Example – 3 : ***Prove that*** $\nabla\left(r^n\right)=nr^{n-2}\vec{r}$ ***and hence show that*** $\nabla^2\left(r^n\right)=n(n+1)\,r^{n-2}$ ***by using vector identities.***

Solution : First show that again find

$$\nabla\times\left(r^n\vec{r}\right)=0$$

Here $r=|\vec{r}|=x^2+y^2+z^2 \;\Rightarrow r^2=x^2+y^2+z^2$

$$2r\frac{\partial r}{\partial x}=2x,\quad or\quad \frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial r}{\partial z}=\frac{z}{r}$$

Now $\nabla\left(r^n\right)=\left(\Sigma\frac{\partial}{\partial x}i\right)\left(r^n\right)$

$$=\sum nr^{n-1}\frac{\partial r}{\partial x}i=\sum nr^{n-1}\left(\frac{x}{r}\right)i$$

$$=\sum nr^{n-2}x\vec{i}=nr^{n-2}\Sigma x\vec{i}=nr^{n-2}\vec{r}$$

Thus $\nabla\left(r^n\right)$ = grad (r^n) $=nr^{n-2}\vec{r}$

$$\nabla\left(r^n\right)=nr^{n-2}\vec{r}$$

Also $\nabla^2\left(r^n\right)=\nabla\cdot\left(\nabla r^n\right)=\nabla\cdot\left(nr^{n-2}\vec{r}\right)$

Consider $\nabla^2\cdot\left(\phi\vec{A}\right)=\phi\left(\nabla\cdot\vec{A}\right)+\nabla\phi\cdot\vec{A}$

$$\therefore\quad \nabla\cdot\left(nr^{n-2}\vec{r}\right)=n\left\{r^{n-2}\nabla\cdot\vec{r}+\nabla\left(r^{n-2}\right)\cdot\vec{r}\right\}$$

$$=n\left\{3r^{n-2}+(n-2)r^{n-4}\vec{r}\cdot\vec{r}\right\}\qquad \therefore\nabla\cdot\vec{r}=3$$

$$=n\left\{3r^{n-2}+(n-2)r^{n-4}r^2\right\}\qquad \therefore\vec{r}\cdot\vec{r}=r^2$$

$$=n\left\{3r^{n-2}+(n-2)r^{n-2}\right\}=nr^{n-2}\left\{3+(n-2)\right\}$$

Thus $\nabla^2(r^n)=n(n+1)\,r^{n-2}$

Again show that $\nabla\times(\vec{r}\vec{r})=0$

Or, $\quad r^n\vec{r}=r^n\Sigma xi=\Sigma\left(r^n x\right)\vec{i}$

$$\nabla\times\left(r^n\vec{r}\right)=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ r^n x & r^n y & r^n z\end{vmatrix}=\sum\vec{i}\left\{\frac{\partial}{\partial y}\left(r^n z\right)-\frac{\partial}{\partial z}\left(r^n y\right)\right\}$$

$$= \sum \vec{i}\left[nr^{n-1}\frac{\partial r}{\partial y}z - nr^{n-1}\frac{\partial r}{\partial z}y \right]$$

$$= \sum \vec{i}\left[nr^{n-1}\frac{y}{r}.z - nr^{n-1}\frac{z}{r}.y \right]$$

$$= \sum \vec{i}\left[nr^{n-1}yz - nr^{n-2}yz \right] = 0$$

Thus $\nabla \times \left(r^n \vec{r}\right) = curl\left(r^n \vec{r}\right) = \vec{o}$

Example– 4 : (i) If $\vec{F} = (\vec{a}\times\vec{r})r^n$***, show that***

$$\textbf{curl } \vec{F} = (n+2)r^n\vec{a} - nr^{n-2}(\vec{a}\cdot\vec{r})\vec{r}$$

(ii) If $\vec{F} = \dfrac{\vec{a}\times\vec{r}}{r^2}$***, show that***

$$\nabla \times \vec{F} = \frac{2-n}{r^n}\vec{a} + \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\vec{r}$$

Solution : **(i)** $\text{curl } \vec{F} = \text{curl}\left\{r^n(\vec{a}\times\vec{r})\right\} = \nabla r^n \times (\vec{a}\times\vec{r}) + r^n\nabla\times(\vec{a}\times\vec{r})$

$$= nr^{n-2}\vec{r}\times(\vec{a}\times\vec{r}) + r^n 2\,\vec{a}\left(\text{since } \nabla\times(a\times\vec{r})\right) = 2\vec{a}$$

$$= nr^{n-2}\left\{r^2\vec{a} - (\vec{r}\cdot\vec{a})\vec{r} + 2r^n\vec{a}\right\} = (n+2)r^n\vec{a} - nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r}$$

(ii) $\vec{F} = \dfrac{\vec{a}\times\vec{r}}{r^n}$

$$\therefore \nabla\times\vec{F} = \nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^n}\right) = \nabla\left(\frac{1}{r^n}\right)\times(\vec{a}\times\vec{r}) + \frac{1}{r^n}\nabla\times(\vec{a}\times\vec{r})$$

$$= -nr^{-n}r^2\vec{r}\times(\vec{a}\times\vec{r}) + \frac{1}{r^n}2\vec{a} = \frac{-n}{r^{n+2}}\left[r^2\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}\right] + \frac{2\vec{a}}{r^n}$$

$$= \frac{-na}{r^n} + \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}} + \frac{2\vec{a}}{r^n} = \frac{(2-n)\vec{a}}{r^n} + \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\,\vec{r}$$

Example – 5 : If $V = \hat{r}$***, show that*** $\nabla\times V = 0$

Solution : Here $\vec{V} = \hat{r} = \dfrac{\vec{r}}{r} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{r} & \dfrac{y}{r} & \dfrac{z}{r} \end{vmatrix}$$

$$= \vec{i}\left[\left(-\frac{1}{2}\frac{z\cdot 2y}{r^3}+\frac{1}{2}\frac{y}{r^3}:2z\right)\right]+\hat{j}\left[-\frac{1}{2}\frac{x\cdot 2z}{r^3}+\frac{1}{2}\frac{z}{r^3}\cdot 2x\right]+\hat{k}\left[-\frac{1}{2}\frac{y\cdot 2x}{r^3}+\frac{1}{2}\frac{x\cdot 2y}{r^3}\right]$$

$= 0 + 0 + 0 = 0$

Example – 6 : Prove that $\nabla^2\left(r^n\vec{r}\right) = n(n+3)\, r^{n-2}\vec{r}$

Solution : $\nabla^2\left(r^n\vec{r}\right)$ involves Laplacian of a vector point function and hence we recall the identity

$$\text{curl}\left(\text{curl}\ \vec{A}\right) = \text{grad}\left(\text{div}\ \vec{A}\right)\ \nabla^2 - \vec{A}$$

$$\therefore\ \nabla^2\left(-\vec{A}\right) = \text{grad}\left(\text{div}\ \vec{A}\right) - \text{curl}\left(\text{curl}\ \vec{A}\right)$$

Now $\nabla^2\left(r^n\vec{r}\right) = \text{grad}\left\{div\left(r^n\vec{r}\right)\right\} - \text{curl}\left\{curl\left(r^n\vec{r}\right)\right\}$

But div $\left(r^n\vec{r}\right) = (n+3)\, r^n$ and curl $\left(r^n\vec{r}\right) = \vec{0}$

$\therefore\ \nabla^2\left(r^n\vec{r}\right) = \text{grad}\ \{(n+3)\, r^n\} = (n+3)\ \text{grad}\ (r^n)$

But grad $(r^n) = n\ r^{n-2}\,\vec{r}$

Thus $\nabla^2\left(r^n\vec{r}\right) = (n+3)n\ r^{n-2}\vec{r} = n(n+3)r^{n-2}\vec{r}$

Example – 7 : **Show that**

(a) ***div*** $[(\vec{r}\times a)\times b] = -2\,(a\cdot b)$ ***(b)*** ***grad*** $[\vec{r}, a, b] = a\times b$

(c) ***curl*** $(\vec{r}\times a) = -2a$ ***(d)*** ***div*** $(\vec{r}\times a) = 0$

(e) ***grad*** $(a\cdot\vec{r}) = a,$

Where a and b are a constant vector and r is the radius vector.

Solution : **(a)** div $((\vec{r}\times a)\times b) = \text{div}\ [(\vec{r}\cdot b)a - (a\cdot b)\,\vec{r}\,]$

$$= \text{div}\left[\left(xb_1+yb_2+zb_3\right)a - \left(a\cdot b\right)\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\right]$$

if $r = x\hat{i}+y\hat{j}+z\hat{k},\ a = a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$$= \frac{\partial}{\partial x}\left[a_1\left(xb_1+yb_2+zb_3\right)\right]+\frac{\partial}{\partial y}\left[a_2\left(xb_1+yb_2+zb_3\right)\right]+\frac{\partial}{\partial z}\left[a_3\left(xb_1+yb_2+zb_3\right)\right]$$

$$-\left(a\cdot b\right)\left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right) = -2\,(a\cdot b)$$

$= a_1 b_1 + a_2 b_2 + a_3 b_3 - (3\ a\ .\ b) = (a.b) - 3\ (a\ b) = 2(a, b)$

(b) grad [r, a, b] = ∇ [r·(a×b)]

$$= \nabla\left[x(a\times b)_{\hat{i}-\text{component}} + y(a\times b)_{\hat{j}-\text{component}} + z(a\times b)_{\hat{k}-\text{component}}\right]$$

$$= \hat{i}\,(a\times b)_{\hat{i}-\text{component}} + \hat{j}\,(a\times b)_{\hat{j}-\text{component}} + \hat{k}\,(a\times b)_{\hat{k}-\text{component}}$$

$$= (a \times b)$$

(c) curl (r × a) $= \sum \hat{i} \times \dfrac{\partial}{\partial x}(r\times a) = \sum \hat{i} \times \left[\dfrac{\partial r}{\partial x} \times a\right]$

$$= \sum \hat{i} \times \left(\hat{i}\times a\right) = \sum\left(\left(\hat{i}\cdot a\right)\hat{i} - \left(\hat{i}\cdot\hat{i}\right)a\right) = a - 3a = -2a$$

(d) div (r × a) = a · curl r – r · curl a

$$\text{Now curl } r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(0-0) = 0$$

and curl a = 0 (∵ a is a constant vector)

∴ div (r × a) = 0

(e) grad (a·r) = ∇ $(a_1x + a_2y + a_3z)$ if $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $r = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \text{grad}\,(a\cdot r) = \sum \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) = \sum \hat{i}\,a_1 = a$$

Example – 8 : Evaluate the following :

(i) $\textit{div}\left[(xy\sin z)\hat{i} + \left(y^2\sin x\right)\hat{j} + \left(z^2\sin xy\right)\hat{k}\right]$ ***at the point*** $\left(0, \dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

(ii) ***Curl Curl of*** $\vec{V} = \left(2xz^2\right)\hat{i} - yz\hat{j} + \left(3xz^3\right)\hat{k}$ ***at*** $(1, 1, 1)$

Solution : **(i)** $\textit{div}\left[(xy\sin z)\hat{i} + \left(y^2\sin x\right)\hat{j} + \left(z^2\sin xy\right)\hat{k}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(xy\sin z)\hat{i} + \left(y^2\sin x\right)\hat{j} + \left(z^2\sin xy\right)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(xy\sin z) + \frac{\partial}{\partial y}\left(y^2\sin x\right) + \frac{\partial}{\partial z}\left(z^2\sin xy\right) = y\sin z + 2y\sin x + 2z\sin xy$$

At the pt. $\left(0, \dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$$\text{div}\left\{(xy\sin z)\hat{i} + \left(y^2\sin x\right)\hat{j} + \left(z^2\sin xy\right)\hat{k}\right\} = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}$$

(ii) $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + (3xz^3)\hat{k}$

$$Curl\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}$$

$$= \hat{i}\{y\} - \hat{j}\{3z^3 - 4xz\} + \hat{k}\{0\} \quad = y\hat{i} + (-3z^3 - 4xz)\hat{j}$$

$$Curl\,Curl\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -3z^3 + 4xz & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-3z^3 + 4xz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-3z^3 + 4xz) - \frac{\partial}{\partial y}(y)\right\}$$

$$= -\hat{i}(-9z^2 + 4x) + \hat{j}(0) + \hat{k}(4z - 1) = (9z^2 - 4x)\hat{i} + (4z - 1)\hat{k}$$

At (1, 1, 1) crul curl $\vec{V} = 5\hat{i} + 3\hat{k}$.

Example – 9 : If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ ***prove that***

(i) $\mathbf{Div}(\mathbf{r}^n\vec{\mathbf{r}}) = (\mathbf{n}+3)\mathbf{r}^n$

(ii) ***Find the value of n for which vector*** $(r^n\vec{r})$ ***is solenoidal vector.***

Hence declare that $\dfrac{\vec{r}}{r^3}$ ***is solenoidal vector.***

Solution : Let $\vec{F} = r^n\vec{r}$

(i) $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(r^n\vec{r})$

Now, $r^n\vec{r} = (x^2 + y^2 + z^2)^{\frac{n}{2}}(x\vec{i} + y\vec{j} + z\hat{k})$

$$\left[\because r = \vec{r} = \sqrt{x^2 + y^2 + z^2},\, r^n = (x^2 + y^2 + z^2)^{\frac{n}{2}}\right]$$

Now $= \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\left[(x^2 + y^2 + z^2)^{\frac{n}{2}}(x\hat{i} + y\hat{j} + z\hat{k})\right]$

$$= \frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} x\right\} + \frac{\partial}{\partial y}\left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} y\right\} + \frac{\partial}{\partial z}\left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} z\right\}$$

$$= \left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} + \frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1} \cdot 2x^2\right\} + \left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} + \frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1} \cdot 2y^2\right\}$$

$$+\left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} + \frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1} \cdot 2z^2\right\}$$

$$= 3\left(x^2+y^2+z^2\right)^{\frac{n}{2}} + \frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1} \cdot 2\left(x^2+y^2+z^2\right)$$

$$= 3\left(x^2+y^2+z^2\right)^{\frac{n}{2}} + n\left(x^2+y^2+z^2\right)^{\frac{n}{2}} = (3+n)\left(x^2+y^2+z^2\right)^{\frac{n}{2}} = (3+n)r^n$$

$$\therefore \ \operatorname{div}\left(r^n \vec{r}\right) = (n+3)r^n$$

(ii) Given $\vec{F} = r^n \vec{r}$ is solenoidal $\therefore \operatorname{div} \vec{F} = 0$

$$\therefore (3+n)\left(x^2+y^2+z^2\right)^{\frac{n}{2}} = 0$$

$\Rightarrow n + 3 = 0$ or $n = -3$

Put $n = -3$ is $r^n\vec{r}$ we see that div $\left(r^{-3}\vec{r}\right) = 0 \ \therefore \dfrac{\vec{r}}{r^3}$ is solenoidal vector.

Example – 10 : A fluid motion is given by $\vec{\upsilon} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\vec{j} + \left(xy\cos z + y^2\right)\hat{k}$***. Is the motion irrotational ? If so find the velocity potential.***

Solution : $\vec{\upsilon} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\vec{j} + \left(xy\cos z + y^2\right)\hat{k}$

$$curl\ \vec{\upsilon} = \begin{vmatrix} \hat{i} & \vec{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left(xy\cos z + y^2\right) - \frac{\partial}{\partial z}(x\sin z + 2yz)\right\} + \vec{j}\left\{\frac{\partial}{\partial z}(y\sin z - \sin x) - \frac{\partial}{\partial x}\left(xy\cos z + y^2\right)\right\}$$

$$+\hat{k}\left\{\frac{\partial}{\partial x}(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x)\right\}$$

$$= \hat{i}\{x\cos z + 2y - x\cos z - 2y\} + \vec{j}\{y\cos z - y\cos z\} + \hat{k}\{\sin z - \sin z\}$$

$$= \hat{i}\cdot 0 + \vec{j}\cdot 0 + \hat{k}\cdot 0 = \vec{0}$$

$\because$ curl $\vec{\upsilon} = \vec{0}$ $\therefore$ Motion is irrotational.

Let u be the velocity potential.

$\therefore$ Motion is irrotational.

$\therefore \vec{\upsilon} = \text{grad } u$.

$$= (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\vec{j} + (xy\cos z + y^2)\hat{k} = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$$

$$\therefore \frac{\partial u}{\partial x} = y\sin z - \sin x, \frac{\partial u}{\partial y} = x\sin z + 2yz \text{ and } \frac{\partial u}{\partial z} = xy\cos z + y^2$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$

$= (y \sin z\, dx + x \sin z\, dy + xy \cos z\, dz) + (- \sin x\, dx) + (2yz\, dy + y^2\, dz)$

$= d[xy\sin z] + d(\cos x) + d(y^2z)$ (*by inspection*)

$= d\,[xy \sin z + \cos x + y^2z]$

Integrate, $u = xy \sin z + \cos x + y^2z + c$ which is the required velocity potential.

Example – 11 : Show that $\vec{F} = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)\,j + (2y^2z + xy)\,k$ is a conservative force field. Find its scalar potential

Solution : We have to show that curl $\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy^2 + yz) & (2x^2y + xz + 2yz^2) & (2y^2z + xy) \end{vmatrix}$$

$= i(4yz + x - x - 4yz) - j\,(y - y) + k\,(4xy + z - 4xy - z) = \vec{0}$

$\therefore$ $\vec{F}$ is conservative.

Now we have to find ϕ such $\nabla\phi = \vec{F}$

i.e, $\dfrac{\partial \phi}{\partial x}i + \dfrac{\partial \phi}{\partial y}j + \dfrac{\partial \phi}{\partial z}k = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)k$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2xy^2 + yz \quad \therefore \quad \Rightarrow \phi = \int (2xy^2 + yz)dx + f_1(y,z)$$

i.e, $\phi = x^2y^2 + xyz + f_1\,(y\,,z)$ ... (1)

$$\frac{\partial \phi}{\partial y} = 2x^2y + xz + 2yz^2$$

$$\therefore \quad \phi = \int (2x^2y + xz + 2yz^2)dy + f_2(x,z)$$

i.e, $\phi = x^2y^2 + xyz + y^2z^2 + f_2\,(x\,,z)$ ... (2)

$$\Rightarrow \frac{\partial \phi}{\partial z} = 2y^2z + xy \quad \therefore \quad \phi = \int (2y^2z + xy)dz + f_3(x,y)$$

i.e, $\phi = y^2z^2 + xyz + f_3\,(x\,,y)$ ... (3)

Let us choose $f_1(y,z) = y^2z^2, f_2(x, z) = 0, f_3(x, y) = x^2y^2$

Thus $\phi = x^2y^2 + y^2z^2 + xyz$ is the required scalar potential.

Example – 12 : Prove that

$$\textit{(i)}\quad \nabla^2 f(\mathbf{r}) = f''(\mathbf{r}) + \frac{\mathbf{2}}{\mathbf{r}} f'(\mathbf{r}) \qquad \textit{(ii)}\quad \nabla^2(r^n) = n(n+1)r^{n-2}$$

Solution : **(i)** $\nabla^2 f(r) = \nabla\cdot\{\nabla f(r)\} = \text{div}\{\text{grad} f(r)\}$

$$\text{grad } f(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r) = \hat{i}\frac{\partial}{\partial x} f(r) + \hat{j}\frac{\partial}{\partial y} f(r) + \hat{k}\frac{\partial}{\partial z} f(r)$$

$$= \hat{i} f'(r)\frac{\partial r}{\partial x} + \hat{j} f'(r)\frac{\partial r}{\partial y} + \hat{k} f'(r)\frac{\partial r}{\partial z} = f'(r)\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$$

$$= f'(r)\,\text{grad r}$$

$$\because \nabla^2 f(r) = \text{div}\left[f'(r)\,\text{grad r}\right] \qquad \because \text{grad r} = \frac{\vec{r}}{r}$$

$$= \text{div}\left[f'(r)\frac{\vec{r}}{r}\right] \qquad \because \text{if } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = \sqrt{x^2+y^2+z^2}$$

$$= \text{div}\left[\frac{f'(r)}{r}\vec{r}\right] \qquad \text{grad r} = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r}$$

which is of the type div $(\phi\vec{a})$

where $\phi = \dfrac{f'(r)}{r}$ and $\vec{a} = \vec{r}$

$$= \frac{f'(r)}{r}\text{div } \vec{r} + \vec{r}\cdot\text{grad } \frac{f'(r)}{r}$$

Since div $\vec{r} = 3$

$$= \frac{f'(r)}{r}3 + \vec{r}\cdot\left[\frac{d}{dr}\left(\frac{f'(r)}{r}\right)\text{grad r}\right] \quad \text{as proved above grad} f(r) = f'(r)\text{ grad r}$$

$$= \frac{3f'(r)}{r} + \vec{r}\cdot\left\{\frac{rf''(r) - f'(r)}{r^2}\right\}\frac{\vec{r}}{r} \qquad \text{Replace } f(r) \text{ by } \frac{f'(r)}{r}$$

$$= \frac{3f'(r)}{r} + \left\{\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right\}\vec{r}\cdot\vec{r} = \frac{3f'(r)}{r} + \left\{f''(r) - \frac{f'(r)}{r}\right\}\frac{r^2}{r^2}$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r} = f''(r) + \frac{2}{r}f'(r)$$

(ii) $\nabla^2 r^n = \nabla\cdot\nabla r^n = \text{div grad } r^n$

$$\text{grad } r^n = \hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial}{\partial z}r^n = \hat{i}\cdot nr^{n-1}\frac{\partial r}{\partial x} + \hat{j}nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left\{\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right\} = nr^{n-1}\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

Now $\nabla^2 r^n = \text{div}\left[\left(nr^{n-2}\right)\vec{r}\right]$ which is of the type div $(\phi\vec{a})$

$$= \left(nr^{n-2}\right)\text{div}\vec{r} + \vec{r}\cdot\text{grad}\left(nr^{n-2}\right) = \left(nr^{n-2}\right)3 + \vec{r}\cdot n\,\text{grad}\, r^{n-2}$$

$$= 3nr^{n-2} + n\vec{r}\cdot(n-2)r^{n-4}\vec{r}$$ ($\because$ grad $r^n = nr^{n-2}\vec{r}$ change n to $n-2$

grad $r^{n-2} = (n-2)r^{n-4}\vec{r}$)

$$= 3nr^{n-2} + n(n-2)r^{n-4}(\vec{r}\cdot\vec{r}) = 3n\, r^{n-2} + n\,(n-2)\, r^{n-4}\cdot r^2$$

$$= 3n\, r^{n-2} + n\,(n-2)\, r^{n-2} = (3n + n^2 - 2n)\, r^{n-2} = (n + n^2)\, r^{n-2} = n\,(n+1)\, r^{n-2}$$

Example – 13 : Prove that curl $(\phi\overline{F}) = \phi(\Delta \times \overline{F}) + \Delta\phi \times \overline{F}$ **.**

Solution : Let ϕ and $\vec{A} = a_1 i + a_2 j + a_3 k$ be respectively scalar and vector point functions of x, y, z.

$$\therefore \quad \phi\vec{A} = (\phi a_1)i + (\phi a_2)j + (\phi a_3)k$$

Now curl $\left(\phi\vec{A}\right) = \Delta \times \left(\phi\vec{A}\right) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix}$

$$= \Sigma i\left\{\frac{\partial}{\partial y}(\phi a_3) - \frac{\partial}{\partial z}(\phi a_2)\right\}$$

$$= \Sigma i\left\{\left(\phi\frac{\partial a_3}{\partial y} + \frac{\partial\phi}{\partial y}a^3\right) - \left(\phi\frac{\partial a_2}{\partial z} + \frac{\partial\phi}{\partial z}a_2\right)\right\}$$

$$= \phi\Sigma\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)i + \Sigma\left(\frac{\partial\phi}{\partial y}a_3 - \frac{\partial\phi}{\partial z}a_2\right)i$$

$$= \phi\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_{33} \end{vmatrix} + \begin{vmatrix} i & j & k \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \phi\left(\Delta \times \vec{A}\right) + \Delta\phi \times \vec{A}$$

Thus curl $\left(\phi\vec{A}\right) = \phi\left(curl\,\vec{A}\right) + \Delta\phi \times \vec{A}$

Example – 14 :* *If $\vec{F}$ = grad $(x^3y + y^3z + z^3x - x^2y^2z^2)$ find div $\vec{F}$ and curl $\vec{F}$ at (1, 2, 3).

Solution : If $\vec{F}$ = grad $(x^3y + y^3z + z^3x + x^2y^2z^2)$

$$= \Sigma i \frac{\partial}{\partial x}\left(x^3y + y^3z + z^3x - x^2y^2z^2\right)$$

$$= (3x^3y + z^3 - 2xy^2z^2)\, i + (x^3 + 3y^2z - 2x^2yz^2)\, j + (y^3 + 3xz^2 - 2x^2y^2z)k$$

i.e., $\vec{F} = F_1 i + F_2 j + F_3 k$ (say)

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \cdot \vec{F} = 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6xz - 2x^2y^2$$

$$\nabla \cdot \vec{F}\,]_{(1,2,3)} = -32$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(3x^2y + z^3 - 2xy^2z^2\right) & \left(x^3 + 3y^2z - 2yx^2z^2\right) & \left(y^3 + 3z^2x - 2zx^2y^2\right) \end{vmatrix}$$

$$= i\left[\left(3y^2 - 4x^2yz\right) - \left(3y^2 - 4x^2yz\right)\right] - j\left[\left(3z^2 - 4xy^2z\right) - \left(3z^2 - 4xy^2z\right)\right]$$

$$+ k\left[\left(3x^2 - 4xyz^2\right) - \left(3x^2 - 4xyz^2\right)\right] = \vec{0}$$

Thus div $\vec{F} = -32$ and curl $\vec{F} = \vec{0}$.

Exercise – 3

1. A particle move along a curve whose parametric equations are : $x = e^{-t}$, y = 2 cos 3t, z= 2 sin 3t where t is the time. Find the velocity and acceleration at any time t and also their magnitudes at t = 0.
2. If ‘ $\vec{r}$ ’ is a unit vector. Prove that $\left|\hat{r} \times \dfrac{d\hat{r}}{dt}\right| = \left|\dfrac{d\hat{r}}{dt}\right|$.
3. Find the directional derivative of $4xz^3 - 3x^2y^2z^2$ at (2, –1, 2) along the z – axis.
4. Find the directional derivative of $\phi = xy^2 + yz^3$ at the point (1, –2, –1) in the direction of the normal to the surface $x \log z - y^2 = 4$ at (–1, 2, 1)
5. Find the value of the constants a and b such that the surfaces $ax^2 - byz = (a + 2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point (1, –1, 2)

6. What is the directional derivative of the function $2xy + z^2$ at the point $(1, -1, 3)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$?

7. If $\vec{a}$ is a constant vector prove that $\nabla\left(a \cdot \nabla \frac{1}{r}\right) = -\frac{\vec{a}}{r^3} + \frac{3(\vec{a} \cdot \vec{r})\vec{r}}{r^5}$

8. For a solenoidal vector $\vec{F}$, show that curl curl curl curl $\vec{F} = \nabla^4 \vec{F}$.

9. Find Curl $\vec{F}$, where $\vec{F} = \nabla (x^3 + y^3 + z^3 - 3xyz)$

10. Show that, div curl curl $\left(f\,\vec{a} + \nabla^2 div\left(f\,\vec{a}\right)\right) = \vec{a}.\text{gard}\,\nabla^2 f$

11. If a is a constant vector and r denotes the position vector of any point in space and if $f = (a \times r)\, r^n$, show that div $f = 0$ and curl $f = (n + 2)\, r^n\, a - n\, r^{n-2}\, (a \cdot r)\, r$.

12. Find the condition that the vector field

    $\vec{F} = (xyz)^b\, (x^a i + y^a j + z^a k)$ is both solenoidal and irrotational.

13. Find the value of the constant 'a' such that $\vec{F} = (axy - z^3)\, i + (a - 2)\, x^2 j + (1 - a)\, xz^2\, k$ is irrotational and hence find a scalar function ϕ such that $\vec{F} = \nabla\phi$

14. Prove that div curl $\vec{F} = \nabla \cdot \nabla \times \vec{F} = 0$.