MODULE – IV

[Vector integral calculus : Line Integrals, Green Theorem, Surfaceintegrals, Gauss theorem and Stokes Theorem (without Proof)]

Vector Integration Applications

STUDY MATERIAL

Prepared by :

Dr. Chittaranjan Mallick
Department of Mathematics,
P.M.E.C.
Berhampur

CHAPTER – 5

Vector Integration Applications

5.1 : Introduction

The concept of a surface integral is a natural generalization of the concept of a double integral. There we integrate over a region in a plane and here we integrate over a piecewise smooth surface in space. Any Integral which is to be evaluated over a surface is called a surface integral.

The definition of a surface integral is parallel to that of a double integral. Here we consider a portion S of a surface. We assume that S has finite area and is simple, i.e., *S* has no points at which it intersects or touches itself. Let $f(x, y, z)$ be a function which is defined and continuous on *S*. We sub-divide *S* into *n* parts $S_1, S_2, \ldots, S_n$ of areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, respectively. Let $P(x_k, y_k, z_k)$ be an arbitrary point in each part S_k. Let us form the sum

$$J_n = \sum_{x=1}^{n} f(x_k, y_k, z_k)\,\Delta A_k$$

Now we let *n* tend to infinity in such a way that the largest part out of $S_1, S_2, \ldots, S_n$ shrinks to a point. Then the infinite sequence $J_1, J_2, \ldots, J_n$ has a limit is independent of the choice of subdivisions and points P_k. This limit is called the Surface Integral of F (x, y, z) over S and is denoted by

$$\iint_S f(x, y, z)\, dS\,.$$

Thus, $\displaystyle\iint_S f(x,y,z)\, dS = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta A_k$, provided the limit exists.

Surface Integrals:

Any integral which is to be evaluated over a surface is called a surface integral.

Suppose that a surface is bounded by a simple closed curve *C* as in the figure given bleow.

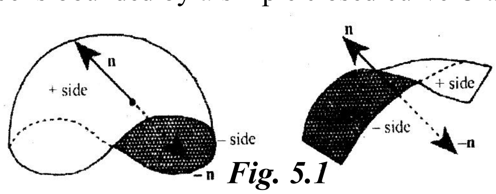

Fig. 5.1

Now the surface can be thought of as to have two sides separated by C of which one is arbitrarily chosen positive and the other, the negative side. In what follows, we denote the positive side by S. In the case of a closed surface, usually the outer side is taken as the positive side. The unit normal at any point to the surface drawn in the positive side is denoted by **n**.

Consider a plane surfae whose area is A. When one side of this surface is chosen as the positive side S with **n** as the unit normal drawn to this side, the vector area of this positive side can be specified by $A\mathbf{n}$. Also the area dS of one side of an indinitely small bit of a general surface can be specified vectorially as $dS\mathbf{n}$ or simply as dS, where **n** is the unit normal drawn to this side.

Let $f(x, y, z)$ be a vector function of position defined over the positive side S of a surface. Let P be any point on the surface S and let **n** be the unit vector at P in the direction of the outward drawn normal to the surface at P. The **f. n** is the normal component of **f** at P. The integral of **f. n** over S is defined as

$$\iint_S \mathbf{f.n}\, dS$$

This is an example of a surface integral (sometimes know as flux).

Remark - 1 : Another example of surface integral is

$$\iint_S \mathbf{f} \times \mathbf{n}\, dS$$

Remark - 2 : With the different of surface area dS, we have associated a vector $d\mathbf{S}$ (called vector area) whose magnitude is dS and whose direction is that of **n**.

Then we have $\quad d\mathbf{S} = \mathbf{n}\, ds$

and $$\iint_S \mathbf{f.n}\, dS = \iint_S \mathbf{f}\, dS$$

Remark - 3 : If $\mathbf{f.\ n} = \phi(x, y, z)$

then $$\iint_S \mathbf{f.n}\, dS = \iint_S \phi(x, y, z)\, dS$$

In particular if $\phi(x, y, z) = 1$, then $\iint_S dS$ will give the surface area of S.

Evaluation of the Surface Integral :

In order to evaluate Surface Integrals, it is convenient to express them as double integrals taken over the orghogonal projection of the surface S on one of the coordinate planes. But this is possible only if any line perpendicular to the coordinate plane chosen meets the surface S in not more than one point. If the surface does not satisfy this condition, then it can be subdivided into surface which do satisfy this condition.

Suppose the surface S is such that any line perpendicular to the xy-plane meets S in no more than one point. Then the equation of the surface S can be written in the form $z = f(x, y)$.

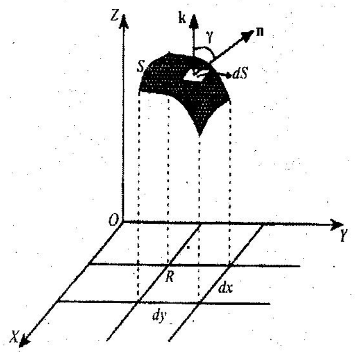

Fig. 5.2

Let R be the orghogonal projection of S on the xy-olane. If γ is the acute angle which the udirected normal **n** at P (x, y, z) to the surface S makes with the z-axis, then it can be shown that

$$\cos\gamma \, dS = dx, dy,$$

where dS is the small element of area of surface S at the point P (x, y)

$$dS = \frac{dx\,dy}{\cos\gamma} = \frac{dx\,dy}{|\mathbf{n.k}|}$$

where **k** is the unit vector along the z - axis.

Hence $$\iint_S \mathbf{f.n}\,dS = \iint_R \mathbf{f.n}\,\frac{dx\,dy}{|\mathbf{n.k}|}$$

For simplification of S.I. we have to take projections of the surface on the coordinate planes suitably.

(a) When S is projected on xy-plane then

$\int_S \vec{F}\cdot\vec{n}\,dS, = \iint_{S_1}\left(\vec{F}\cdot\vec{n}\right)\frac{dx\,dy}{\vec{n}\cdot\vec{k}}$ where S_1 is the projection of S on xy-plane whose normal is $\vec{k}$.

(b) When S is projected on yz-plane then

$\int_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_2}\left(\vec{F}\cdot\vec{n}\right).\frac{dy\,dz}{\vec{n}\cdot\vec{i}}$,where $S_2$is the projection of S on yz-plane whose normal is $\vec{i}$.

(c) When S is projected on zx-plane, then

$\int_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_3}\left(\vec{F}\cdot\vec{n}\right)\frac{dz\,dx}{\vec{n}\cdot\vec{j}}$, where S_3 is the projection of S on zx-plane whose normal is $\vec{j}$.

Physically, if $\vec{F}$ represents the velocity of a fluid, then the total outward flux of $\vec{F}$ across a closed surface is the surface integral $\int_S \vec{F}\cdot\vec{n}\,dS$. If the flux vanishes, then $\vec{F}$ is said to be solenoidal.

Volume Integrals :

Let V be the volume bounded by the surface S. Also let $f(x, y, z)$ be a single valued function of position defined over V. Now subdivide V into **n** elements of volumes Δv_1, Δv_2 .... Δv_n. In each part Δv_i, let us choose an arbitrary point (x_i, y_i, z_i). Then the limit of the sum $\Sigma(x_i, y_i, z_i)\Delta v_i$ as $n \to \infty$ and $\Delta v_i \to 0$, if it exists is called the denoted by $\iiint f(x,y,z)dV$.

If we subdivide the volume into small cuboids by drawing lines parallel to the three co-ordinate axes, then $dV = dx\,dy\,dz$ denote the element of volume. As volume element dV is a scalar, there are only two possible volume, integral. Any integral which is to be evaluated over a volume is called a volume integral.

$$\iiint_V \phi\, dV \text{ or } \iiint_V \vec{F}\, dV = \iiint_V f(x,y,z)dx\,dy\,dz$$

The results of integration are respectively a scalar and a vector.

Let $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ be any continuously differentiable vector function and a closed surface in space enclosing a volume V, then the volume integrals are defined as follows :

(A) $$\int_V \vec{F}\cdot dV = \int_V \left[f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right] dV$$

$$= \vec{i}\int_V f_1 dV + \vec{j}\int_V f_2 dV + f_3\vec{k}\int_V f_3\, dV$$

(B) $$\int_V \left(\nabla\cdot\vec{F}\right)dV = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

(In (A) generally single integral is used due to single differential dV. If we consider $dV = dx\,dy\,dz$ then triple integrals are to be used.)

Other form is $\int_V \left(\nabla\times\vec{F}\right)dV$.

It is often convenient to convert multiple integrals into others with fewer integral signs. This is done by two important theorems, viz., Gauss' divergence theorem and Stokes theorem of vector analysis.

Note - 1. If we sub-divide the volume V into small cuboids by drawing planes parallel to the co-ordinate planes then $dv = dx\,dy\,dz$

$$\iiint_V \phi dV = \iiint_V \phi(x,y,z)\,dx\,dy\,dz$$

Note 2. If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then

$$\iiint_V \vec{F}\,dV = \hat{i}\iiint_V F_1(x,y,z)dx\,dy\,dz + \hat{j}\iiint_V F_2(x,y,z)dx\,dy\,dz + \hat{k}\iiint_V F_3(x,y,z)dx\,dy\,dz$$

Illustrative Examples

Example – 1 : Evaluate $\int \vec{F}\cdot\vec{n}\,ds$ ***where s is the curved surface of the cylinder*** $x^2 + y^2 = a^2$, $0 \le z \le 2a$.

Solution : Let $\phi = x^2 + y^2 - a^2$

$$\therefore \nabla\phi = 2x\hat{i} + 2y\hat{j}$$

$$\therefore \vec{n} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\hat{i} + y\hat{j}}{a} \quad \left(\because x^2 + y^2 = a^2\right)$$

$$\vec{F}\cdot\vec{n} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\cdot\frac{x\hat{i} + y\hat{i}}{a} = \frac{x^2 + y^2}{a} = \frac{a^2}{a} = a$$

$$ds = \frac{dz\cdot dy}{\vec{n}\cdot\hat{i}} = \frac{dz\,dy}{\left(\dfrac{x\hat{i} + y\hat{j}}{a}\right)i} = \frac{dz\,dy}{x}.a$$

$$\therefore \int_s \vec{F}\cdot\vec{n}\,ds = \int_0^{2a}\int_{-a}^{a} a\frac{dz\,dy}{x} = \int_0^{2a}\int_{-a}^{a}\frac{1}{\sqrt{a^2 - y^2}}dy\,dz = \int_0^{2a}\left[sin^{-1}\left(\frac{y}{a}\right)\right]_{-a}^{a}\cdot dz = \int_0^{2a}\pi\,dz = 2\pi a$$

Example – 2 : Evaluate $\int_v \nabla.\vec{F}\,dV$ ***where*** $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ ***and V is the region bounded by the surfaces*** $x = 0, x = 2, y = 0, y = 6, z = x^2$ ***and*** $z = 4$.

Solution : The integration over the region ν is covered by means of the element *dx dy dz* as follows :

(a) Keeping *x*, *y* fixed and integrating with respect to *z* from $z = x^2$ to $z = 4$ (from P to Q in the elementary column) ***(fig 5.3)***

(b) Then by keeping *x* fixed and integrating with respect to *y* from $y = 0$ to $y = 6$ (from R to S in the elementary state), and

(c) Finally integrating with respect to *x* from $x = 0$ to $x = 2$ (from A to B in the given volume.)

Thus the required integrals is

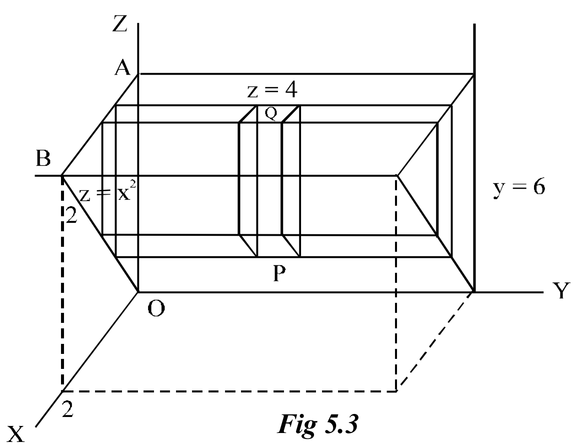

Fig 5.3

$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}\left(x\hat{i} + y\hat{i} + z\hat{k}\right)dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[xz\hat{i} + yz\hat{j} + \frac{z^2}{2}\hat{k}\right]_{z=x^2}^{4} dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left[x\left(4 - x^2\right)\hat{i} + y\left(4 - x^2\right)\hat{j} + \frac{1}{2}\left(16 - x^4\right)\hat{k}\right]dx\,dy$$

$$= \int_{x=0}^{2} \left[x(4-x^2) y\hat{i} + \frac{y^2}{2}(4-x^2)\hat{j} + \frac{1}{2}(16-x^4) y\hat{k} \right]_{y=0}^{y=6} dx$$

$$= \int_{x=0}^{2} \left[x(4-x^2)6\hat{i} + 18(4-x^2)\hat{j} + \frac{1}{2}(16-x^4)6\hat{k} \right] dx$$

$$= \left[6\left(4\frac{x^2}{2} - \frac{x^4}{4}\right)\hat{i} + 18\left(4x - \frac{x^3}{3}\right)\hat{j} + \frac{1}{2}\left(16x - \frac{x^5}{5}\right)6\hat{k} \right]_0^2$$

$$= 6\left(4\frac{4}{2} - \frac{16}{4}\right)\hat{i} + 18\left(8 - \frac{8}{3}\right)\hat{j} + 3\left(32 - \frac{32}{5}\right)6\hat{k} = 6\cdot 4\hat{i} + 18\cdot\frac{16}{3}\hat{j} + 3\cdot\frac{128}{5}\hat{k}$$

$$= 24\hat{i} + 96\hat{j} + \frac{384}{5}\hat{k} = \frac{24}{5}\left[5\hat{i} + 20\hat{j} + 16\hat{k}\right]$$

Example – 3 : Evaluate $\int_V (2x+y)dV$ ***, where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$.***

Solution : The integration over the region V is covered by means of the element $dx\ dy\ dz$ as follows :

(a) Keeping x, y fixed and integrating with respect to z from $z = 0$ to $z = 4 - x^2$ from P to Q in the elementary column. ***(fig 5.4)***

(b) Then by keeping x fixed and integrating with respect to y from $y = 0$ to $y = 2$ from R to S in the elementary slab and

(c) Finally integrating with respect to x from $x = 0$ to $x = 2$.

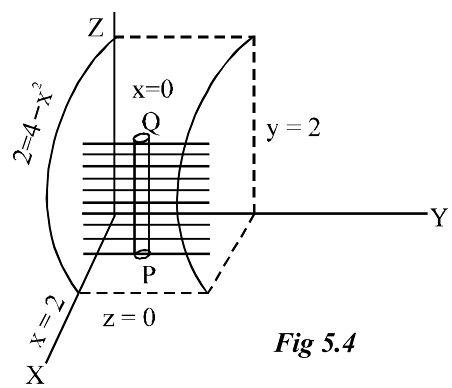

Fig 5.4

Thus, the required integral is $\int_V (2x+y)\, dV$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x+y) dx\, dy\, dz = \int_{x=0}^{2}\int_{y=0}^{2} (2x+y)\left[z\right]^{4-x^2} dx\, dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (2x+y)(4-x^2) dx\, dy = \int_{x=0}^{2} \left\{ 2x(4-x^2)\left[y\right]_0^2 + (4-x^2)\left[\frac{y^2}{2}\right]_0^2 \right\} dx$$

$$= \int_{x=0}^{2} \left\{ 4x(4-x^2) + (4-x^2)\cdot 2 \right\} dx = \int_{x=0}^{2} \left(16x - 4x^3 + 8 - 2x^2\right) dx$$

$$= \left[16\frac{x^2}{2} - 4\frac{x^4}{4} + 8x - 2\frac{x^3}{3} \right]_0^2 = 16\frac{4}{2} - 4\frac{16}{4} + 16 - 2\cdot\frac{8}{3} = 32 - 16 + 16 - \frac{16}{3} = \frac{80}{3}$$

Example – 4 : Evaluate $\int_S \vec{F}\cdot\vec{n}\,dS$ ***where*** $\vec{F} = 6z\vec{i} - 4\vec{j} + y\vec{k}$ ***and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.***

Solution : First to find a unit normal drawn outward on the surfaces S.

Let $\phi = 2x + 3y + 6z - 12$

Then $\vec{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{1}{7}\left(2\vec{i} + 3\vec{j} + 6\vec{k}\right)$ ***(fig 5.5)***

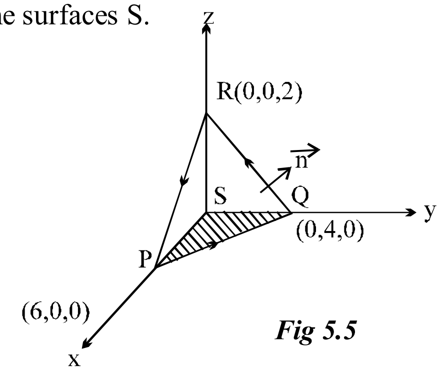

Fig 5.5

$$\vec{F}.\vec{n} = \left(6z\hat{i} - 4\hat{j} + y\hat{k}\right).\frac{1}{7}\left(2\vec{i} + 3\vec{j} + 6\vec{k}\right)$$

$$= \frac{12}{7}z - \frac{12}{7} + \frac{6}{7}y$$

Taking projection on xy-plane

$$\int_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\frac{dx\,dy}{\vec{n}\cdot\vec{k}} = \int_0^6\int_0^{\frac{12-2x}{3}} \left(\frac{12}{7}z - \frac{12}{7} + \frac{6}{7}y\right)\frac{dx\,dy}{\frac{6}{7}}$$

$$= \int_0^6\int_0^{\frac{12-2x}{3}} \left[\frac{12}{7}\left(\frac{12-2x-3y}{6}\right) - \frac{12}{7} + \frac{6}{7}y\right], \frac{7}{6}\,dx\,dy$$

$$= \int_0^6\int_0^{\frac{12-2x}{3}} \left[\frac{12}{7}.\frac{12}{6} - \frac{12}{7}.\frac{2}{6}x - \frac{12}{7}.\frac{3}{6}y - \frac{12}{7} + \frac{6}{7}y\right]\frac{7}{6}dxdy$$

$$= \int_0^6\int_0^{\frac{12-2x}{3}} \left[\frac{24}{7} - \frac{4}{7}x - \frac{6y}{7} - \frac{12}{7} + \frac{6y}{7}\right]\frac{7}{6}dxdy$$

$$= \frac{1}{6}\int_0^6\int_0^{\frac{12-2x}{3}} (12-4x)\,dy\,dx = \frac{1}{6}\int_0^6 \left[(12-4x)y\right]_0^{\frac{12-2x}{3}} dx$$

$$= \frac{1}{18}\int_0^6 (12-4x)(12-2x)dx$$

$$= \frac{4}{9}\int_0^6 \left(18 - 9x + x^2\right)dx = \frac{4}{9}\left[18x - \frac{9x^2}{2} + \frac{x^2}{3}\right]_0^6 = \frac{4}{9}\times 18 = 8$$

Example – 5 : Let $\vec{F} = 2xz\vec{i} - x\vec{j} + y^2\vec{k}$. ***Evaluate*** $\iiint_V \vec{F}\cdot dV$ ***where V is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, $z = 4$.***

Solution : To cover the region V we first keep x and y fixed and integrate from $z = x^2$ to $z = 4$. Then keep x fixed and integrate from $y = 0$ to $y = 6$, finally, integrate from $x = 0$ to $x = 2$. Also we have $dV = dxdydz$

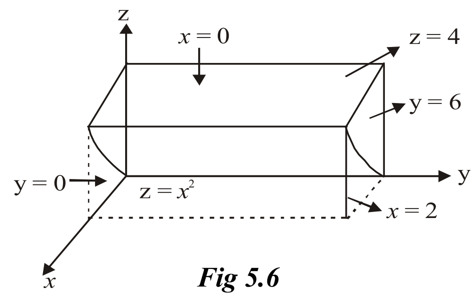

Fig 5.6

Required vector volume integral $= \iiint_V \vec{F} dV$

$\iiint_V \vec{F}\, dx\, dy\, dz$ ***(fig 5.6)***

$$= \int_{x=0}^{x=2} \left[\int_{y=0}^{y=6} \left\{ \int_{z=x^2}^{z=4} \left(2xz\hat{i} - x\hat{j} + y^2\hat{k} \right) dz \right\} dy \right] dx$$

$$= \int_{x=0}^{x=2} \left[\int_0^6 (16x - x^2)\hat{i} + \left(x^3 - 4x \right)\hat{j} + 72\left(4 - x^2 \right) y^2\hat{k})dy \right] dx$$

$$= \int_{x=0}^{x=2} \left[\left(96x - 6x^5 \right)\hat{i} + \left(6x^3 - 24x \right)\hat{j} + 72\left(4 - x^2 \right)\hat{k} \right] dx$$

$$= \vec{i} \int_0^2 \int_0^6 \int_{x^2}^4 2xz\, dz\, dy\, dx - \vec{j} \int_0^2 \int_0^6 \int_{x^2}^4 x\, dz\, dy\, dx + \vec{k} \int_0^2 \int_0^6 \int_{x^2}^4 y^2\, dz\, dy\, dx$$

$$= 128\vec{i} - 24\vec{j} + 348\vec{k}$$

5.2 : Transformation of Surface Integral Into Line Integrals – Stokes Theorem

Transformation of surface integrals into line integrals and conversely is done with the help of a theorem known as **Stoke's theorem,** given by Gorge Gabriel Stokes (1819–1903), an Irish mathematician and physicist, who made important contribution to the theory of infinite series and several branches of theoretical physics. Stokes theorem is an extension of Green's theorem in vector form to surfaces and curves in three dimensions. Under suitable restrictions

(i) on the vector ***F***,

(ii) on the boundry curve ***C***, and

(iii) on the surface *S* bounded by *C*.

Stokes theorem connects line integral to a surface integral.

In stoke's Theorem, we require that the surface *S* is **orientable.** By orientable, we mean that it is possible to consistently assign a unique direction, called positive, at each point of *S*,

and that there exists a unit normal $\hat{n}$ pointing in this direction. As we move about over the surface *S* without touching its boundry, the direction cosines of the unit vector $\hat{n}$ should

vary continuously and when we return to the straight position, $\hat{n}$ should return to its initial direction.

The work performed by a vector field on a particle that traverses a simple closed, piecewise smooth curve C in the positive direction can be obtained by integrating the normal component of the curl over an oriented surface S bounded by C.

Stokes Theorem :

Statement : If S be an *open surface* bounded by a *closed curve* C and $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ be any continuously differentiable vector function, then (fig 5.7)

$$\int_C \vec{F}\, d\vec{r} = \int\int_s curl\,\vec{F}.\,\vec{n}\, ds = \iint_S (\nabla \times F).\hat{n}ds = \oint_C \vec{F}.d\vec{r}$$

where $\vec{n}$ is the unit external normal at any point of S.

Cartesian form of stokes theorem in space :

The line integral of the tangential component of a vector function $\vec{F}$ (finite and differentiable), around a simple closed curve C is equal to the surface integral of the normal component of curve of $\vec{F}$ taken over any surface S having C as its boundary.

Proof : Let the outward drawn normal vector $\vec{n} = l\vec{i} + m\vec{j} + n\vec{k}$ be the unit external normal at any point S. Here $\vec{n}$ make angles α, β, γ with positive directions of x, y, z axes such that, let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$.

$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. Also $\vec{r} = x\vec{i} + y\vec{i} + z\vec{k}$, and $d\vec{r} = dx\vec{i} + dy\vec{i} + dz\vec{k}$

i.e. $\vec{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$.

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \times \vec{F}$$

$$= \vec{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \vec{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \vec{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\therefore \text{Curl}\ \vec{F}.\vec{n} = l\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + m\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + n\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

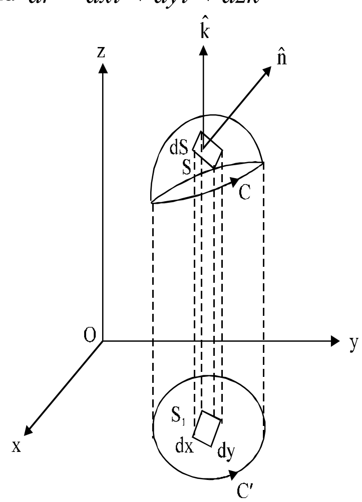

Fig 5.7

Also $\vec{F}.d\vec{r} = \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = f_1dx + f_2dy + f_3dz$

$\therefore$ Stoke's Theorem takes the form

$$\oint_C (f_1dx + f_2dy + f_3dz) = \int\int_S l\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + m\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + n\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \quad ...(i)$$

$$= \iint_S \left\{\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)dydz + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)dzdx + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dxdy\right\}$$

Let z = ϕ (*x, y*) be the equation of the surface *S* whose projection on the *xy*-plane is S_1. Then the projection of *C* on the *xy*-plane is C_1 which is the boundary of S_1.

$$\therefore \quad \oint_C f_1(x,y,z)\,dx = \oint_{C_1} f_1[x,y,\phi(x,y)]\,dx = \oint_{C_1}\{f_1[x,y,\phi(x,y)]\,dx + 0.dy\}$$

$$= -\int\int_{S_1} \frac{\partial}{\partial y} f_1[x,y,\phi(x,y)]\,dx\,dy = -\iint_{S_1}\left[\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}.\frac{\partial \phi(x,y)}{\partial y}\right]dx\,dy \quad \text{....(ii)}$$

The direction cosines of the normal to the surface $z = \phi$ (*x, y*) are given by

$$\frac{l}{-\frac{\partial \phi(x,y)}{\partial x}} = \frac{m}{-\frac{\partial \phi(x,y)}{\partial y}} = \frac{n}{1} \qquad \text{....(iii)}$$

Also, *dx dy* = projection of *ds* on the *xy*-plane = $ds.\ \hat{n} \Rightarrow ds = \frac{dx\,dy}{\hat{n}}$

From (ii)

$$\oint_C f_1[x,y,\phi(x,y)]\,dx = -\int\int_{S_1}\left[\frac{\partial f_1}{\partial y} - \frac{\partial f_1}{\partial z}\frac{m}{n}\right].\hat{n}ds = \int\int_{S_1}\left[m\frac{\partial f_1}{\partial z} - n\frac{\partial f_1}{\partial y}\right]ds \quad \text{...........(iv)}$$

Similarly, we can prove that

$$\int_C f_2[x,y,\phi(x,y)]\,dy = \int\int_{S_1}\left(n\frac{\partial f_2}{\partial x} - l\frac{\partial f_2}{\partial z}\right)ds \quad \text{.............(v)}$$

$$\int_C f_3[x,y,\phi(x,y)]\,dz = \int\int_{S_1}\left(l\frac{\partial f_3}{\partial y} - m\frac{\partial f_3}{\partial x}\right)ds \quad \text{..............(vi)}$$

Adding (iv), (v) and (vi) we get the result of Stroke's Theorem.

$$\oint_C (f_1dx + f_2dy + f_3dz) = \int\int_{S_1}\left(m\frac{\partial f_1}{\partial z} - n\frac{\partial f_1}{\partial y}\right) + \int\int_{S_1}\left(n\frac{\partial f_2}{\partial x} - l\frac{\partial f_2}{\partial z}\right) + \int\int_{S_1}\left(l\frac{\partial f_3}{\partial y} - m\frac{\partial f_3}{m}\right)ds$$

$$= \int\int_S\left\{\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)cos\,\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)cos\gamma\right\}ds$$

Hence theorem is proved.

Note : **Green's theorem in a plane as a special case of Stoke's Theorem.**

Cartesian form of stokes Theorem for plane :

Let us take a system of Cartesian rectangular coordinate axes such that the plane of the given surface *S* in the *xy*-plane and *z*-axis lies along the direction of th normal vector **n**.

Here, the normal vector is constant.

Let, $f = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore \oint_C \boldsymbol{f}.d\boldsymbol{r} = \oint_C \boldsymbol{f}.\frac{d\boldsymbol{r}}{ds}ds = \oint_C \boldsymbol{f}.\mathbf{t}\, ds$$

(where $\frac{d\boldsymbol{r}}{ds} = \mathbf{t}$ is the unit tangent vector C)

$$= \oint_C \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right).\left(\hat{i}\frac{dx}{ds} + \hat{j}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}\right)ds$$

$$= \oint_C \left(f_1\frac{dx}{ds} + f_2\frac{dy}{ds}\right)ds$$

$$= \oint_C \left(f_1 dx + f_2 dy\right)$$

$\left(\because \frac{dz}{ds} = 0, \text{ as the tangent at any point lies in the } xy \text{ plane}\right)$

Now, $\oint_C$ curl $\boldsymbol{f}.\,\mathbf{n}\, dS = \oint_C$ curl $\boldsymbol{f}.\,\mathbf{k}\, dS$

$$= \oint_C \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dx\, dy$$

$\therefore$ Stoke's theorem for plane becomes

$$\oint_C \left(f_1 dx + f_2 dy\right) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dx\, dy$$

Let us, now prove it.

Let the region S be such that any line parallel to either axis meet the boundary C in at the most two points.

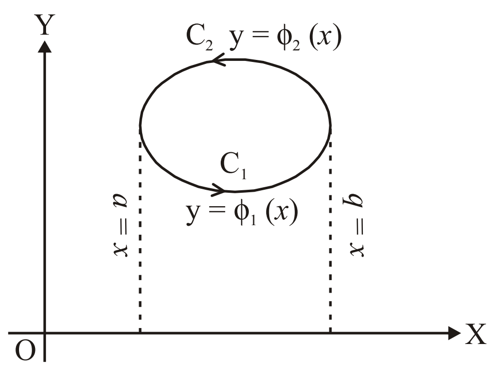

Fig. 5.8

Let the region be $x = a,\ x = b,\ y = \phi_1(x),\ y = \phi_2(x)$, where $\phi_2(x) \geq \phi_1(x)$.

The boundary C is split up into two areas C_1 and C_2 as shown.

$$\therefore \iint_S \frac{\partial f_1}{\partial y} dx\, dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f_1}{\partial y} dy \right] dx$$

$$= \int_a^b f_1(x, \phi_2(x)) - f_1(x, \phi_1(x))\, dx = -\int_{C_2} f_1(x, y)\, dx - \int_{C_1} f_1(x, y)\, dx$$

$$= -\oint_C f_1(x, y)\, dx \qquad \text{... (i)}$$

Similarly,

$$\iint_S \frac{\partial f_2}{\partial x} dx\, dy = -\oint_C f_2(x, y)\, dy \qquad \text{... (ii)}$$

$\therefore$ From (i) and (ii)

$$= \oint_C (f_2 dy + f_1 dx) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\, dy$$

Hence the result.

If the region *S* is such that a line parallel to either axes meets the boundary *C* in more than two points, we divide *S* into a finite number of sub-regions such athat the boundary of each is met in at the most two points by any line parallel to either axis. By applying the above result to each sub-region and then adding the results, we find that the theorem is true for *S*. This is because the line integrals along the boundary curves will cancel in pairs.

Let $\vec{F} = f_1 \hat{i} + f_2 \hat{j}$ be a vector point function which is continuously differentiable in a region ***S*** of the *xy* plane bounded by the closed curve *C*.

Then $\int_C \vec{F}.d\vec{r} = \int_C (f_1 \hat{i} + f_2 \hat{j})(dx\, \hat{i} + dy\, \hat{j}) = \int_C f_1\, dx + f_2 dy$

and curl $\vec{F}.\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & o \\ f_1 & f_2 & o \end{vmatrix} \hat{k} = \left\{ (o\, \hat{i} - o\hat{j}) + \hat{k} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \right\} \hat{k} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$

Hence Stoke's theorem $\int_C \vec{F}.d\vec{r} = \int\int_S Curl\, \vec{F}.\hat{n} ds$ takes the form

$$\int_C f_1 dx + f_2 dy = \int\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\, dy \text{ which is Green's theorem.}$$

Illustrative Examples

Example – 1: Verify Stoke's theorem for the function $\vec{F} = x^2 \hat{i} - xy\hat{j}$ integrated round the square in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = a.

Solution : Here $\vec{F} = x^2 \hat{i} - xy\hat{j}$

The curve C is the boundary of the square OABC in the plane z = 0, so that the sides OA, AB, BC, CO and bounded by the lines $x = 0$,

$y = 0, x = a, y = a.$

Here $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\therefore \vec{F}.d\vec{r} = x^2 dx - xy\,dy$$

Now $$\oint_C \vec{F}.d\vec{r} = \oint_C \left(x^2 dx - xy\,dy\right) = \int_{OABC} \left(x^2 dx - xydy\right)$$

$$\oint_{OA}\left(x^2dx - xydy\right) + \oint_{AB}\left(x^2dx - xydx\right) + \oint_{BC}\left(x^2dx - xydx\right) + \oint_{CO}\left(x^2dx - xydx\right)$$

$$= \int_{\substack{x=0\\ at\ y=0}}^{x=a} \left(x^2dx - xy\,dy\right) + \int_{at\ x=a,\ y=0}^{y=a}\left(x^2dx - xy\,dy\right)$$

(a long OA) $y = 0, dy = 0$ (a long AB) $x = a, dx = 0$

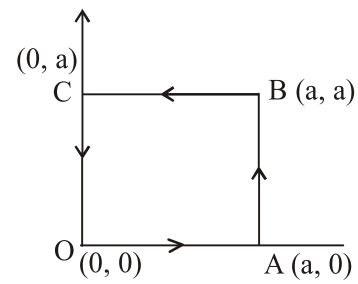

Fig 5.9

$$+ \int_{\substack{x=a\\ at\ y=a}}^{0} \left(x^2dx - xy\,dy\right) + \int_{\substack{y=0\\ at\ x=0}}^{0}\left(x^2dx - xy\,dy\right)$$

(A long BC) $y = a, \therefore dy = 0$ (A long CO) $x = a, \therefore dx = 0$

$$= \int_{x=0}^{a} x^2dx - a\int_{y=0}^{0} y\,dy + \int_{x=a}^{0} x^2dx + \int_{y=a}^{0}\left(0\,dx - 0\,dy\right)$$

$$= \left|\frac{x^3}{3}\right|_0^a - a\left|\frac{y^2}{2}\right|_0^a + \left|\frac{x^3}{3}\right|_a^0 + 0 = \frac{a^3}{3} - a.\frac{a^2}{2} - \frac{a^3}{3} = -\frac{a^3}{2} \qquad \ldots (1)$$

For surface integral,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix} = \hat{k}\left(-y - 0\right) = -y\,\hat{k}$$

Now positive unit normal to the plane z = 0 is $\hat{n} = \hat{k}$.

Also element of area $dS = dx\,dy.$

$$\therefore \text{cure } \vec{F}.\hat{n} = -y\,\hat{k}.\hat{k} = -y$$

Thus $$\iint_S curl\ F \cdot \hat{n} dS = \int_{x=0}^{a}\int_{y=0}^{a} -y\,dx\,dy$$

$$= -\int_0^a \left[\frac{y^2}{2}\right]_0^a dx = -\int_0^a \frac{a^2}{2}dx = \frac{-a^2}{2}\left[x\right]_0^a = \frac{-a^2}{2}.a = \frac{-a^3}{2}$$

Equality of (1) and (2) varifies stokes theorem.

Example – 2 : Verify Stoke's Theorem for the vector field $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ ***integrated round the rectangle in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a, y = b.***

Solution : Let C denote the boundary of the rectangle OPQR, then C consists of four lines OP, PQ, QR, RO in the plane Z = 0, so that the equation $x = 0$, $y = 0$, $x = a$ and $y = b$.

$\vec{r} = x\vec{i} + y\vec{j}$, $d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xy\, dy$

$$\therefore \quad \oint_C \vec{F}.d\vec{r} = \int_{OPQR} \left[(x^2 - y^2)dx + 2xy\, dy\right]$$

$$= \left[\int_{OP} + \int_{PQ} + \int_{QR} + \int_{RO}\right]\left[(x^2 - y^2)dx + 2xydy\right]$$

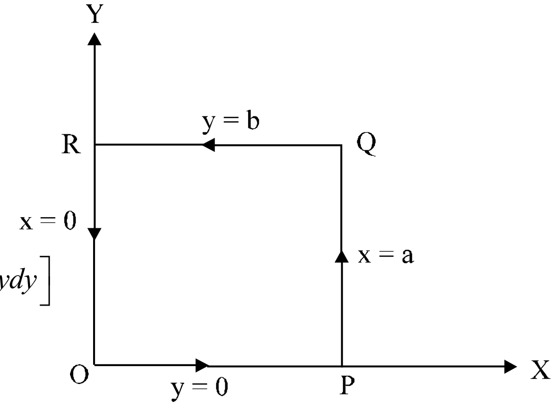

Fig. 5.10

1. Along OP : $y = 0$, $dy = 0$, $x : 0 \to$ a

$$\int_{OP} \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx = \frac{a^3}{3}$$

2. Along PQ : $x = a$, dx = 0, y : $0 \to b$

$$\oint_{PQ} \vec{F} \cdot d\vec{r} = \int_0^b 2ay\, dy = 2a\left[\frac{y^2}{2}\right]_0^6 = ab^2$$

3. Along QR : $y = b$, $dy = 0$, $x : a \to 0$

$$\oint_{OR} \vec{F} \cdot d\vec{r} = \int_a^0 (x^2 - b^2)dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0 = ab^2 - \frac{a^3}{3}$$

4. Along RO : $x = 0$, $dx = 0$, $y : b \to 0$

$$\int_{RO} \vec{F} \cdot d\vec{r} = 0 \qquad (\because \vec{F} \cdot d\vec{r} = 0)$$

Along (1), (2), (3) and (4)

$$\therefore \text{ L.H.S. } = \oint_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0 = 2ab^2$$

$$\text{Now, } Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\vec{k}$$

For the surface, S, $\vec{n} = \vec{k}$, cure $\vec{F}.\hat{n} = 4y\hat{k}.\hat{k} = 4y$

$$\text{R.H.S.} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \int_0^a\int_0^b 4y\,dy\,dx = \int_0^a 4\cdot\left[\frac{y^2}{2}\right]_0^b dx = 2b^2\int_0^a dx = 2ab^2$$

$\therefore$ L.H.S. = R.H.S. Hence the Stoke's theorem is verified.

Example –3 : Verify Stoke's theorem for $\vec{F} = \left(x^2 + y^2\right)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines x = ± a, y = 0, y = b.

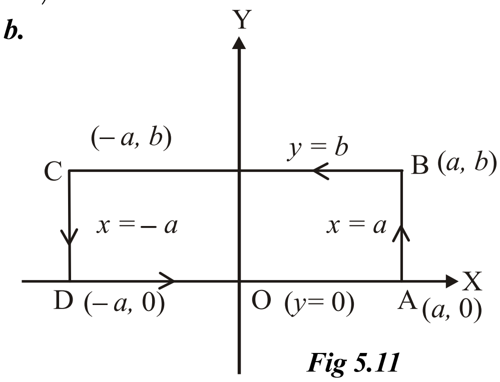

Fig 5.11

Solution : $\vec{F} = \left(x^2 + y^2\right)\hat{i} - 2xy\hat{j}\ ,\ \vec{r} = x\hat{i} + y\hat{j}$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\oint_C \vec{F}\cdot\overrightarrow{dr} = \oint_C \left(\left(x^2 + y^2\right)\hat{i} - 2xy\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right)$$

$$= \oint_C \left\{\left(x^2 + y^2\right)dx - 2xy\,dy\right\}$$

where C consists of AB, BC, CD, DA.

along AB, $x = a \therefore dx = 0$ and y varies from 0 to b

along BC, $y = b \therefore dy = 0$ and x varies from a to $-a$

along CD, $x = -a \therefore dx = 0$ and y varies from b to 0

along DA, $y = 0 \therefore dy = 0$ and x varies from $-a$ to a

$$\therefore \oint_C \left\{\left(x^2 + y^2\right)dx - 2xy\,dy\right\} = \int_{AB}\left(x^2 + y^2\right)dx - 2xy\,dy$$

$$+ \int_{BC}\left(x^2 + y^2\right)dx - 2xydy + \int_{CD}\left(x^2 + y^2\right)dx - 2xy\,dy + \int_{DA}\left(x^2 + y^2\right)dx - 2xy\,dy$$

$$= \int_0^b 0 - 2ay\,dy + \int_a^{-a}\left(x^2 + b^2\right)dx + \int_b^0 2ay\,dy + \int_{-a}^a x^2dx$$

$$= -ay^2\Big|_0^b + \left\{\frac{x^3}{3} + b^2x\right\}_a^{-a} + ay^2\Big|_b^0 + \frac{x^3}{3}\Big|_{-a}^a = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$\therefore \oint_C \left(x^2 + y^2\right)dx - 2xy\,dy = -4ab^2 \qquad \text{....(1)}$$

Now, find $\iint_S Curl\,\vec{F}\cdot\hat{n}\,dS$

$$Curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = \hat{i}(0) + \hat{j}(0) + \hat{k}(-2y - 2y) = -4y\,\hat{k}$$

$\hat{n} = \hat{k}$ $\because$ Rectangle in xy plane.

$\therefore Curl\ \vec{F} \cdot \hat{n} = -4y\hat{k} \cdot \hat{k} = -4y$

$$\iint_S Curl\ \vec{F} \cdot \hat{n} dS = \iint_S -4y\ dS = \iint_R -4y \frac{dx\ dy}{|\hat{n} \cdot \hat{k}|} = \iint_R -4y\ dx\ dy$$

$$= \int_{x=-a}^{a} \int_{y=0}^{b} -4y\ dy\ dx = \int_{-a}^{a} \left[-2y^2\right]_0^b dx = \int_{-a}^{a} -2b^2 dx = \left[-2b^2 x\right]_{-a}^{a} = -4ab^2$$

$$\therefore \iint_S Curl\ \vec{F} \cdot \hat{n}\ dS = -4ab^2 \qquad ...(2)$$

(1) and (2) are same $\therefore$ Stoke's theorem is verified.

Example – 4 : If ϕ is a scalar point function, using Stoke's theorem prove that Curl (grad ϕ)= 0.

Solution : Stoke's theorem is $\iint_S curl\ \vec{F} \cdot \vec{n}\ dS = \int_C \vec{F} \cdot d\vec{r}$ ....(i)

Let $\vec{F}$ as the gradient of a scalar function.

i.e., $\vec{F} = \text{grad}\ \phi = \nabla\phi$

$\therefore$ (i) becomes

$$\iint_S curl\,(grad\ \phi) \cdot \vec{n}\ dS = \oint_C \nabla\phi \cdot d\vec{r} \qquad ...(ii)$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}\ ,\ d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

and $\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

$$\therefore \nabla\phi \cdot d\vec{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$\therefore$ From (ii), we get $\int_S curl\,(grad\ \phi) \cdot \vec{n}\ dS = \int_C d\phi = \left[\phi\,(x, y, z)\right]_C = 0$ ...(iii)

($\because$ C is a closed curve, the lower and upper limits of the definite integral will be the same). The result (iii) i.e., the surface integral is zero, is true for any surface S. Hence the integrand must be indentically zero i.e., curl (grad ϕ) = 0.

Hence Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S Curl\ \vec{F} \cdot \hat{n}\ dS$ takes the form.

$\int_C f_1 dx + f_2 dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\ dy$ which is Green's theorem in a plane.

***Example – 5 :** Verify stokes theorem for a vector field $\vec{F} = (x^2 - y^2)\,\hat{i} + 2xy\,\hat{j}$ in the rectangular region of the xy- plane bounded by the lines x = 0, x = 1, y = 0, y = 2.*

Solution : For the verification of stokes theorem, we shall show that

$$\oint_C \vec{F}.\vec{d}r = \iint_s curl\ \vec{F}.\hat{n}\ dS$$

$$\vec{F}.d\vec{r} = ((x^2 - y^2)\hat{i} + 2xy\,\hat{j}).\,(dx\hat{i} + dy\hat{j})$$

$$= (x^2 - y^2)\,dx + 2xy\,dy$$

Now $$\iint_S \vec{F}.\vec{d}r = \oint_C \left[(x^2 - y^2)dx + 2xy\,dy\right],$$

where C consists of OA, AB, BC and CO

$$\therefore \quad \oint_C \vec{F}.d\vec{r} = \oint_{OA}\left[(x^2 - y^2)dx + 2xydy\right] + \oint_{AB}\left[(x^2 - y^2)dx + 2xydy\right]$$

$$+ \oint_{BC}\left[(x^2 - y^2)dx + 2xydy\right] + \oint_{CO}\left[\left(x^2 - y^2\right)dx + 2xydy\right] \quad ....(1)$$

Along OA, $y = 0 \therefore dy = 0$ and x varies from 0 to 1.
Along AB, $x = 1 \therefore dx = 0$ and y varies from 0 to 2.
Along BC, $y = 2 \therefore dy = 0$ and x varies from 1 to 0.
Along CO, $x = 0 \therefore dx = 0$ and y varies from 2 to 0.
$\therefore$ from (1),

$$\oint_C \vec{F}.d\vec{r} = \int_0^1 x^2dx + \int_0^2 \left[0 + 2(1)ydy\right] + \int_1^0 (x^2 - 4)dx + \int_2^0 0\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[y^2\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_1^0 + 0,$$

$$= \frac{1}{3} + 4 + \left[0 - \frac{1}{3} + 4\right] = 8 \quad ...(2)$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (2y + 2y)\hat{k} = 4y\hat{k}$$

Also for the surface $S, \hat{n} = \hat{k}$

$$\therefore \iint_s curl\,\vec{F}.\hat{n}dS = \int_{x=0}^{1}\int_{y=0}^{2} 4y\,\hat{k}.\hat{k}\frac{dxdy}{|\hat{k}.\hat{n}|} = \int_{x=0}^{1}\int_{y=0}^{2} 4y\,dx\,dy = 4\int_0^1 dx\int_0^2 y\,dy$$

$$=4[x]_0^1\left[\frac{y^2}{2}\right]_0^2=4(1)\left(\frac{4}{2}\right)=8 \qquad .....(3)$$

From (2) and (3), we have

$$\oint_C \vec{F}.d\vec{r}=\iint_S curl\ \vec{F}.\hat{n}\,dS,$$ which verifies stokes theorem.

Transformation of Volume Integrals into Surface Integrals and Conversely.

5.3 : Gauss Divergence Theorem

Statement : Let 'V' be a closed bounded region in space whose boundary is a piece-wise smooth orientable surface S. Let $F\ (x, y, z)$ be a vector function, which is countinuous and has continuous first partial derivatives in some domain containing R. Then

$$\iiint_V div\ \vec{F}\ dV=\iint_S \vec{F}\,.\,\vec{n}\ ds \qquad .....(1)$$

Where $F\,.\,\hat{n}$ is the component of F is the direction of the outward normal of S with respect to V and $\hat{n}$ is the outer unit normal vector of S.

In other world, the volume integral of divergence of a vector point function $\vec{F}$ taken over the volume V enclosed by a surface S, is equal to the surface Integral of the component of $\vec{F}$ taken over a closed surface S.

In compoennt of F =[F_1, F_2, F_3] and of outer normal vector $\vec{n}=[cos\alpha, cos\beta, cos\gamma]$ of S of (1) becomes

$$\iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz=\iint_S\left(F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma\right)ds$$

$$=\iint_S F_1dydz+F_2dzdx+F_3dxdy$$

Proof : Triple integrals can be transformed into surface integrals over the boundary surface of a region in space conversely. The transformation is done by divergence theorem, which involves the divergence of a vector function F = [F_1, F_2, F_3] = $F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ namely

$$div\vec{\nabla}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

If we write F and $\hat{n}$ in terms of components, say,

$F=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ and $\hat{n}=cos\,\alpha\,\hat{i}+cos\,\beta\,\hat{j}+cos\,\gamma\,\hat{k}$,

and let $\hat{n}ds=(\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k})ds$

where α, β and γ are the angles between $\hat{n}$ and the positive directions of x, y and z axes, respectively, then formula (1) takes the form

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\,dy\,dz = \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma \right) dS \quad ..... (2)$$

Therefore $dS \cos\alpha$, $dS \cos\beta$ *and* $dS \cos\gamma$ are the orthogonal projections of the element area dS on the yz-plane, the zx-plane and the xy-plane respectively. If we make a subdivision of S by planes parallel to the yz-plane the zx- plane and the xy-plane, then the projections on the co-ordinate planes will be rectangles with sides dy and dz on the yz-plane, dz and dx on the zx-plane and dx and dy on the xy-plane.

Hence the projected surface elements are $dy\,dz$, $dz\,dx$ and $dxdy$

$$\iint_S \vec{F}.\hat{n}ds = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy \right)$$

Since S is an orientable surface, then, by definition, this means

$$\iint_S F_1 \cos\alpha\, dS = \iint_S F_1\, dy\,dz \quad ..... (3)$$

$$\iint_S F_2 \cos\beta\, dS = \iint_S F_2\, dz\,dx \quad ..... (4)$$

$$\iint_S F_3 \cos\gamma\, dS = \iint_S F_3\, dx\,dy \quad ..... (5)$$

Then relation (2), with the help of relations (3, 4, 5) may be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial x} \right) dx\,dy\,dz = \iint_S \left(F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy \right) \quad ..... (6)$$

It is clear that relation (6) is true, if the following three relations hold simultaneously :

$$\iiint_V \frac{\partial F_1}{\partial x} dx\,dy\,dz = \iint_S F_1\, dy\,dz \quad ..... (7)$$

$$\iiint_V \frac{\partial F_2}{\partial y} dx\,dy\,dz = \iint_S F_2\, dz\,dx \quad ..... (8)$$

$$\iiint_V \frac{\partial F_3}{\partial z} dx\,dy\,dz = \iint_S F_3\, dx\,dy \quad ..... (9)$$

Proof of Gauss Divergence Theorem :

Let S be such a surface that a line parallel to z-axis meets it in two points only. Denote the lower and upper positions of S by S_1 and S_2 and let their equations be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. *(fig 5.12)*

We denote the projection of S on xy-plane by R.

Then

$$\iiint_V \frac{\partial F_3}{\partial z} dx\,dy\,dz = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z} dz \right] dx\,dy$$

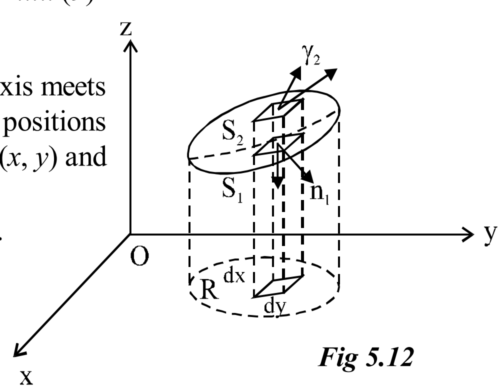

Fig 5.12

$$= \iint_R \left| F_3(x, y, z) \right|_{f_1}^{f_2} dx\, dy$$

$$= \iint_R \left[F_3(x, y, f_2) - F_3(x, y, f_1) \right] dx\, dy \text{ ...... (10)}$$

Now for the upper position S_2, $dx\, dy = \cos \gamma_2\, dS_2 = \hat{k} \cdot \hat{n}\ dS_2$

and for the lower position S_1, $dx\, dy = \cos \gamma_1\, dS_1 = -\hat{k} \cdot \hat{n}\ dS_1$

a negative sign being taken as the normal to dS_1 makes an obtuse angle $\pi - \gamma_1$ with $\hat{k}$. Therefore,

$$\iint_R F_3(x, y, f_2) dx\, dy = \iint_{S_2} F_3\, \hat{k} \cdot \hat{n}\, dS_2 \text{ and } \iint_R F_3(x, y, f_1) dx\, dy = \iint_{S_1} F_3\, \hat{k} \cdot \hat{n}\, dS_1$$

Substituting in (10) we get

$$\iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_{S_2} u_3\, \hat{k} \cdot \hat{n}\, dS_2 + \iint_{S_1} u_3\, \hat{k} \cdot \hat{n}\, dS_1 = \iint_S F_3\, \hat{k} \cdot \hat{n}\, dS \quad \text{..... (11)}$$

$$\iiint_V \frac{\partial F_1}{\partial x} dx\, dy\, dz = \iint_S F_1\, \hat{k} \cdot \hat{n}\, dS \quad \text{..... (12)}$$

$$\iiint_V \frac{\partial F_2}{\partial y} dx\, dy\, dz = \iint_S F_2\, \hat{k} \cdot \hat{n}\, dS \quad \text{..... (13)}$$

Adding (11), (12) and (13) we obtain the result (6).Hence divergence theorem is proved.

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy \right)$$

Illustrative Examples

Example – 1 : Verify Gauss's Divergence Theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$ and $z = 2$

Solution: R.H.S=Given $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$, div $\vec{F} = 2z$

$$\iiint_V div \cdot \vec{F}\, dV = \iiint_V 2z\, dx\, dy\, dz = 2 \int_0^{2\pi}\int_0^3\int_0^2 zr\, dr\, d\theta\, dz$$

(Taking on cylindrical coordinate system) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$= 2 \cdot \int_0^{2\pi}\int_0^3 r \left[\int_0^2 z\, dz \right] dr\, d\theta = 4 \int_0^{2\pi}\int_0^3 r\, dr \cdot d\theta$$

$$= 4\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^3 d\theta = 18\int_0^{2\pi} d\theta = 18, (2\pi) = 36\pi$$

For S.I. = $\int\limits_{\substack{S_1 \\ (base)}} + \int\limits_{\substack{S_2 \\ (top)}} + \int\limits_{\substack{S_3 \\ (comex\ protion)}}$

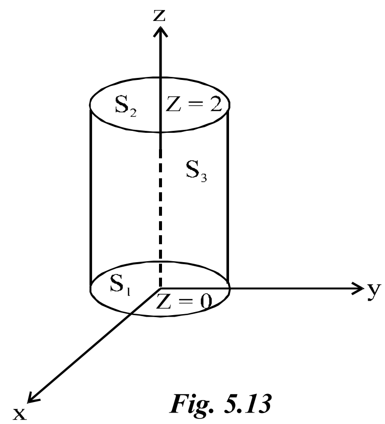

Fig. 5.13

On $S_1, z = 0,\ \vec{F} = y\vec{i} + x\vec{j}$ and $\vec{n} = -\vec{k}$

$\therefore \quad \vec{F}\cdot\vec{n} = 0 \Rightarrow \int\limits_{S_1} \vec{F}\cdot\vec{n}\, dS = 0$

On S_2, $\vec{F} = y\vec{i} + x\vec{j} + 4\vec{k}, \vec{n} = \vec{k} \quad \therefore\ \vec{F}\cdot\vec{n} = 4$

$\therefore \quad \int\limits_{S_2} \vec{F}\cdot\vec{n}\, dS = \int\limits_{S_2} 4\, .\, dS = 4\ S_2 = 4\, .\, \pi\, .\, 3^2 = 36\pi\ (x^2 + y^2 = 9)$

On S_3, A perpendicular to $x^2 + y^2 = 9$ has the direction grad ϕ.

Let $\phi = x^2 + y^2 - 9$

$$\vec{n} = \frac{grad\ \phi}{|grad\ \phi|} = \frac{2x\,\vec{i} + 2y\,\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\,\vec{i} + 2y\,\vec{j}}{\sqrt{4.9}} = \frac{1}{3}\left(x\vec{i} + y\vec{j}\right)$$

$$\vec{F}\cdot\vec{n} = \left(y\vec{i} + x\vec{j} + z\vec{k}\right).\frac{1}{3}\left(x\vec{i} + y\vec{j}\right) = \frac{1}{3}\left(xy + xy\right) = \frac{2}{3}xy$$

$\int\limits_{S_3} \vec{F}\cdot\vec{n}\, dS = \frac{2}{3}\int\limits_{S_3} xy\, dS = \frac{2}{3}\int\limits_{S'_3} xy\, .\frac{dy\, dz}{\vec{n}\cdot\vec{i}}$ (Projecting S_3 on yz-plane which is a rectangle)

$$\vec{n}.\vec{i} = \frac{1}{3}\left(x\vec{i} + y\vec{j}\right).\hat{i} = \frac{x}{3}$$

$$= \frac{2}{3}\int\limits_{S'_3} xy\, .\frac{dy\, dz}{x/3} = 2\int\limits_{S'_3} y\ \ dy\, dz = 2\int_0^2\int_{-3}^3 y\, dy\, dz = 0 \quad \left(\because \int_{-3}^3 \text{y dy} = 0\right)$$

$\therefore$ L.H.S = 0 + 36π + 0 = 36π = R.H.S.

Hence Divergence theorem is verified.

Example – 2 : Use Divergence theorem to prove the following :

(i) $\iint\limits_S \boldsymbol{Curl\ \vec{F}\, .\, \hat{n}\, dS = 0}$ ***(ii)*** $\iint\limits_S \boldsymbol{\vec{r}\, .\, \hat{n}\, dS = 3V}$

(iii) $\oint\limits_S \nabla r^2\, \vec{dS}$***, where S is any closed surface enclosing a volume V.***

Solution : (i) By Divergence Theorem, we have $\iint\limits_S Curl\ \vec{F}\, .\, \hat{n}\, dS$

$$= \iiint_V div\left(Curl\ \vec{F}\right) dV = \iiint_V \left(\nabla . \nabla \times \vec{F}\right) dV$$

Since $\nabla . \nabla \times \vec{F} = 0$

$$\therefore \quad \iint_S Curl\ \vec{F} . \hat{n}\, dS = 0$$

(ii) By the Divergence Theorem, we have

$$\iint_S \vec{r} . \hat{n}\, dS = \iiint_V div\ \vec{r}\ dV$$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $div\ \vec{r} = \nabla . \vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\therefore \iint_S \vec{r} . \hat{n}\, dS = \iiint_V 3\, dV = 3V$$

(iii) $\oint_S \nabla r^2\, \overrightarrow{dS} = \iiint_V div\left(\nabla r^2\right) dV = \iiint \nabla^2 (r^2)\, dV$

Now, $\nabla^2 . r^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2 + y^2 + z^2) = \frac{\partial^2}{\partial x^2}\left(x^2\right) + \frac{\partial^2}{\partial y^2}\left(y^2\right) + \frac{\partial^2}{\partial z^2}\left(z^2\right)$

$= 2 + 2 + 2 = 6$

$$\therefore \oint_S \nabla r^2\, \overrightarrow{dS} = \iiint_V 6\, dV = 6V .$$

Example – 3: Verify divergence theorem for $\vec{F} = \left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}$ ***taken over the rectangular parallelopiped*** $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

Solution : [For verification of divergence theorem we shall evaluate the volume and surface integrals separately and show that they are equal]

Here $\vec{F} = \left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}$

Now $div\ \vec{F} = \frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right) = 2\,(x + y + z)$

$$\therefore \quad \iiint_V div\ \vec{F} dV = \iiint_V 2\left(x + y + z\right) dV = \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} 2(x + y + z)\, dx\, dy\, dz$$

$$= 2\int_0^c\int_0^b \left\{\frac{x^2}{2} + (y + z)x\right\}_0^a dy\, dz = 2\int_0^c\int_0^b \left\{\frac{a^2}{2} + \left(y + z\right)a\right\} dy\, dz$$

$$= 2\int_0^c \left\{\frac{a^2}{2}y + a . \frac{y^2}{2} + ayz\right\}_0^b dz = 2\int_0^c \left\{\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right\} dz$$

$$= 2\left\{\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right\}_0^c = \{a^2bc + ab^2c + abc^2\} = abc(a+b+c) \;....\;(1)$$

To evaluate surface integral, the surface S consists of six surfaces given below.

	Equation	dS	Outward normal $(\hat{n})$
$S_1 \to OAC'B$	$z = 0$	$dx\,dy$	$-\hat{k}$
$S_2 \to CB'PA'$	$z = c$	$dx\,dy$	$\hat{k}$
$S_3 \to OBA'C$	$x = 0$	$dy\,dz$	$-\hat{i}$
$S_4 \to AC'PB'$	$x = a$	$dy\,dz$	$\hat{i}$
$S_5 \to OCB'A$	$y = 0$	$dx\,dz$	$-\hat{j}$
$S_6 \to BA'PC'$	$y = b$	$dx\,dz$	$\hat{j}$

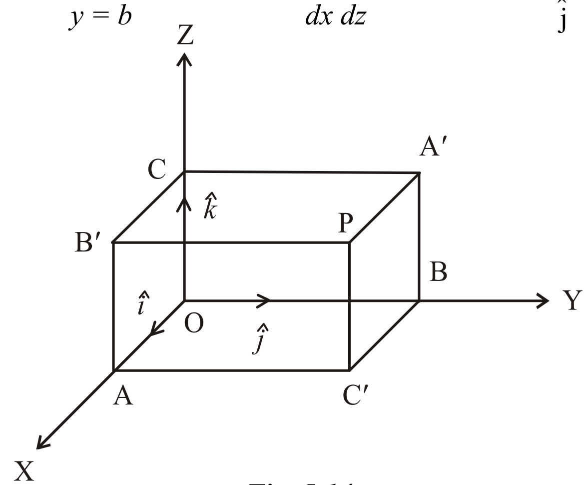

Fig. 5.14

$$\iint_S \vec{F}\,.\,\hat{n}\,dS = \iint_{S_1} \vec{F}\,.\,\hat{n}\,dS + \iint_{S_2} \vec{F}\,.\,\hat{n}\,dS + \iint_{S_3} \vec{F}\,.\,\hat{n}\,dS + \iint_{S_4} \vec{F}\,.\,\hat{n}\,dS + \iint_{S_5} \vec{F}\,.\,\hat{n}\,dS + \iint_{S_6} \vec{F}\,.\,\hat{n}\,dS \quad ...(2)$$

$$\iint_{S_1} \vec{F}\,.\,\hat{n}\,dS = \iint\left\{\left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}\right\}.\left(-\hat{k}\right)dx\,dy$$

$$= \iint\left(xy - z^2\right)dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b} xy\,dx\,dy = \int_{x=0}^{a}\left[\frac{xy^2}{2}\right]_0^b dx = \frac{b^2}{2}.\left[\frac{x^2}{2}\right]_0^a = \frac{a^2b^2}{4} \quad (\because\ z = 0)$$

$$\iint_{S_2} \vec{F}\,.\,\hat{n}\,dS = \iint\left\{\left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}\right\}.\hat{k}\,dx\,dy$$

$$= \iint\left(z^2 - xy\right)dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b}(c^2 - xy)\,dy\,dx \qquad (\because\ z = c)$$

$$= \int_{x=0}^{a}\left(c^2y - \frac{xy^2}{2}\right)\Bigg|_0^b dx = \int_0^a\left(c^2b - \frac{xb^2}{2}\right)dx = \left[c^2bx - \frac{x^2b^2}{4}\right]_0^a = abc^2 - \frac{a^2b^2}{4}$$

$$\iint_{S_3} \vec{F}\,.\,\hat{n}\,dS = \iint\left\{\left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}\right\}.\left(-\hat{i}\right)dy\,dz$$

$$= \iint\left(yz - x^2\right)dy\,dz = \iint yz\,dy\,dz = \int_{y=0}^{b}\int_{z=0}^{c} yz\,dz\,dy$$

$$= \int_0^b\left[\frac{yz^2}{2}\right]_0^c dy = \int_0^b \frac{c^2y}{2}dy = \left[\frac{c^2y^2}{4}\right]_0^b = \frac{b^2c^2}{4} \quad (\because\ x = 0)$$

$$\iint_{S_4} \vec{F}\,.\,\hat{n}\,dS = \iint\left\{\left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}\right\}.\hat{i}\,dy\,dz$$

$$= \iint\left(x^2 - yz\right)dy\,dz = \int_{z=0}^{c}\int_{y=0}^{b}\left(a^2 - yz\right)dy\,dz \quad (\because\ x = a)$$

$$= \int_0^c\left\{a^2y - \frac{zy^2}{2}\right\}_0^b dz = \int_0^c\left(a^2b - \frac{b^2z}{2}\right)dz = \left[a^2bz - \frac{b^2}{4}z^2\right]_0^c = a^2bc - \frac{b^2c^2}{4}$$

$$\iint_{S_5} \vec{F}\,.\,\hat{n}\,dS = \iint\left\{\left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}\right\}.\left(-\hat{j}\right)dx\,dz$$

$$= \iint\left(xz - y^2\right)dx\,dz = \int_{z=0}^{c}\int_{x=0}^{a}\left(xz\,dx\,dz\right)$$

$$= \int_0^c\left[\frac{zx^2}{2}\right]_0^a dz = \int_0^c \frac{a^2}{2}z\,dz = \left[\frac{a^2z^2}{4}\right]_0^c = \frac{a^2c^2}{4} \quad (\because y = 0)$$

$$\iint_{S_6} \vec{F}\,.\,\hat{n}\,dS = \iint\left\{\left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}\right\}.\hat{j}\,dx\,dz$$

$$= \iint\left(y^2 - zx\right)dx\,dz = \int_{z=0}^{c}\int_{x=0}^{a}\left(b^2 - zx\right)dx\,dz \qquad (\because\ y = b)$$

$$= \int_0^c\left[b^2x - \frac{zx^2}{2}\right]_0^a dz = \int_0^c\left(b^2a - \frac{za^2}{2}\right)\left[b^2az - \frac{a^2z^2}{4}\right]_0^c$$

$$= b^2ac - \frac{a^2c^2}{4} = ab^2c - \frac{a^2c^2}{4}$$

Substituting the value of S_1, S_2, S_3, S_4, S_5, S_6 in (2)

$$\iint_S \vec{F}\,.\,\hat{n}\, dS = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + ab^2c - \frac{a^2c^2}{4} - \frac{a^2c^2}{4}$$

$$= abc^2 + a^2bc + ab^2c = abc\,(a + b + c) = \iiint_V div\,\vec{F}\,dV \qquad (\because \text{ of I})$$

Hence $\iint_S \vec{F}\,.\,\hat{n}\,dS = \iiint_V div\,\vec{F}\,dV$

$\therefore$ Divergence theorem is verified.

Example – 4 : If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ prove that $\iint_S \left(x^4 + y^4 + z^4\right) dS = 4\pi a^6$.

Solution : First of all convert the integral $\iint_S \left(x^4 + y^4 + z^4\right) dS$ into the form $\iint_S \vec{F}.\hat{n}\,dS$

where $\hat{n}$ is a unit vector normal to the surface $\phi\,(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$

i.e., $\hat{n} = \dfrac{grad.\phi}{|grad\,\phi|}$

$$grad\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\text{grad}\,\phi| = 2\sqrt{x^2 + y^2 + z^2} = 2a, \;\therefore\; \hat{n} = \frac{2\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{2a} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, now we take $\vec{F}\,.\,\hat{n} = x^4 + y^4 + z^4$

$$\Rightarrow \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right).\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} = x^4 + y^4 + z^4$$

$$\frac{1}{a}(f_1x + f_2y + f_3z) = x^4 + y^4 + z^4$$

$$\therefore \quad \frac{f_1x}{a} = x^4,\; \frac{f_2y}{a} = y^4,\; \frac{f_3z}{a} = z^4$$

$$\therefore \quad f_1 = ax^3, f_2 = ay^3, f_3 = az^3$$

$$\vec{F} = a\left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right)$$

Now, $\iint_S \left(x^4 + y^4 + z^4\right) dS = \iint_S \vec{F}\,.\,\hat{n}\,dS = \iiint_V div\,\vec{F}\,dV = \iiint_V \nabla\,.\,a\left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right) dV$

$$= a\iiint 3\left(x^2 + y^2 + z^2\right) dV = 3a\iiint a^2 dV = 3a^3V = 3a^3\,.\,\frac{4}{3}\pi a^3 = 4\pi a^6.$$

Exercise – 5

1. Evaluate $\int_S \vec{F}.\vec{n}\,ds$ where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.
2. If $\vec{F} = \left(2x^2 - 3z\right)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ show that $\int \nabla.\vec{F}dV$ over the region bounded by the co-ordinate planes and the plane $2x + 2y + z = 4$ is $\frac{8}{3}$.
3. Verify Stoke's theorem for the vector field $\vec{F} = \left(2x - y\right)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane.
4. Let S be the region bounded by the ellipse C : $4x^2 + y^2 = 4$ in the plane z = 1 and let $\hat{n} = \hat{k}$ and $F = x^2\hat{i} + 2x\hat{j} + z^2\hat{k}$. Use Stoke's theorem to find the value of $\oint_C \vec{F}.dr$.
5. Verify Divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ taken over the cube bounded by $x = 0$, $x = 1$, y = 0, y = 1, z = 0 and z = 1.
6. Evaluate $\iint_S \vec{A}.\vec{dS}$, where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2 = a^2$.

❒ ☸ ❒