Example – 5 : ***For what value of λ, the equation***

$$x + y + z = 1$$
$$x + 2y + 4z = \lambda$$
$$x + 4y + 10z = \lambda^2$$

has a solution and solve them completely in each case.

Solution : Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$$(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix} \quad (R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{bmatrix} (R_3 \to R_3 - 3R_2) \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

$$(K) \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$ [No. of non zero rows]

The equation will be consistent if $\rho(A) = \rho(k) = 2$

It is only possible if $\lambda^2 - 3\lambda + 2 = 0$

or $(\lambda - 1)(\lambda - 2) = 0$ of $\lambda = 1, 2$

i.e. the equation are only consistent if $\lambda = 1$ or $\lambda = 2$

Since rank of K = 2 = 2 (number of unknowns the system has infinite number of solution.)

Case (i) for $\lambda = 1$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $x + y + z = 1$ ....(i)

$y + 3z = 0$ .....(ii)

From (ii) $y = -3z$

From (i) $x = 1 + 2z$

Let z = k

$\therefore$ $x = 1 + 2k,\ y = -3k,\ z = k$

Where k is an arbitary parameter.

Case (ii) for $\lambda = 2$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ $x + y + z = 1$ .... (iii)

$y + 3z = 1$ .... (iv)

From (iv) $y = 1 - 3z$

From (iii) $x = 2z$

$\therefore$ $x = 2k,\ y = 1 - 3k,\ z = k$

where k is an arbitary parameter.

1.13 : System of Linear Homogeneous Equations

Consider the homogeneous linear equations

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots + a_{2n}x_n = 0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots \qquad \dots\dots (1)$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = 0$$

Find the rank 'r' of the coefficient matrix A by reducing it to the triangular form by elementary row operations

(i) If r = n, the equations (1) have only a trivial zero solution

$x_1 = x_2 \dots\dots\dots\dots x_n = 0$

If r < n, the equations (1) have (n – r) linearly independent solutions.

The number of linearly independent of solutions is (n – r) means of the variables the values of the remaining variable can be uniquely found. Thus the equations (iii) will have an **infinite number of solutions**.

(2) When m < n (i.e. the number of equations is less than the number of variables), the solution is always other than $x_1 = x_2 \dots\dots x_n = 0$. The number of solution is infinite.

(3) When m = n (i.e. the number of equations the number of variables), the necessary and sufficient condition for solution other than $x_1 = x_2 \dots\dots = x_n = 0$, is that the determinant of the coefficient matrix is zero. In this case the equations are said to be consistent and such a solution is called **non-trivial solution**.

(For homogeneous system) since all b's are zero, $r = r_1$, then

(a) If r = n $\Rightarrow$ the equations have only **trivial solution**. i.e. $x_1 = x_2 = x_3 \dots\dots\dots x_n = 0$

(b) If r < n $\Rightarrow$ the equations have an infinite number of non-trivial solutions.

To obtain infinite solution, set (n – r) variables any arbitary value and solve for the remaining unknows. In a homogenous system if the number of unknowns, it **has a non-trivial solution**.

Illustrative Examples

Example – 1 : ***Solve the following system of equations :***

$$x - y + z = 0$$
$$x + 2y - z = 0$$
$$2x + y + 3z = 0$$

Solution : Writing the equations in the form AX = 0, we have

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix} \quad (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - 2R_1)$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix} \quad (R_3 \to R_3 - R_2)$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix} \quad \left(R_2 \to R_2 \times \frac{1}{3}, R_3 \to R_3 \times \frac{1}{3}\right)$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\rho(A) = 3$ = number of variables and hence the given equations have only trivial solutions x = y = z = 0

Example – 2 : ***Solve***

$$x - y + 2z - 3w = 0$$
$$3x + 2y - 4z + w = 0$$
$$4x - 2y + 9w = 0$$

Solution : Writing in matrix form, we have

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. AX = 0

where $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}$ $(R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 4R_1)$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 2 & -8 & 21 \end{bmatrix} \left(R_2 \to R_2 \times \frac{1}{5}\right) \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -8 & 21 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \end{bmatrix}$$

i.e., rank of A is 3.

Since rank = no. of equations, so we cannot omit any equation, but can write them in the following form;

$x - y + 2z - 3w = 0$

$y - 2z + 2w = 0$

$-4z + 17w = 0$ (let $w = k$)

i.e., $4z = 17w$

$z = \frac{17w}{4}$, $z = \frac{17k}{4}$, $y = \frac{13k}{2}$, $x = k$ be the solutions, where K being any parameter.

Example – 3 : ***Determine the values of λ for which the following set of equation may possess non-trival solution.***

$$\boldsymbol{3x_1 + x_2 - \lambda x_3 = 0}$$
$$\boldsymbol{4x_1 - 2x_2 - 3x_3 = 0}$$
$$\boldsymbol{2\lambda x_1 + 4x_2 + \lambda x_3 = 0}$$

Solution : For each permissible values of λ, determine the general solution.

For Non-trival solution of homogeneous system, rank of coeffecient matrix <no. of unknowns.

i.e., Rank of A < 3.

i.e., $\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + 8\lambda - 9 = 0 \Rightarrow (\lambda + 9)(\lambda - 1) = 0$

$\Rightarrow \lambda = 1$ and -9.

For $\lambda = 1$. The given system reduces to

$$3x_1 + x_2 - x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2x_1 + 4x_2 + x_3 = 0$$

Now, Rank of A = 2 < 3 $\left(\because \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = -10 \neq 0\right)$

$\therefore$ System has infinite number of solution.

Let x_1 = k and from the first two equations we get

$x_2 - x_3 = -3k$ and $-2x_2 - 3x_3 = -4k$

On solving $x_2 = -k$ and $x_3 = 2k$ where k is any real number.

For $\lambda = -9$. The given system reduces to

$$3x_1 + x_2 + 9x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$-18x_1 + 4x_2 - 9x_3 = 0$$

Now Rank of A = 2 < 3

$\therefore$ System has infinite number of solution.

Let x_1 = 3k and from the first two equations we get

$x_2 + 9x_3 = -9k$ and $-2x_2 - 3x_3 = -12k$

On solving $x_2 = 9k$ and $x_3 = -2k$.

Example – 4 : ***Find the values of λ for which the equations***

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$
$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$
$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

are consistent, and find the ratios of x : y : z when λ has the smallest of these values. What happens when λ has the greater of these values.

Solution : The given equations will be consistent, if

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0 \quad [R_2 \to R_2 - R_1]$$

or if, $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0 \quad [C_3 \to C_3 + C_2)$$

or if, $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix} = 0$$

or if, $(\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda-1) \end{vmatrix} = 0$ or if, $2(\lambda - 3)\,[(\lambda - 1)(3\lambda - 1) - (5\lambda + 1)] = 0$

or if, $6\lambda(\lambda - 3)^2 = 0$ or if, $\lambda = 0$ or 3.

(a) When $\lambda = 0$, the equations become $-x + y = 0$ ....(i)

$$-x - 2y + 3z = 0 \text{ ......(ii)}$$

$$2x + y - 3z = 0 \text{ .....(iii)}$$

Solving (ii) and (iii), we get $\dfrac{x}{6-3} = \dfrac{y}{6-3} = \dfrac{z}{-1+4}$. Hence x = y = z.

(b) When $\lambda = 3$, equations becomes identical.

Example – 5 : ***Show that there are 3 real values of λ for which the equations $(a - \lambda)x + by + cz = 0$, $bx + (c - \lambda)y + az = 0$, $cx + ay + (b - \lambda)z = 0$ are simultaneous true and that the product of these values of λ is***

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution : The given three equations are

$$(a - \lambda)x + by + cz = 0 \text{ ....(1)}$$

$$bx + (c - \lambda)y + az = 0 \text{ ....(2)}$$

$$cx + ay + (b - \lambda)z = 0 \text{ ....(3)}$$

As per the given condition the 'λ' has 3 real values. So the system of equation has non-trival solution. So its rank must be < 3. So all the third order determinant must be zero.

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a - \lambda)\,[(c - \lambda)\,(b - \lambda) - a^2] - b\,[b\,(b - \lambda) - ca] + c\,[ab - c\,(c - \lambda)] = 0$$

$$\Rightarrow (a - \lambda)\,(bc - c\lambda - b\lambda + \lambda^2 - a^2) - b\,(b^2 - b\lambda - ca) + c\,(ab - c^2 + c\lambda) = 0$$

$$\Rightarrow abc - ac\lambda - ab\lambda + a\lambda^2 - a^3 - \lambda bc + c\lambda^2 + b\lambda^2 - \lambda^3 + a^2\lambda - b^3 + b^2\lambda$$
$$+ abc + abc - c^3 + c^2\lambda = 0$$

$$\Rightarrow -a^3 - b^3 - c^3 - \lambda^3 + \lambda^2(a + b + c) + \lambda(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc = 0$$

$$\Rightarrow \begin{vmatrix} a+b+c-\lambda & b & c \\ a+b+c-\lambda & c-\lambda & a \\ a+b+c-\lambda & a & b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow a+b+c-\lambda \begin{vmatrix} 1 & b & c \\ 1 & c-\lambda & a \\ 1 & a & b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow a+b+c-\lambda \begin{vmatrix} 1 & b & c \\ 0 & c-b-\lambda & a-c \\ 0 & a-c+\lambda & b-a-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a + b + c - \lambda) \{(c - b - \lambda)(b - a - \lambda) - (a - c)(a - c + \lambda)\} = 0$

$\Rightarrow (a + b + c - \lambda)(bc - ca - c\lambda - b^2 + ab + b\lambda - b\lambda + a\lambda + \lambda^2 - a^2 + ac$
$- a\lambda + ac - c^2 + c\lambda) = 0$

$\Rightarrow (a + b + c - \lambda)(- a^2 - b^2 - c^2 + ab + bc + ca + \lambda^2) = 0$

$\Rightarrow a + b + c = \lambda$ or $a^2 + b^2 + c^2 - ab - bc - ca = \lambda^2$

So the value of λ is $\lambda = a + b + c$

and $\lambda^2 = a^2 + b^2 + c^2 - ab - bc - ca$

The product of these two values is

$\lambda^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc$ ....(4)

As per the question the product of the real value is,

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= a(bc - a^2) - b(b^2 - ca) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= - a^3 - b^3 - c^3 + 3abc$

$= a^3 + b^3 + c^3 - 3abc$ ....(5)

$\therefore$ Equation (4) and equation (5) are same (Proved).

1.14 : Solution of a system of non-homogeneous equation

Direct methods : These methods are based on the elimination of variables to transform the set of equations to a triangular form.

Gauss elimination method :

We have seen that any system of linear algebraic equations can be solved by the use of determinants. If the order of the determinat is large, then the evaluation, becomes tedious. To avoid these unnecessary computations, various mathematicians have tried to develop simpler and less time consuming procedures, and various methods for solving system of linear equations.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_nx_n = b_n \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

One of the most important methods is so called **Gaussian ellimination method.**

Gauss elimination method is a direct method which consists of transforming the given system of simultaneous linear equations to an equivalent upper triangular system. From this transformed system the required solution can be obtained by the method of back substitution.

Consider the system of '*n*' equation and '*n*' unknowns given by AX = b, where A is the coefficient matrix '*b*' is the right hand side matrix and X is the unknown variables. With the help of A and B we have constructed a new type of matrix i.e., Augmented matrix. The augmented matrix is

$$K = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_n & b_n \end{array}\right]$$

To transform the system to an equivalent upper triangular system, we use the following elementary row operations.

First step : Elimination of x_1, from the second, third .....n[th] equation.

We assume that the order of the elimination and order of unknwon in each equations are such that $a_{11} \neq 0$. The variable x_1 can be eliminated from the second equation by subtracting (a_{21}/a_{11}) times the first equation from second equation (a_{31}/a_{11}) times the first equation from third equation.

Or, i.e., $R_i \longrightarrow R_i - \dfrac{a_{i1}}{a_{11}} R_1$; $i = 2, 3, \ldots\ldots\, n$ makes all the entries a_{21}, a_{31} .... a_{n1} in the first column zero.

Here the first equation is the pivotal equation. a_{11} ($\neq 0$) is called pivot and $\dfrac{-a_{i1}}{a_{11}}$ for $i = 2, 3 \ldots$ are called multipliers for first elimination. If $a_{11} = 0$, we interchange the first row with another suitable row so as to have $a_{11} \neq 0$.

This gives new system as follows :

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b'_2 \\ a'_{22}\, x_2 + \ldots + a'_{2n}\, x_n = b'_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a'_{n2}\, x_2 + \ldots + a'_{nn}\, x_n = b'_n \end{array}\right\} \qquad \ldots(2)$$

Second Step (Elimination x_2 from the third, ...n[th] equations in (2)) :

If the coefficient a'_{22} ....a'_{nn} in (2) are not all zero, we may assume that the order of equation and the unknowns is such that $a'_{22} \neq 0$. Then we eliminate x_2 from the third, fourth ......n^{th} equations of (2) by subtracting.

i.e., $R_i \longrightarrow R_i - \dfrac{a_{2i}}{a_{22}} R_2$; $i = 3,4,\ldots\ldots.n$.

This makes all entries a_{32}, a_{42} ....a_{n2} on the second column zero.

Step – 3 : By successive elimination, we arrive at a single equation in the unknown x_n which can be solved and substituting this in the preceeding equation, we obtain the value of x_{n-1}.

In general $R_i \longrightarrow R_{i-1} - \dfrac{a_{ik}}{a_{kk}} R_k$; $i = k + 1, k + 2 \ldots.n$.

will make all entries $a_{k+1,n}$ .... $a_{n,k}$ in the k^{th} column zero.

Also when elimination is complete, the system takes the Triangular form

$$\left.\begin{aligned} C_{11}x_1 + C_{12}x_2 + \dots + C_{1n}x_n &= d_1 \\ C_{22}x_2 + \dots + C_{2n}x_n &= d_2 \\ \dots \quad \dots \quad \dots \quad \dots & \\ C_{nn}x_n &= d_n \end{aligned}\right\} \quad \dots\dots\dots\dots (3)$$

Hence the given system of equation is reduced to the form UX = b, where 'U' is an upper triangular matrix. The required solution can be obtained by the method of back substitution.

The Gauss- Elimination method can be generalized to find the solution of 'n' simultaneous equation in n-unknows.

Illustrative Examples

Example – 1 : ***Solve the equations x + y = 2 and 2x + 3y = 5 by Gauss elimination method.***

Solution : The given system of equations can be written as $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

The augmented matrix is $K = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right]$

we note $a_{11} = 1 \neq 0$ is the pivot. The first equation is the pivot equations is multiplier for the second.

$$K \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad (R_1 \to R \text{ and } R_2 \to R_2 - 2R_1)$$

which is an upper triangular system.

$$\therefore\ x + y = 2$$
$$y = 1$$

Using backward substitution we get $y = 1$ and $x = 1$.

Example – 2 : ***Solve the following system of equations using Gaussian elimination method***

x + y + z = 9

2x – 3y + 4z = 13

3x + 4y + 5z = 40.

Solution : The given set of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

The augmented matrix is

$$K = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right] \quad (R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right] \qquad \left(R_3 \to R_3 + \frac{1}{5}R_2\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \frac{12}{5} & 12 \end{array}\right]$$

which is an upper triangular system. The given system of equations reduces to the system.

$$x + y + z = 9$$
$$-5y + 2z = -5$$
$$\frac{12}{5}z = 12$$

Now by back substitution method we obtain the solution $x = 1$, $y = 3$, $z = 5$.

Example – 3 : ***Solve the equations*** $-x_1 + x_2 + 2x_3 = 2$, $3x_1 - x_2 + x_3 = 6$ ***and*** $-x_1 + 3x_2 + 4x_3 = 4$ ***by Gauss elimination method.***

Solution : The given equation is in Augmented form is

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right] (R_2 \to R_2 + 3R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right] (R_3 \to R_3 - R_2)$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

which is an upper triangular system, which can be solved by Back substitution method

$-x_1 - x_2 + 2x_3 = 2$ ......(i)

$2x_2 + 7x_3 = 12$ ....(ii)

$-5x_3 = -10$ ....(iii)

From (iii) $\therefore x_3 = 2$

From (ii) $2x_2 + 14 = 12 \Rightarrow 2x_2 = -2 \Rightarrow x_2 = -1$.

From (i) $-x_1 - 1 + 4 = 2 \Rightarrow -x_1 = 2 - 3 \Rightarrow x_1 = 1$

We get $x_1 = 2$, $x_2 = -1$ and $x_3 = 2$.

Example – 4 : ***Solve by Gauss elimination method.***

$$9x + 4y + 3z = -1$$
$$5x + y + 2z = 1$$
$$7x + 3y + 4z = 1$$

Solution : The given equation can be augmented form

$$\left[\begin{array}{ccc|c} 9 & 4 & 3 & -1 \\ 5 & 1 & 2 & 1 \\ 7 & 3 & 4 & 1 \end{array}\right]$$

Step – I : Elimination of x from the 2nd, 3rd using 1st by subtracting $\frac{5}{9}$ times of the first equation from the 2nd equation and $\frac{7}{9}$ times of the first equation from the 3rd.

$$\left[\begin{array}{ccc|c} 9 & 4 & 3 & -1 \\ 0 & -11/9 & 1/3 & 14/9 \\ 0 & -1/9 & 5/3 & 16/9 \end{array}\right]$$

Step – II : Eliminate y from the 3rd equation using 2nd by subtracting $\dfrac{\frac{-1}{9}}{\frac{11}{-9}} = \dfrac{1}{11}$ times of the 2nd from 3rd equation.

$$\begin{bmatrix} 9 & 4 & 3 & -1 \\ 0 & -11/9 & 1/3 & 14/9 \\ 0 & 0 & 18/11 & 18/11 \end{bmatrix}$$

$$9x + 4y + 3z = -1$$

$$\frac{-11}{9}y + \frac{1}{3}z = \frac{14}{9}$$

$$\frac{18}{11}z = \frac{18}{11}$$

Beginning from the last, we have $z = 1$, $y = -1$, $x = 0$.

Example – 5 : ***$x - 3y + 2z = 2$, $5x - 15y + 7z = 10$***

Solution : The given equation is in augmented form

$$\begin{bmatrix} 1 & -3 & 2 & 2 \\ 5 & -15 & 7 & 10 \end{bmatrix}$$

Step – 1 : Elimination of x from the 2nd equation

$$\begin{bmatrix} 1 & -3 & 2 & 2 \\ 0 & 0 & -3 & 0 \end{bmatrix} R_2 = R_2 - 5R_1$$

$x - 3y + 2z = 2$ and $-3z = 0$

So starting from the last we have, $z = 0$. Putting the value of z we have

$x - 3y = 2 \Rightarrow x = 2 + 3y$

So, for different values of y, we get different values of x, so the sytem has infinite many solutions.

Note – 1 : $\begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 6 & 3 & 9 \\ 0 & 0 & 0 & 5 \end{bmatrix} \rightarrow$ The system has no solution.

Note – 2 : $\begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 6 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$ The system has infinite solution.

1.15 : Vector Sapce

Definition : A non-empty set V is called a vector space or a linear space over a field F if in V there are define two algebraic operations (Called vector addtion and multiplication by scalars) are as follows :

(1) $u + \upsilon \in$ V, $\forall\ u, \upsilon \in$ V (Closure law)

(2) $u + \upsilon = \upsilon +$ u, $\forall\ u, \upsilon \in$ V (Commutative law)

(3) $(u + \upsilon) + \omega =$ u + (v + ω), $\forall\ u, \upsilon, \omega \in$ V (Associative law)

(4) There exists an element denoted by 0 ∈ V such that

$\mathbf{0} + u = u + \mathbf{0} = u \ \forall\ u \in v$

0 is the additive identity element in V.

(5) For each u ∈ V, there exists an element $-u \in$ V such that

$u + (-u) = 0.$

$-u$ is the additive inverse of u.

(6) $\alpha\, u \in$ V, $\forall\ \alpha \in$ F and $\forall\ u \in$ V.

(7) $\alpha\,(u + \upsilon) = \alpha u + \alpha\upsilon$, $\forall\ \alpha \in$ F and $\forall\ u, \upsilon \in$ V.

(8) $(\alpha + \beta)u = \alpha u + \beta u$, $\forall\ \alpha, \beta \in$ F and $\forall\ u \in$ V.

(9) $(\alpha\beta)u = \alpha\,(\beta u)$, $\forall\ \alpha, \beta \in$ F and $\forall\ u \in$ V.

(10) $1.u = u$, $\forall\ u \in$ V.

An element of v is called a vector. An element of F is called a scalar. 0 is called the additive identity element or the zero elements of V. It is also called a null or zero vector.

The zero element of F is denoted by 0. The unit element of F is 1.

From (1) it follows that V is closed under addition.

Note :

1. V under vector addition forms an abelian group. Since + is commutative, $\alpha + 0 = 0 + \alpha = \alpha$ and $\alpha + (-\alpha) = 0$.
2. Scalar multiplication is not a binary operation, but it is a function from F × V into V.

1.16 : Linear Combination of Vectors

Definition : Let V be a vector space over a field F. A finite sum of the form $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$, where $\alpha_1, \alpha_2, ..., \alpha_n \in F$ and $u_1, u_2, ...u_n \in$ V is called a linear combination of $u_1, u_2, ...u_n$.

Example – 1: ***Write the vector*** $v = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ ***in the vector space of 2 × 2 matrices as a linear combination of*** $v_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Solution : Let $V = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$; $\alpha_1, \alpha_2, \alpha_3 \in R$. ....(1)

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ -\alpha_2 & -\alpha_1 \end{bmatrix}$$

By definition of equality of two matrices, we have

$$\alpha_1 + \alpha_2 + \alpha_3 = 3,\ \alpha_1 + \alpha_2 - \alpha_3 = -1$$
$$-\alpha_2 = 1,\ \alpha_1 = 2$$

Solving these, we get $\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = 2$, Putting these values of $\alpha_1, \alpha_2, \alpha_3$ in eq^n(1).

$$V = 2v_1 - v_2 + 2v_3.$$

Example – 2: ***For what value of k (if any) the vector $\alpha = (1, -2, k)$ can be expressed as a linear combination of vectors $\alpha_1 = (3, 0, -2)$ and $\alpha_2 = (2, -1, -5)$ in R^3.***

Solution : Because vector $\alpha = (1, -2, k)$ is a linear combination of $\alpha_1 = (3, 0, -2)$ and $\alpha_2 = (2, -1, -5)$; therefore there exist scalars a and b such that

$$\alpha = a\alpha_1 + b\alpha_2$$

or $(1, -2, k) = a(3, 0, -2) + b(2, -1, -5) = (3a + 2b, -b, -2a - 5b)$

Equating corresponding entires, we have

$3a + 2b = 1$ ....(1), $-b = -2$ or $b = 2$ ....(2)

$-2a - 5b = k$ ....(3)

Putting $b = 2$ from (2) in (1),

$3a + 4 = 1$ or $3a = -3$ or $a = -1$

Putting $a = -1$, $b = 2$ in equation (3), we have $2 - 10 = k$ or $k = -8$.

Definition : Let V be a vector space over a field F and S is a subset of V. Then span of S denoted by $[S]$ is the set of linear combination of vectors of S.

$[S] = \{\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_k u_k \mid \alpha_1, \alpha_2, ..., \alpha_k \in F$ and $u_1, u_2, ... u_k \in S$, and $k \in Z\}$

Example – 3 : Let S = {(1, 2, 0), (2, 3, 1)}

Then $[S] = \{\alpha(1, 2, 0) + \beta(2, 3, 1) \mid \alpha, \beta \in R\}$

$= \{(\alpha + 2\beta, 2\alpha + 3\beta, \beta) \mid \alpha, \beta \in R\}$

1.17 : Linear independenece and dependence vectors

Any quantity with n component is called a vector. Thus the columns or rows of a matrix are vectors. Any ordered n-tuple of numbers is called n-vectors. We mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2 ...... x_n$ by any n numbers then the ordered n-touple

$X=(x_1,x_2................x_n)$ is called an n-vector. Similarly if $y_1,y_2......y_n$ be any n numbers then the orderd n touple $Y = (y_1,y_2, ...y_n)$ is called an n-vector. The n numbers $x_1, x_2...........x_n$ and $y_1,y_2...........y_n$ are called components of the n-vector, $X=(x_1,x_2................x_n)$ and $Y = (y_1,y_2, ...y_n)$. A vector may be written either as a row vector or as a column vector. If A be a matrix of order $m\times n$, then each row of A will be an m-vector and each column of A will be n-vector.

Definition :

Let V be a vector space over a field F and $\upsilon_1, \upsilon_2, ..., \upsilon_n \in V$. Then the set $\{\upsilon_1, \upsilon_2, ..., \upsilon_n\}$ is said to be linearly independent if $\alpha_1\upsilon_1 + \alpha_2\upsilon_2 + ...+ \alpha_n\upsilon_n = \mathbf{0}$,

where $\alpha_1, \alpha_2, ..., \alpha_n \in F \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = \mathbf{0}$

(The set $\{\upsilon_1, \upsilon_2, ..., \upsilon_n\}$ is linearly independent means the vectors $\upsilon_1, \upsilon_2, ..., \upsilon_n$ are linearly independent.)

Note : An infinite set S of vectors in V are said to be the linearly independent of every subset of S is linearly indepedent.

Definition :

Let V be a vector space over a field F and $\upsilon_1, \upsilon_2, ..., \upsilon_n \in V$. The set $\{\upsilon_1, \upsilon_2, ..., \upsilon_n\}$ is said to be linearly dependent if it is not linearly independent, i.e., if there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n$, **not all zero such** that $\alpha_1\upsilon_1 + \alpha_2\upsilon_2 + ... + \alpha_n\upsilon_n = 0$. An empty set is linearly independent.

Definition :

Let V be a vector space over F. Vectors $v_1, v_2,........, v_n \in V$, are said to be linearly dependent (L.D.) over F if there exist scalars $\alpha_1, \alpha_2, ..., \alpha_n$ in F, not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0.$$

Here, 0 on the right hand side indicates the null vector.

Vectors which are not linearly dpendent are called linearly independent (L.I.).

In fact, vectors $v_1, v_2,....., v_n$ are linearly independent if and only if $\alpha_1 v_1 + \alpha_2 v_2 + ...+ \alpha_n v_n = 0, \alpha_i \in F$

implies $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$.

i.e., zero solution is the only solution.

If $S = \{v_1, v_2, ..., v_n\}$, then we say that the set S is L.I. or L.D. according as the vectors $v_1, v_2, ..., v_n$ are L.I. or L.D.

An infinite subset S of V is said to be L.I if every finite subset of S is L.I.

Theorem – 1 : Row-equivalent matrices have the same rank.

Theorem – 2 : ‘p’ vectors with ‘n’ components each are linearly independent if the matrix with these vectors as row vectors has rank p, but they are linearly dependent if that rank is less than p.

Theorem – 3 : Rank in terms of column vectors

The rank ‘r’ of a matrix ‘A’ equals the maximum number of linearly independent column vectors of A.

Hence A and its transpose A^T have the same rank.

Theorem – 4 : ‘p’ vectors with $n < p$ components are always linearly dependent.

Illustrative Examples

Example – 1: ***Examine the linear independence of the following sets of vectors :***

(i) $\{(1, 0), (0, 1)\}$ in R^2

(ii) $\{(1, 1, 1), (1, 2, 3), (0, 1, 2)\}$ in R^3

(iii) $\{(1, 1, 1), (1, 2, 3), (3, 3, 4)\}$ in R^3

(iv) $\{(1, 2, 3), (1, 0, 0), (0, 2, 3)\}$ in R^3.

Solution :

(i) Suppose $\alpha (1, 0) + \beta (0, 1) = 0$, for some scalars α, β.

$\Rightarrow (\alpha, 0) + (0, \beta) = 0 = (0, 0)$

$\Rightarrow \alpha = 0, \beta = 0$ (by definition of equality of two vectors is the only solution)

$\therefore$ (1, 0) and (0, 1) are linearly independent,

(ii) Suppose $\alpha (1, 1, 1) + \beta (1, 2, 3) + \gamma (0, 1, 2) = 0$, for some scalars α, β, γ

$\Rightarrow (\alpha, \alpha, \alpha) + (\beta, 2\beta, 3\beta) + (0, \gamma, 2\gamma) = 0 = (0, 0, 0)$

$\Rightarrow \quad \alpha + \beta = 0 \quad$ ...(i)

$\alpha + 2\beta + \gamma = 0 \quad$ ...(ii)

and $\quad \alpha + 3\beta + 2\gamma = 0 \quad$ ...(iii)

Let us solve (i), (ii) and (iii) simultaneously,

From (i), $\quad \beta = -\alpha$

From (ii), $\quad \gamma = \alpha$

From (iii), $\alpha - 3\alpha + 2\alpha = 0$, which is satisfied for each α

$\therefore \beta = -\alpha, \gamma = a$ is a solution for every value of α

In particular, for $\alpha = 1, \beta = -1, \gamma = 1$ is a solution. Hence the given vectors are linearly dependent.

$(1, 1, 1) - (1, 2, 3) + (0, 1, 2) = 0$ is in fact a linear relation between them.

(iii) Suppose $\alpha (1, 1, 1) + \beta (1, 2, 3) + \gamma (3, 3, 4) = 0$ for some scalars α, β, γ.

$\Rightarrow \quad \alpha + \beta + 3\gamma = 0$

$\alpha + 2\beta + 3\gamma = 0$

and $\quad \alpha + 3\beta + 4\gamma = 0$

Matrix form of these equations is

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A necessary condition for this system of homogeneous equations $AX = 0$ in three unknowns to have a non-zero solution is that A is singular

$$\text{i.e., } |A| = 0 \quad \text{or} \quad \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 0$$

But, $\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1\,(8-9) - 1\,(4-3) + 3\,(3-2) = 1 \neq 0$

$\therefore$ These give the only solution $\alpha = 0, \beta = 0, \gamma = 0$.

Hence, the vectors (1, 1, 1), (1, 2, 3) and (3, 3, 4) are linearly independent.

Alternately, one may solve the above three equations simultaneously and see that $\alpha = 0$ $\beta = 0, \gamma = 0$ is the only solution.

(iv) Since $1\,(1, 2, 3) + (-1)\,(1, 0, 0) + (-1)\,(0, 2, 3) = 0$ and the coefficients $1, -1, -1$ are non-zero.

$\therefore$ the vectors are linearly dependent.

Example–2: ***Show that the set of vectors {(1, 2, 0), (0, 3, 1), (–1, 0, 1)} in V_3 (Q) is L.I. (where Q is the field of rationals).***

Solution : Suppose $\alpha\,(1, 2, 0) + \beta\,(0, 3, 1) + \gamma\,(-1, 0, 1) = 0 = (0, 0, 0)$ where $\alpha, \beta, \gamma \in Q$.

$\Rightarrow (\alpha - \gamma, 2\alpha + 3\beta, \beta + \gamma) = (0, 0, 0)$

$\Rightarrow \alpha - \gamma = 0, 2\alpha + 3\beta = 0, \beta + \gamma = 0$

From first and last equations $\alpha = \gamma, \beta = -\gamma$

But $2\alpha + 3\beta = 2\gamma - 3\gamma \neq 0$ unless $\gamma = 0$

$\therefore \quad \alpha = 0, \beta = 0, \gamma = 0$ is the only solution.

Hence, the given vectors are L.I.

Or

Matrix form of the above linear homogeneous equation is

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX = 0$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1\,(3-0) - 0 - (2-0) = 1 \neq 0$$

$\therefore$ Matrix A is non-singular.

Hence $\alpha = 0, \beta = 0, \gamma = 0$ is the only solution.

Hence the given vectors form a L.I. set.

Example – 3 : ***If V is the vector space of all 2 × 3 matrices over R, show that the matrices***

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} \text{ form } a \text{ L.I. set.}$$

Solution : Suppose $\alpha A + \beta B + \gamma C = 0$ where $\alpha, \beta, \gamma \in R$

$$\Rightarrow \alpha \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + \gamma \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\alpha+\beta+4\gamma & \alpha+\beta-\gamma & -\alpha-3\beta+2\gamma \\ 3\alpha-2\beta+\gamma & -2\alpha-2\gamma & 4\alpha+5\beta+3\gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow\ 2\alpha + \beta + 4\gamma = 0,\ \alpha + \beta - \gamma = 0,\ -\alpha - 3\beta + 2\gamma = 0$ ...(1)

and $\quad 3\alpha - 2\beta + \gamma = 0,\ -2\alpha - 2\gamma = 0,\ 4\alpha + 5\beta + 3\gamma = 0$ ...(2)

Solving equations in (1), we get $\alpha = \beta = \gamma = 0$

which obviously satisfy the 3 equations in (2) also.

Hence, the given matrices form a L.I. set.

Example – 4 : ***Show that the vectors u = (1 + i, 2i), v = (1, 1 + i) in $V_2(C)$ are L.D. but in $V_2(R)$ are L.I.***

Solution : Two vectors are dependent if one is a multiple of the other.

Thus u and v are dependent if for some number $\alpha + i\beta \in C$,

$$u = (\alpha + i\beta)v$$

i.e., if $\quad (1 + i, 2i) = (\alpha + i\beta)(1, 1 + i) = (\alpha + i\beta, \alpha - \beta + i(\alpha + \beta))$

if $\quad \alpha + i\beta = 1 + i$ and $\alpha - \beta + i(\alpha + \beta) = 2i$

i.e., if $\quad \alpha = 1, \beta = 1$

Thus, $u = (1 + i)v$. Hence, u, v are L.D.

If $u, v \in v_2(R)$ then u is not a multiple of v because $1 + i \notin R$. Thus, u, v are L.I.

Example – 5 : ***Examine S = {(1, 2, 1) (–1, 1, 0), (5, –1, 2)} is L.I. or L.D.***

Solution : S = {(1, 2, 1), (–1, 1, 0), (5, –1, 2)}

$\alpha_1 (1, 2, 1) + \alpha_2(-1, 1, 0) + \alpha_3(5, -1, 2) = (0, 0, 0)$

$\Rightarrow (\alpha_1 - \alpha_2 + 5\alpha_3, 2\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + 2\alpha_3) = (0, 0, 0)$

$\Rightarrow \alpha_1 - \alpha_2 + 5\alpha_3 = 0,\ 2\alpha_1 + \alpha_2 - \alpha_3 = 0$ and $\alpha_1 + 2\alpha_3 = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

So S is linearly independent

$\alpha_1 = -2\alpha_3$ putting in (2) $-2\alpha_3 - \alpha_2 + 5\alpha_3 = 0$

$3\alpha_3 - \alpha_2 = 0 \Rightarrow \alpha_2 = 3\alpha_3$

$2\alpha_1 + \alpha_2 - \alpha_3 = 0$

$\Rightarrow -4\alpha_3 + 3\alpha_3 - \alpha_3 = 0 \Rightarrow -2\alpha_3 = 0 \Rightarrow \alpha_3 = 0$

$$\text{or,} \quad \begin{vmatrix} 1 & -1 & 5 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= -4 + 6 = 2 \neq 0.$

So the system of solution has a unique solution and that is

$\alpha_1 = \alpha_2 = \alpha_3 = 0.$

Example – 6: ***Show that the following vectors are L.I. or L.D.***

(a) [3, 4] [4, 3] ***(b) P_1 = [2, 2, 8] P_2 = [0, 1, 4]***

(c) [1, 4, 5] [4, 4, 8] [3, – 3, 0]

Solution : (a) Let α & β be two scalars such that $\alpha[3\ 4] + \beta[4\ 3] = [0\ 0]$

$\Rightarrow [3\alpha, 4\alpha] + [4\beta, 3\beta] = [0\ 0]$

$\Rightarrow [\ 3\alpha + 4\beta,\ 4\alpha + 3\beta] = [0\ 0]$

i.e. $3\alpha + 4\beta = 0\ \&\ 4\alpha + 3\beta = 0$

Solving these two we get $\alpha = 0,\ \beta = 0$

$\therefore$ The given vectors are L.I.

(b) Let $P_1 = [2, 2, 8] \qquad P_2 = [0, 1, 4]$

Let α, β be two scalars such that $\alpha P_1 + \beta P_2 = 0$

$\Rightarrow \alpha\,[2, 2, 8] + \beta\,[0, 1, 4] = 0$

$\Rightarrow [2\alpha, 2\alpha, 8\alpha] + [0\beta, 1\beta, 4\beta] = 0$

$\Rightarrow [2\alpha, 2\alpha + \beta, 8\alpha + 4\beta] = [0, 0, 0]$

$\Rightarrow 2\alpha = 0,\ 2\alpha + \beta = 0,\ 8\alpha + 4\beta = 0$

$\Rightarrow \alpha = 0\ \&\ \beta = 0$

$\because$ $P_1\ \&\ P_2$ are L.I.

OR, Let $A = \begin{bmatrix} 2 & 2 & 8 \\ 0 & 1 & 4 \end{bmatrix}$

Consider the submatrix of A is $\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad A_1 = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$

$|A_1| = 2 - 0 = 2 \neq 0$ $\because$ Rank of A = 2 = No. of rows of A.

$\because$ [2, 2, 8] & [0, 1, 4] are L.I.

(c) [1 4 5], [4 4 8], [3 – 3 0]

let $A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 4 & 8 \\ 3 & -3 & 0 \end{bmatrix}$

$|A| = 0$, consider the submatrix of A is $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

let, $A_1 = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$, $|A_1| = 4 - 16 \neq 0$

$\because$ Rank of A = 2

But No. of rows in A = 3

Clearly rank A < No of rows is A

$\therefore$ The set of vectors are L.D.

Example – 7 : ***Verify [1, 4, 5] [4, 4, 8] [3, –3, 0] are L.D.***

Solution : Are the given vectors linearly dependent or independent

$V_1 = [1, 4, 5],\ V_2 = [4, 4, 8],\ V_3 = [3, -3, 0]$

Arranging each 3 dimensional row vector along a row of a matrix we obtain a matrix A of order 3×3.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 4 & 8 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -12 & -12 \\ 0 & -15 & -15 \end{bmatrix} \quad \begin{pmatrix} \because R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim\begin{bmatrix} 1 & 4 & 5 \\ 0 & -12 & -12 \\ 0 & 0 & 0 \end{bmatrix} \qquad [R_3 \to R_3 - \frac{5}{4}R_2]$$

Which is in echelon form. The equivalent matrix has two nonzero rows. Hence $\rho(A) = 2$

Further since $\rho(A) <$ number of vectors.

So the set of given vectors is linearly dependent (L.D.).

Example – 8 : ***Are the following sets of vectors linearly independent or dependent ?***

(a) [1, 0, 0], [1, 1, 0], [1, 1, 1]

(b) [–1, 5, 0], [16, 8, –3], [–64, 56, 9]

Solution : (a) Let $V_1 = [1\ 0\ 0], V_2 = [1\ 1\ 0], V_3 = [1\ 1\ 1]$

Since V_1, V_2 and V_3 are 3 dimensional row vectors.

So by placing each row vector as a row we get a matrix 'A' of order (3 × 3).

$$\text{i.e, } A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} \because R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad (\because R_3 \to R_3 - R_2)$$

Which is in echelon form having three nonzero rows.

So $\rho(A) = 3 =$ Number of vectors.

Hence the given set of vectors are Linearly independent (L.D.).

(b) Let $V_1 = [-1\ 5\ 0], V_2 = [16\ \ 8\ -3], V_3 = [-64\ \ 56\ \ 9]$

Since V_1, V_2 and V_3 are 3 dimensional row vectors.

So by placing each row vector as a row we get a matrix 'A' of order (3×3).

$$\text{i.e, } A = \begin{bmatrix} -1 & 5 & 0 \\ 16 & 8 & -3 \\ -64 & 56 & 9 \end{bmatrix}_{3\times 3}$$

$$\begin{bmatrix} -1 & 5 & 0 \\ 0 & 88 & -3 \\ 0 & -264 & 9 \end{bmatrix}_{3\times 3} \begin{bmatrix} \because R_2 \to R_2 + 16R_1 \\ R_3 \to R_3 - 64R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 5 & 0 \\ 0 & 88 & -3 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \qquad [\because\ R_3 \to R_3 - 3R_2]$$

Which is in echelon form having two nonzero rows.

Here $\rho(A) = 2 <$ Number of vectors.

Hence the given set of vector are Linearly dependent (L.D.).

Example – 9 : *Verify* [1, – 1, 1], [1, 1, –1],[–1, 1, 1], [0, 1, 0] *are L.D.*

Solution : Let $V_1 = [1\ -1\ 1]$, $V_2 = [1\ 1\ -1]$, $V_3 = [-1\ 1\ 1]$, $V_4 = [0\ 1\ 0]$

Since V_1, V_2, V_3 and V_4 are 3 dimensional row vectors.

So by placing each row vector as a row we get a matrix 'A' of order (4×3).

$$\text{i.e, } A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{4\times 3} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}_{4\times 3} \qquad \begin{pmatrix} \because R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{4\times 3} \quad (R_3 \leftrightarrow R_4) \quad \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}_{4\times 3} \quad \left(R_3 \to R_3 - \frac{1}{2}R_2\right)$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3} \quad (R_4 \to R_4 - 2R_3)$$

Which is in echelon form having 3 nonzero rows.

Here $\rho(A) = 3 <$ Number of vectors.

Hence the given set of vectors are Linearly dependent.

Exercise – 1

1. Find the rank of following matrices

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 2 & 2 & -1 & 6 & 3 \\ 1 & 1 & 2 & 3 & -1 \\ 0 & 2 & 5 & 2 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 9 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 1 \end{bmatrix}$

2. Determine the rank of the following matrices.

(a) $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & 1 & -2 & 8 \\ 0 & 5 & 12 & -10 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 1 \end{bmatrix}$

3. Show that the rank of the transpose of the matrix $\begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & -2 \end{bmatrix}$ is same as that of original matrix.

4. Show that $\rho(A) = 2$ and $\rho(B) = 2$, where

$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

5. Under what condition the rank of the following matrix is 3 : $\begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$

6. Investigate the solution of the system of linear equations
$2x + y + 11 = 0,\ 6x + 20\,y - 6z + 3 = 0,\ 6y - 18z + 1 = 0$

7. Investigate for what values of λ and μ the following equations
$2x + 3y + 5z = 9$
$7x + 3y - 2z = 8$
$2x + 3y + \lambda z = \mu$
have (i) no solution (ii) a unique solution and (iii) an infinite number of solution.

8. Test for consistency and solve $5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 10z = 5$

9. Test for consistency and solve $2x - 3y + 7z = 5,\ 3x + y - 3z = 13,\ 2x + 19y - 47z = 32$

10. Show that the following system of equations is inconsistent
$2x - 3y + 7z = 5,\ 5x - 2y + 4z = 18,\ 2x + 19y - 47z = 32$

11. Find the value of k such that the following system of equations is consistent
$2x + y - z = 12,\ x - y - 2z = -3,\ 3y + 3z = k$

12. Test the consistency of the following $x - 5y + 3z = -1,\ 2x - y - z = 5,\ 5x - 7y + z = 2$

13. Show that the following system is consistent
$x + 2y - 5z + 2w = -2$
$3x - y + 2z + 4w = 19$
$2x - 3y + 7z + 2w = 21$

14. Apply rank test to show that the following systems of equations have unique solutions which are trivial.

(a) $x + 2y + 3z = 0$
$2x + 3y + z = 0$
$3x + y + 2z = 0$

(b) $x + 2y + 3z + 4w = 0$
$8x + 5y + z + 4w = 0$
$5x + 6y + 8z + w = 0$
$8x + 3y + 7z + 2w = 0$

15. Determine the values of λ for which the following set of equations may possess non-trivial solution.
$3x_1 + x_2 - \lambda x_3 = 0,\ 4x_1 - 2x_2 - 3x_3 = 0,\ 2\lambda x_1 + 4x_2 + \lambda x_3 = 0$
For each values of λ, determine general solution

16. Solve the equations
(a) $x + y + 3z = 0,\ 3x + 4y + 4z = 0,\ 7x + 10y + 12z = 0$
(b) $4x + 2y + z + 3w = 0,\ 6x + 3y + 4z + 7w = 0,\ 2x + y + w = 0$

Answers

1. (a) 2 (b) 4 (c) 2 (d) 3 **2.** (a) 2 (b) 2 (c) 1 (d) 2
5. $x \neq 1$,
6. The solution of the given system is consistent
7. (i) $\lambda = 5$, and $\mu \neq 9$ no solution
(ii) $\lambda \neq 5$, and μ may be any value, unique solution (iii) $\lambda = 5$, and $\mu = 9$. Infinite solution
9. $x = \frac{7}{11},\ y = \frac{3}{11}$ and $z = 0$
11. k = 18 12. Inconsistent, 15. $\lambda = 1, -9,\ x_1 = 3p,\ x_2 = 9p,\ x_3 = -2p$
16. (a) $x = y = z = 0$, (b) $z = -w,\ y = -2x - w$
which give an infinite number of solution, x and w are parameter.

Objective type Questions with Answers

Short Answertype Questions Carries 2 Marks

1. What do you mean by linearly independent vectors? Are the following vectors linearly independent ?
$[2\ \ -3], [3\ \ 6], [-1\ \ 4]$

Ans. Linearly Independent Vectors :
$v_1, v_2, v_3, \ldots v_n$ is called linearly independent if $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ with $c_1 = c_2 = c_3 = c_n = 0$.

$[2\ \ -3], [3\ \ 6], [-1\ \ 4]$

$$2c_1 + 3c_2 - c_3 = 0 \Rightarrow c_1 = c_3 - 3\frac{c_2}{2} - 3c_1 + 6c_2 + 4c_3 = 0$$

$\Rightarrow 3c_1 = 6c_2 + 4c_3 \;\Rightarrow \frac{3}{2}(c_3 - 3c_2) = 6c_2 + 4c_3$

$3c_3 - 9c_2 = 12c_2 + 8c_3 \;\Rightarrow 5c_3 = -21c_2$

So, they linearly dependent.

2. Explain the conditions for which a system of linear equations will possess more than one solution.

Ans. When the column vectors of the matrix of coefficients of the given system of equns are not linearly independent, the system will possess more than one solution.

3. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ then find A^{-1}.

Ans. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \;\therefore \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 - 2R_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

4. If α, β are sclars such that $A = \alpha\beta + \beta I$ then prove that AB = BA.

Ans. $A = \alpha\beta + \beta I$

$\therefore AB = (\alpha\beta + \beta I)B = \alpha\beta B + \beta I B = \alpha\beta B + \beta B$

and $\therefore BA = B(\alpha\beta + \beta I) = B\alpha\beta + B\beta I = B\alpha\beta + B\beta$

$\therefore AB = BA$

5. What do you mean by Null matrix ?

Ans. The matrix of any order containing only zero as its elements is called a Null matrix.

6. Define rank of a matrix.

Ans. The rank of a matrix is the largest order of the non-vanishing minor of that matrix.

7. Find the rank of the matrix. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$

Ans. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 2 \end{bmatrix}$ $(R_3 - 2R_1)$

From the last matrix we see that none of the rows contians only zeros

$\therefore \quad \rho(A) = 3$

8. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then Find A^n

Ans. $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

9. Find the rank of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Ans. The determinant of this matrix is 1 which is non zero.

So rank of the matrix = order of this matrix = 4.

10. Test for consistency of the equations $x + 2y = 1$ & $7x + 14y = 12$

Ans. $A = \begin{bmatrix} 1 & 2 \\ 7 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \therefore \rho(A) = 1$

$K = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 14 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 10 \end{bmatrix} \therefore \rho(K) = 2$

Since $\rho(A) \neq \rho(K)$, the equations are inconsistent.

11. Find the maximum value of the rank of a 4×5 matrix.

Ans. The maximum value of the rank of a 4×5 matrix is 4 i.e. rank $\leq$ min (rows cloumns)

12. If the rank of the matrix A is 2 find the rank of its transpose A^T.

Ans. We know that $\rho(A) = \rho(A^T) \therefore \rho(A^T) = 2$

13. Determine the rank of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans. Let $|A| = 1, \neq 0$, i.e. rank of A = 3.

14. Find the rank of the matrix $\begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 0 \end{bmatrix}$

Ans. 2 *i.e.* rank $\leq$ min (2, 5)

15. Reduce the matrix $A = \begin{bmatrix} 2 & 6 & 5 \\ 2 & 5 & 4 \\ 5 & 16 & 13 \end{bmatrix}$ to the normal form

Ans. $[I_3]$

16. For the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ find the non singular matrix p

Ans. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

17. For which value of λ and μ, the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have no solution and infinite number of solutions.

Ans. $\lambda = 3, \mu \neq 10$ and $\lambda = 3, \mu = 10$

18. Determine the values of λ for which the following set of equations may possess non-trival solution.

$3x_1 + x_2 - \lambda x_3 = 0,\ 4x_1 - 2x_2 - 3x_3 = 0,\ 2\lambda x_1 + 4x_2 + \lambda x_3 = 0$

Ans. $\lambda = -1, -9$

19. Determine the value of λ so that the equation $2x + y + 2z = 0$, $x + 2y + 3z = 0$, $4x + 3y + \lambda z = 0$ have a non zero solution

Ans. $\lambda = -5$

20. Are the following vectors are linearly independent

$x_1 = (4, -5, 2, 6)$, $x_2 = (2, -2, 1, 3)$, $x_3 = (6, -3, 3, 9)$ and $x_4 = (4, -1, 5, 6)$

Ans. Yes

21. If the following vectors are linearly dependent. Find a relation between then $x_1 = (3, 2, 7)$, $x_2 = (2, 4, 1)$, $x_3 = (1, -2, 6)$

Ans. yes, $(x_1 = x_2 + x_3)$

22. If rank (A) = 2, rank (B) = 3, then rank (AB) = ——

Ans. 5

23. $\begin{bmatrix} a+b & b+c & c+a \\ b+a & c+a & a+b \\ c+a & a+b & b+c \end{bmatrix} = K \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where K is

Ans. 2

Multiple type or dash fill up type questions. carries 2 Marks

1. The system $x + \alpha y = 0$, $y + \alpha z = 0$, $z + \alpha x = 0$ has infinitely many solutions when α = ——

Ans. -1

2. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution,then the possible value of k are —— and ——

Ans. $-1, 1$

3. The value of γ for which the system of equations $2x - y - 2z = 2$, $x - 2y + z = -4$, $x + y + \gamma z = 4$ has no solution is ——.

Ans. -3

4. If the system of equations $x + 2y - 3z = 1$, $(\gamma + 3)z = 3$, $(2\gamma + 1)x + z = 0$ is inconsistent, then the value of γ is equal to ——.

Ans. –3

5. To multiply a matrix by sealar k, which of them are correct ?

(a) any row by k (b) every element by k

(c) any column by k (d) None

Ans. (b)

6. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ then the determinant AB has the value ——.

Ans. 16

7. The system of equation $x + 2y + z = 9$, $2x + y + 3z = 7$ can be expressed as ———.

Ans. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

8. Matrix has a value. This statement

(a) is always true (b) depends upon, the matrices

(c) False (d) None

Ans. (c)

9. If A is a square matrix such that $AA' = I$ then the value of $A'A$ is ——.

Ans. I

10. If every minor of order r of a matix A is zero, then value of $A'A$ is ——.

Ans. less than or equal to r

11. A square matrix A is called orthogonal, which of them are correct ?

(a) $A = A^2$ (b) $A^T = A^{-1}$

(c) $AA^T = I$ (d) None

Ans. (b)

12. If a matrix A is equivalent to a canonical matrix having exactly r non zero rows, then rank A= ———.

Ans. = r

13. If A is an $n \times 1$ non-null matrix and B is an $1 \times n$ non-null matrix, then rank (AB) = ———

Ans. 1

14. The rank of a product of two matrices can not exceed the rank of either matrix i.e. rank (AB) ———.

Ans. ≤ tank (A)

15. The matrices obtained from a unit matrix I after one or more elementary operation are called ——.

Ans. elementary matrix

16. Rank of a unit (identity) matrix of order 4 is ———.

Ans. 4

17. If the system has a solution, it is called ———.

Ans. consistent

18. If $A^2 + A - 1 = 0$, then A^{-1} = ———.

Ans. $I + A$

19. Let A and B, be two matrices, then which conditions are hold ?

(a) $AB = BA$ (b) $AB \neq BA$

(c) $AB < BA$ (d) $AB > BA$

Ans. (b)

20. Let A and B be two matrices, such that $A = 0$, $AB = 0$, the equation always implies that

(a) $B = 0$ (b) $B \neq 0$

(c) $B = -A$ (d) $B = A'$.

Ans. (b)

21. The equations $x + 2y = 1$, $7x + 14y = 12$ are

(a) In consistent (b) consistent

(c) trial (d) None

Ans. (a)

22. If A and B are n rowed orthogonal matrix then which conditions are hold ?

(a) AB is only orthogonal (b) BA is only orthogonal

(c) AB and BA are both orthogonal (d) None

Ans. (c)

23. If the solution of simultaneous equation is 1, 2, 3 then equations are

(a) $x + y + z = 6$, $x - y + 2z = 5$, $x + 3y + 4z = 7$

(b) $x + y + 2z = 6$, $x - y - z = 7$, $x + y - z = 0$

(c) $x + y + z = 6$, $2x + 3y + 4z = 20$, $x - y + z = 2$

(d) None

Ans. (c)

24. The system of equation $kx + 2y - 2z = 1$, $4x + 2ky - z = 2$, $6x + 6y + kz = 3$ has no solution if k = ———.

Ans. 2

25. A matrix A is idempotent if

(a) $A^4 = A$ (b) $A^3 = A$

(c) $A^2 = A$ (d) $A^n = I$

Ans. (c)

26. By applying elementary transformation to a matrix its rank

(a) increases (b) decreases

(c) does not change (d) None

Ans. (c)

27. If A and B be two square matices, then which conditions are hold ?

(a) $R(A + B) = R(A) + R(B)$ (b) $R(A + B) < R(A) + R(B)$

(c) $R(A + B) > R(A) + R(B)$ (d) None

Ans. (d)

28. Given $9A = \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, which conditions are hold ?

(a) $A^{-1} = A^1$ (b) $A^{-1} = \frac{1}{9}A^1$

(c) $2A^1 = A$ (d) None

Ans. (b)

29. A system of linear equations represented by the matrix equation AX = B is consistent if

(a) $r(A) < r(A, B)$ (b) $r(A) > r(A, B)$

(c) $r(A) = r(A, B)$ (d) $r(A) + r(A, B)$ = twice the order of the matrix

Ans. (c)

30. If A and B are square matrices of the same order, then

(a) $(A+B)^2 \neq A^2 + 2AB + B^2$ (b) $(A-B)^2 = A^2 - 2AB + B^2$

(c) $(A+B)(A-B) \neq A^2 - B^2$ (d) None

Ans. (d)

31. Let A and B be any two matrices such that AB =0 and A is non singular

(a) B =0 (b) B is non singular

(c) B =A (d) B is singular

Ans. (d)