat{n}=(4i-2j+3k)\cdot\frac{i-3j+2k}{\sqrt{14}}=\frac{4+6+6}{\sqrt{14}}=\frac{16}{\sqrt{14}}$$

$$\vec{A}\cdot\hat{n}=(4i+2j)\cdot\frac{i-3j+2k}{\sqrt{14}}=\frac{4-6+0}{\sqrt{14}}=\frac{-2}{\sqrt{14}}$$

Thus the required components of velocity and acceleration are $16/\sqrt{14}$ and $-2/\sqrt{14}$ respectively.

Example – 6: Find the tangent vector to the curve whose parametric representation is x = cos t, y = sin t, z = t, $-\pi\le t\le\pi$. Hence, find the unit tangent vector.

Solution. The position vector of a point on the curve $=\vec{r}(t)=\cos t\ \hat{i}+\sin t\ \hat{j}+t\ \hat{k}$. Therefore, the tangent vector is $\dfrac{d\vec{r}}{dt}=-\sin t\ \hat{i}+\cos t\ \hat{j}+\hat{k}$.

We have $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

Hence the unit tangent vector $\frac{(d\hat{r})}{dt} = \left(\frac{-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}}{\sqrt{2}}\right)$

Example – 7 : If $\vec{r} = a\cos t\hat{i} + a\sin t\hat{j} + at\tan\alpha\hat{k}$ ***, find*** $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$ ***and*** $\left[\frac{d\vec{r}}{dt}\,\frac{d^2\vec{r}}{dt^2}\,\frac{d^3\vec{r}}{dt^3}\right]$**.**

Solution : $\vec{r} = a\cos t\hat{i} + a\sin t\hat{j} + at\tan\alpha\hat{k}$

$$\frac{d\vec{r}}{dt} = -a\sin t\hat{i} + a\cos t\hat{j} + a\tan\alpha\hat{k}, \quad \frac{d^2\vec{r}}{dt^2} = -a\cos t\hat{i} - a\sin t\hat{j}$$

$$\frac{d^3\vec{r}}{dt^3} = a\sin t\hat{i} - a\cos t\hat{j}$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = a^2\sin t\tan\alpha\hat{i} - a^2\cos t\tan\alpha\hat{j} + a^2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\tan^2\alpha + a^4\cos^2 t\tan^2\alpha + a^4} = a^2\sqrt{\tan^2\alpha + 1} = a^2\sec\alpha$$

3.14 : Unit tangent vector to the curve $\vec{r} = \vec{f}(s)$

Let $\vec{P}$ and $\vec{Q}$ be any two neighbouring points on the curve AB with p.v. $\vec{r}, \vec{r} + \delta\vec{r}$ respectively w.r.t. fixed point origin 'O'. A is fixed point on the curve from where parametric value of arc length 's' is measured.

Let s and s + δs be corresponding parametric value for pt. 'P' and 'Q' respectively. ***(fig. 3.16)***

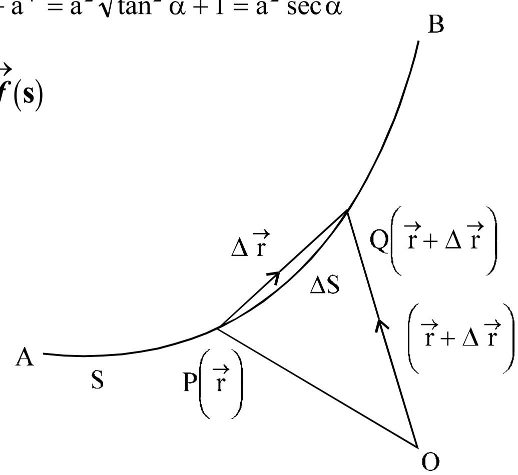


Fig. 3.16

$$\overrightarrow{PQ} = \overrightarrow{\delta r}$$

$\frac{\delta\vec{r}}{\delta s}$ is vector in the direction of $\delta\vec{r}$.

When Q approaches to P through curve.

$$\frac{\text{Chord PQ}}{\text{Arc PQ}} = 1 \text{ As } Q \to P, \text{As } Q \to P \text{ then } \delta S \to 0$$

$$\lim_{Q\to P}\left|\frac{\delta\vec{r}}{\delta s}\right| = 1, \quad \lim_{\delta s\to O}\left|\frac{\delta\vec{r}}{\delta s}\right| = 1, \quad \therefore \frac{d\vec{r}}{ds} = 1$$

$\Rightarrow \dfrac{d\vec{r}}{ds}$ is unit vector.

$\dfrac{d\vec{r}}{ds}$ is vector whose direction is limiting direction of $\delta\vec{r}$ as $\delta s \to 0$.

The limiting direction of $\delta\vec{r}$ as $\delta s \to 0$ is tangent to curve at P.

Let $\hat{t}$ be the unit tangent at P.

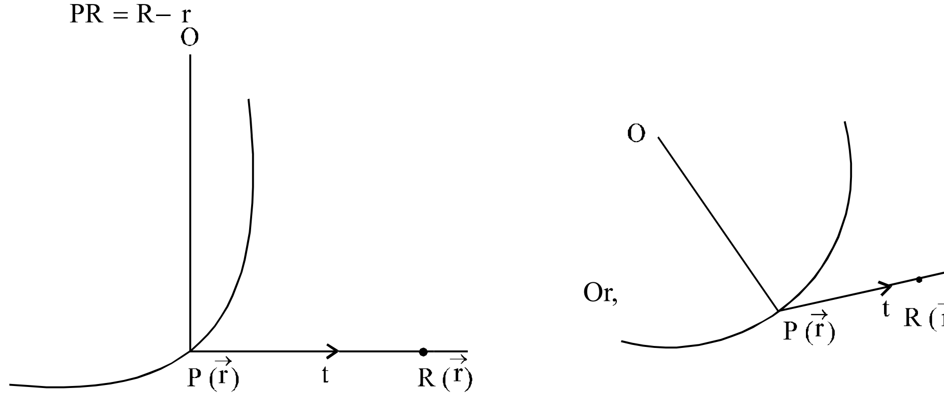
$$\therefore \frac{d\vec{r}}{ds} = 1 \cdot \hat{t}, \quad \therefore \boxed{\frac{d\vec{r}}{ds} = \hat{t}} \text{ unit tangent to the curve at P.}$$

$\boxed{r' = \hat{t}}$ (′ dash denotes derivative w.r.t.s)

Equation of tangent :

Let R be any point on the tangent line at P whose position vector is $\vec{R}$ w.r.t. fixed point 'O'. ***(fig. 3.17)***

$$\overrightarrow{PR} = \vec{R} - \vec{r}$$



(Fig. 3.17)

$\overrightarrow{PR}$ is along the direction of $\hat{t}$.

$$\therefore \vec{R} - \vec{r} = c\hat{t}$$

$$\therefore \boxed{\vec{R} = \vec{r} + c\hat{t}} \text{ Where 'c' is some scalar.}$$

Equation of curve : It is locus of a point whose Cartesian co-ordinate i.e., x, y, z. P.v. of any point with some fixed point is a function of single variable parameter 'u' is called space curve. ***(fig. 3.18)***

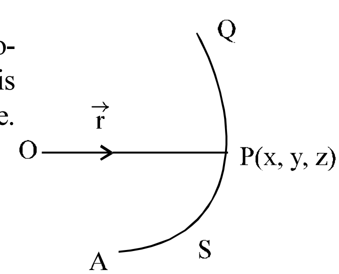


Fig. 3.18

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Where $x = f_1(u)$, $y = f_2(u)$, $z = f_3(u)$

Let A be a fixed point on the curve from where arc length is measured to corresponding point.