CHAPTER – 4

Vector Integration

4.1 : Introduction

The concept of a line integral is a simple and natural generalization of the concept of a definite integral. $\int_a^b f(x)dx$

In the line integral we do not integrate the integrand along x-axis from a to b; instead, we integrate along a curve in place or in space and the integrand is a function defined at the points of that curve. We can define a line integral in a way similar to that of a definite integral.

1. **Line Integrals:** Any integral which is to be evaluated along a curve is called a line integral.

Let $\vec{r} = \vec{r}$ (t) be the equation of a curve. If ϕ and A are the scalar and vector fields respectively and dr is the vector increment of length, then we may encounter the integrals.

$\int_C \phi d\vec{r}$ ....(1) $\int_C A \cdot d\vec{r}$ ....(2) $\int_C A \times d\vec{r}$ .....(3)

each of which being known as line integral along the curve C that may be open or closed. The results of integration are respectively a vector, a scalar and a vector.

To compute any of the integrals, the method of attack will be to reduce the vector integrals, into scalar integrals with which one is assumed to be familiar.

As ϕ is a scalar function and $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$, integral (1) immediately reduces to

$$\int_C \phi dr = \hat{i}\int_C \phi(x,y,z)dx + \hat{j}\int_C \phi(x,y,z)dy + \hat{k}\int_C \phi(x,y,z)dz \quad ....(4)$$

The three integrals on R.H.S. of (4) are ordinary scalar integrals and to avoid complications we shall assume that they are Riemann integrals. The integrals with respect to x cannot be evaluated unless y and z are known in terms of x and similarly for the integrals with respect to y and z. This simply means that the path of integration C must be specified. Unless the integrand has special properties that lead the integral to depend only on the value of end points, the value will depend upon the particular choice of C.

The line integrals (2) and (3) may be interpreted in the similar fashion and like integral (1) they are dependent, in general, on the choice of the path.

Line integral (2) is most commonly used in vector analysis, it is called the line integral of the tangential component of vector A along the curve *C*. If A (vector function of position) represents the force $\vec{F}$ acting on the particle, then the line integral i.e., $\int_C \vec{F} \cdot d\vec{r}$ represents the work done by the force.

Scalar line Integrals :

Thus, the scalar line integral of a vector function $\vec{F}$ along a curve C from *A* to *B* is definite integral of the scalar resolute of $\vec{F}$ in the direction of the tangent to the curve.

Let a continuous curve *C* in space and $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ be a vector point function. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then the line integral is defined as

$$\int_C \vec{F} \cdot d\vec{r} \text{ or } \int_A^B \vec{F} \cdot d\vec{r}$$

be position vector of a point (*x, y, z*) on the curve. If S denotes the arc. length of the curve C from a fixed point on it to the point (*x, y, z*), then $\frac{d\vec{r}}{ds}$ = t is unit vector along the tangent to the curve at the point $\vec{r}$. The component of the vector $\vec{F}$ along the tangent is $\vec{F}\frac{d\vec{r}}{ds}$. The integral of $\vec{F}\frac{d\vec{r}}{ds}$ along C from the point A to B, written as $\int_A^B \left(\vec{F}\frac{d\vec{r}}{ds}\right)ds = \int_A^B \vec{F}.d\vec{r} = \int_C \vec{F}.d\vec{r}$ is called the line integral or move particularly the tangent line integral (or scalar integral) of $\vec{F}$ along *C*.

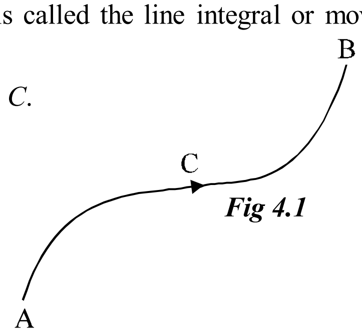

Fig 4.1

Now

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right) \cdot \left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$$

$= \int_C \left(f_1dx + f_2dy + f_3dz\right)$, which is an ordinary integration.

Other type of line integral $\int_C \vec{F} \times d\vec{r}$ which gives a vector. When C is a closed curve then the line integral is written as $\oint_C \vec{F} \cdot d\vec{r}$.

Work done by a force :

Suppose a force $\vec{F}$ acts upon a particle. Let the particle be displaced along a given path *C*, in space. If $\vec{r}$ denotes the position vector of a point on *C*, then $\frac{d\vec{r}}{ds}$ is a unit vector along the tangent of the force $\vec{F}$ along the tangent to C is $F.\ \dfrac{d\vec{r}}{ds}$. Therefore the work done by $\vec{F}$ along C is

$$\left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right)ds \text{ .i.e. } \left(\vec{F}\cdot d\vec{r}\right)$$

The total work W done by $\vec{F}$ in this displacement along C is given by the line integral

$$W = \int_C \vec{F}\cdot d\vec{r}\ ,$$

the integration being taken in the sense of the displacement.

Now if $\vec{\mathrm{F}}$ represents force acting on a particle moving from A to B (displacement), then the line integral represents work done i.e., $\mathrm{W.D.} = \int_{\mathrm{A}}^{\mathrm{B}} \vec{\mathrm{F}}\cdot d\vec{\mathrm{r}}$.

Theorem – 1 :The necessary and sufficient condition that the line integral $\int_A^B \vec{F}\cdot d\vec{r}$ be indepedent of the path joining A and B is that F is the gradient of some scalar function ϕ.

Alt : The necessary and sufficient condition that $\int_{\mathrm{C}} \vec{\mathrm{F}}\cdot d\vec{\mathrm{r}}$ be indepedent of the path is that curl $\vec{\mathrm{F}}$ vanishes identically.

Proof : The condition is sufficient.

Let $\vec{F}$ be the gradient of a scalar function ϕ. i.e., $\vec{F} = \nabla\phi$ ·

The $\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r} = \left(\hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}dy + \hat{k}dz\right)$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$\therefore \quad \int_A^B \vec{F}.d\vec{r} = \int_A^B d\phi = [\phi]_A^B = \phi(B) - \phi(A).$$

Here the value ϕ(B) – ϕ(A) depends only on the terminal values A and B and not on the path of the curve. Hence the condition is sufficient.

The condition in necessary. Now let us suppose that $\int_A^B \vec{F}.d\vec{r}$ depends only on the end values A and B and not on the path of integration.

$\therefore \int_A^B \vec{F}.d\vec{r} = \phi(B) - \phi(A)$, for some function, say $\phi = [\phi]_A^B = \int_A^B d\phi$.

So, $\vec{F}\cdot d\vec{r} = d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}dy + \hat{k}dz\right) = \nabla\phi.d\vec{r}$$

Since this is true for all curves between A and B, $F = \nabla\phi$. Hence the condition is necessary.

Theorem – 2 : The necessary and sufficient condition that the line integral $\int_A^B \vec{F}\cdot d\vec{r}$ be indepedent of the path joining A and B in a given region is that $\int_C \vec{F}.d\vec{r} = 0$, for all closed paths in the region.

Proof : *The condition is necessary.*

Firstly suppose that $\int_A^B \vec{F}\cdot d\vec{r}$ is independent of the path joining A and B. Let $AP_1\,BP_2\,A$ be a closed curve in the region as shown in the *(Fig 4.2)*.

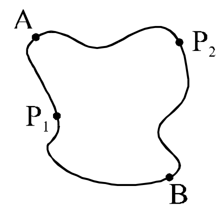

Fig 4.2

Then $\oint_C \vec{F}.d\vec{r} = \int_{AP_1BP_2A} \vec{F}.d\vec{r}$

$$= \int_{AP_1B} \vec{F}.d\vec{r} + \int_{BP_2A} \vec{F}.d\vec{r} = \int_{AP_1B} \vec{F}.d\vec{r} - \int_{AP_2B} \vec{F}.d\vec{r} = 0$$

since the integral from A to B along the path through P_1 is the same as the integral from A to B along the path through P_2 by the hypothesis.

Hence if $\int_A^B \vec{F}.d\vec{r}$ is independent of the path, $\oint \vec{F}.d\vec{r} = 0$ for all closed paths in the region. Thus the condition is necessary.

The condition is sufficient. Let $\oint_C \vec{F}.d\vec{r} = 0$, for all closed paths in the given region. Let AP_1B and AP_2B be two paths joining A and B. Then AP_1BP_2A is a closed path.

$\therefore \quad \int_{AP_1BP_2A} \vec{F}.d\vec{r} = 0 \quad$ i.e., $\quad \int_{AP_1B} \vec{F}.d\vec{r} + \int_{BP_2A} \vec{F}.d\vec{r} = 0$

i.e., $\int_{AP_1B} \vec{F}.d\vec{r} - \int_{AP_2B} \vec{F}.d\vec{r} = 0$ i.e., $\int_{AP_1B} \vec{F}.d\vec{r} = \int_{AP_2B} \vec{F}.d\vec{r}$

Hence if $\oint \vec{F}.d\vec{r} = 0$ for all closed paths in the region, then $\int_A^B \vec{F}.d\vec{r}$ is independent of the path joining A and B.

Corollary : If $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path of integration then $\int_C \vec{F}\cdot d\vec{r}$ along any closed path is zero.

Circulation : If C is a simple closed curve (i.e., a curve which does not intersect itself anywhere), then the tangent line integral of $\vec{F}$ around C is called the circulation of $\vec{F}$ about C and is written as

$$\oint_C \vec{F}.d\vec{r} = \oint_C \left(\vec{F}_1\, dx + \vec{F}_2\, dy + \vec{F}_3\, dz \right).$$

Note : If $\vec{F}$ represents the velocity of a fluid particle then the $\oint_C \vec{F}\cdot d\vec{r}$ is called the circulating of $\vec{F}$ around C. When the circulation is zero in a region, then $\vec{F}$ is said to be irrotational in that region.

Note – 1: If the scalar point is replaced by vector products the corresponding line integral is $\int_C \vec{F} \times d\vec{r}$ which is a vector.

Note –2: If the vector function $\vec{F}$ is replaced by a scalar function ϕ, then the corresponding line integral is denoted as $\int_C \phi d\vec{r}$ which is a vector.

Note–3 : If $\vec{F}(x,y,z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dx} + f_3 \frac{dz}{dt} \right) dt$$

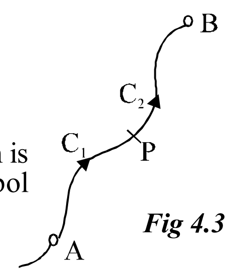

Fig 4.3

Note 4 : If C is a closed curve, then the integral sign $\int_C$ is replaced by $\oint_C$.

In all the above line integrals, it is assumed that the path of integration is piecewise smooth. For a line integral over a closed path C, the symbol $\oint_C \left(\text{instead of } \int_C \right)$ is sometimes used in the literature. *(fig 4.3)*

From the definition of line integral, it follows that the following properties are valid for line integrals :

(a) $\int_C k\, w\, ds = k \int_C w\, ds$ (k constant)

(b) $\int_C (f + g) ds = \int_C f\, ds + \int_C g\, ds$

(orientation of C is the same in all the three integrals.)

(c) $\int_C w\, ds = \int_{C_1} w\, ds, + \int_{C_2} w\, ds$, where the path C is subdivided into two arcs C_1 and C_2, which have the same orientation as C.

Note that if the sense of integration along a curve C is reversed, the value of the line integral is multiplied by –1.

The question is now arises is – How is a line integral evaluated ? We shall now answer this question.

Additivity :

Line integrals have the useful property that if a curve C is made by joining a finite number of curves C_1, C_2 ....C_n end to end, then the integral of a function over C is the sum of the integrals over the curve that make it up.

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds + \ldots\ldots + \int_{C_n} f ds$$

4.2 : Mass and Moment Calculations

Mass and moment formulas for coil springs, thin rods and wires lying along a smooth curve C in space. The distribution is described by a continuous density function $\delta(x, y, z)$ (mass per unit length.)

Mass $= M = \int_C \delta(x,y,z)ds \ (\delta = \delta(x,y,z) = density)$.

First moments about the co-ordinate planes :

$$M_{yz} = \int_C x\delta ds,\ M_{zx} = \int_C y\delta ds,\ M_{xy} = \int_C z\delta ds$$

Co-ordinates of the centre of mass :

$$\bar{x} = M_{yz}/M,\ \bar{y} = M_{zx}/M,\ \bar{z} = M_{xy}/M.$$

Moments of Inertia about axes and other lines :

$$I_x = \int_C \left(y^2 + z^2\right)\delta ds,\ I_y = \int_C \left(z^2 + x^2\right)\delta ds$$

$$I_z = \int_C \left(x^2 + y^2\right)\delta ds,\ I_L = \int_C r^2\ \delta ds$$

$r(x, y, z)$ = distance from the point (x, y, z) to the line L.

Radius of gyration about a line L, $R_L = \sqrt{I_L/M}$.

Illustration : Finding Mass, Centre of Mass moment of inertia, radius of gyration of a coil spring lies along the helix $r(t) = (cost)\hat{i} + (sint)\hat{j} + t\hat{k}$, $0 \le t \le 2\pi$. The spring's density is a constant $\delta = 1$.

Solution : The symmetries involved, the cetntre of mass lies at the point $(0, 0, \pi)$ on z-axis.

We first find $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

$$= \sqrt{(-sin\,t)^2 + (cos\,t)^2 + 1} = \sqrt{2}$$

$$M = \int_{Helix} \delta ds = \int_0^{2\pi} 1.\sqrt{2}dt = \sqrt{2}[t]_0^{2\pi} = 2\sqrt{2}\pi$$

$$I_Z = \int_{Helix} \left(x^2 + y^2\right)\delta ds = \int_0^{2\pi} \left(cos^2 t + sin^2 t\right)1.\sqrt{2}dt = \sqrt{2}\int_0^{2\pi} dt = \sqrt{2}[t]_0^{2\pi} = 2\sqrt{2}\pi$$

$$R_Z = \sqrt{I_z/M} = \sqrt{\frac{2\sqrt{2}\pi}{2\sqrt{2}\pi}} = 1.$$

4.3 : *Work Done by a Force over a Curve in Space*

Supppose that the vector field $\vec{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ represents a force through out a region in space and that

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k} \qquad a \le t \le b$$

is a smooth curve in the region. Then the integral of **F.T.** the scalar component of $\vec{F}$ in the direction of the curves unit tangent vector over the curve is called the work done by $\vec{F}$ over the curve from a to b. To evaluate the work integral along a smooth curve $r(t)$, takes following steps.

→ Evaluate $\vec{F}$ on the curve as a function of parameter t.

→ Find $\frac{d\vec{r}}{dt}$.

→ Integral $\vec{F}\cdot\frac{d\vec{r}}{dt}$ from $t = a$ to $t = b$.

Six different ways to write the work integrals :

$$W = \int_{t=a}^{t=b} \vec{F}\, Tds = \int_{t=a}^{t=b} \vec{F}\, dr \quad \text{(Compact differential form)}$$

$$= \int_b^a \vec{F}.\frac{dr}{dt}\cdot dt \quad \text{(Parameter } t \text{ and velocity vector } \frac{d\vec{r}}{dt}\text{)}$$

$$= \int_a^b \left(M\frac{dg}{dt} + N\frac{dh}{dt} + P\frac{dk}{dt} \right) dt \qquad \text{(Component function)}$$

$$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt$$

$$= \int_a^b Mdx + Ndy + Pdz$$

Flow Integrals and circulation for velocity fields :

If $r(t)$ is a smooth curve in the domain of continuous velocity field F the flow along the curve $t = a$ to $t = b$ is flow $= \int_a^b F.Tds$.

The integral in this case is called a flow integral. If the curve is a closed loop, the flow is called the circulation around the curve.

4.4 : Flux Across a Plane Curve

If C is a smooth closed curve in the domain of a continuously vector field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ in the plane and if $\hat{n}$ is the outward. Pointing unit normal vector on C, the flux of F across C is

$$\text{Flux of F across } C = \int_C F.\hat{n}ds$$

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$, then $\vec{F}.\hat{n} = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$

Hence, $\int_C \vec{F}.nd\vec{s} = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right)$

$\oint_C Mdy - Ndx$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$ $a \le t \le b$, that traces 'C' centre clockwise exactly once.

Illustrative Examples

Example – 1 : A Fluid's velocity field is F = xi + zj + yk. Find the circulation along the helix $\boldsymbol{r(t) = cost\hat{i} + sint\hat{j} + t\hat{k}}$***, 0 ≤ t ≤ π/2.***

Solution : Evaluate $\vec{F}$ on the curve,

$\vec{F} = x\hat{i} + x\hat{j} + y\hat{k} = (cos\,t)\hat{i} + t\hat{j} + (sin\,t)\hat{k}$ and then find $\frac{d\vec{r}}{dt}$.

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

We integrate $\vec{F}.\frac{d\vec{r}}{dt}$ from $t = 0$ to $t = \pi/2$.

$$\vec{F}.\frac{d\vec{r}}{dt} = (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1)$$

$$= -\,sint.cost + t\,cost + sin\,t$$

$$\text{Flow} = \int_{t=a}^{t=b} \left(\vec{F}.\frac{dr}{dt} \right) dt = \int_0^{\pi/2} (-\sin t.\cos t + t\cos t + \sin t)\,dt$$

$$= \left[\frac{\cos^2 t}{2} + t\sin t \right]_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2}.$$

Example–2 : Find the circulation of the field $\boldsymbol{\vec{F} = (x - y)\hat{i} + x\hat{j}}$ ***around the circle*** $\boldsymbol{r(t) = (cost)\hat{i} + (sint)\hat{j}}$***, 0 ≤ t ≤ 2π.***

Solution : First we evaluate

$$\vec{F} = (x - y)\hat{i} + x\hat{j} = (cos\,t - sin\,t)\hat{i} + (cos\,t)\hat{j}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F}.\frac{d\vec{r}}{dt} = -\sin t.\cos t + \left(\sin^2 t + \cos^2 t\right) = 1 - sint.\,cost$$

$$\text{Circulation } = \int_0^{2\pi} F.\frac{dr}{dt}dt = \int_0^{2\pi} (1 - sin\,t.cos\,t)\,dt$$

$$= \left[t - \frac{sin^2 t}{2} \right]_0^{2\pi} = 2\pi$$

Example – 3 : Find the flux of $\vec{F} = (x-y)\hat{i} + x\hat{j}$ ***across the circle*** $x^2 + y^2 = 1$***, in the xy plane.***

Solution : The parameterization $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$, $0 \le t \le 2\pi$

M = $x - y = cost - sint$ $\qquad$ $dy = d\,(sint) = cos\; dt$

$N = x$ $\qquad$ $dx = d\,(cost) = -\,sint\; dt,$

we find

$$\text{Flux} = \int_C Mdy - Ndx = \int_0^{2\pi}\left(cos^2 t - sin\,t\,cos\,t + sin\,t\,cos\,t\right)dt$$

$$= \int_0^{2\pi} cos^2\,tdt = \int_0^{2\pi}\left(\frac{1+cos\,2t}{2}\right)dt = \left[\frac{t}{2} + \frac{sin\,t}{4}\right]_0^{2\pi} = \pi.$$

The flux of F across the circle is π.

Example – 4 : Evaluate $\int_C (x^2 + y^2 + z^2)^2\, ds$ ***, where C is the arc of circular helix.***

$\vec{r}(t) = cost\;\hat{i} + sint\;\hat{j} + 3\,t\,\hat{k}$ ***from A*** **(1, 0, 0)** ***to B*** **(1, 0, 2π).**

Solution : Here $\vec{r}(t) = \cos t\;\hat{i} + \sin t\;\hat{j} + 3\,t\,\hat{k}$,

$$\therefore \dot{r}(t) = -\sin t\;\hat{i} + \cos t\;\hat{j} + 3\,\hat{k} = \frac{d\vec{r}}{dt}$$

$$\text{Now } \frac{ds}{dt} = \sqrt{\dot{r}\cdot\dot{r}} = \sqrt{\sin^2 t + \cos^2 t + 9} = \sqrt{10} = \sqrt{|\dot{r}|^2}$$

On C, $(x^2 + y^2 + z^2)^2 = [\cos^2 t + \sin^2 t + 9t^2]^2 = (1 + 9t^2)^2$

Also A (1, 0, 0) and B (1, 0, 2 π) on C correspond to $0 \le t \le 2\pi$.

$$\text{Thus, we have } \int_C \left(x^2 + y^2 + z^2\right)^2 = \int_0^{2\pi}\left(1 + 9t^2\right)^2 \cdot \frac{ds}{dt}\,dt$$

$$= \sqrt{10}\int_0^{2\pi}\left(1 + 81t^4 + 18t^2\right)dt = \sqrt{10}\left[t + 81\frac{t^5}{5} + 18\frac{t^3}{3}\right]_0^{2\pi} = \sqrt{10}\left[2\pi + 6(2\pi)^3 + \frac{81}{5}(2\pi)^5\right]$$

= 506400

Example – 5 : Evaluate the line integral $\int_C \left[x^2 ydx + (x-z)dy + x\;y\;z\;dz\right]$ ***where C is the arc of the parabola*** $y = x^2$ ***in the plane*** $z = 2$ ***from A (0, 0, 2) to B (1, 1, 2).***

Solution : Since on C, $y = x^2$ and $z = 2$ (constant),

$\therefore$ on C, $dy = 2x\,dx$ and $dz = 0$.

It follows that on C, the integral of the last term in the given integrand is zero.

Thus, $\int_C \left[x^2 y dx + (x-z)dy + x\, y\, z\, dz\right] = \int_0^1 \left[x^2 \cdot x^2 dx + (x-2)\cdot 2x\, dx\right]$

($\because y = x^2$ and z = 2 on C)

$$= \int_0^1 \left(x^4 + 2x^2 - 4x\right)dx = \left|\frac{x^5}{5} + 2\frac{x^3}{3} - 4\frac{x^2}{2}\right|_0^1$$

$$= \frac{1}{5} + \frac{2}{3} - 2 = -\frac{17}{15} \text{ (fig 4.4)}$$

Let us now take up an example in which we consider integration of a line integral over different paths with the same end points.

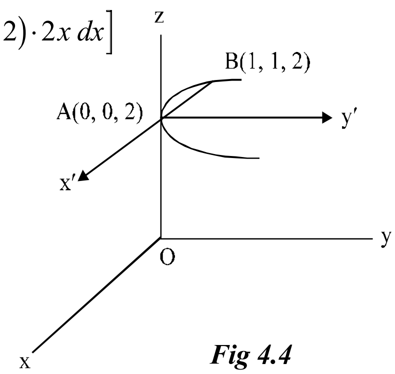

Fig 4.4

Example – 6 : Let C be the line segment from A (0, 0) to B (1, 1) and let f (x, y) = x + y².

Evaluate $\int_C f(x,y)\, ds$ ***when***

(a) C is characterised by x = t, y = t, 0 ≤ t ≤ 1.

(b) C has parametric representation x = sin t, y = sin t, 0 ≤ t ≤ $\frac{\pi}{2}$.

Solution : (a) If $x = t$, $y = t$, $0 \le t \le 1$, then

$$\int_C f(x,y)\, ds = \int_0^1 \left(t + t^2\right)\cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \left(t + t^2\right)\cdot \sqrt{1\cdot + 1}\,dt = \sqrt{2}\left|\frac{t^2}{2} + \frac{t^3}{3}\right|_0^1 = \frac{5\sqrt{2}}{6}$$

(b) If $x = \sin t$, $y = \sin t$, $0 \le t \le \frac{\pi}{2}$, then

$$\int_C f(x,y)\, ds = \int_C \left(x + y^2\right)\cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\sin t + \sin^2 t\right)\sqrt{\cos^2 t + \cos^2 t\, dt} = \sqrt{2}\int_0^{\frac{\pi}{2}} \left(\sin t \cos t + \sin^2 t \cos t\right)dt$$

$$= \sqrt{2}\left|\frac{\sin^2 t}{2} + \frac{\sin^3 t}{3}\right|_0^{\frac{\pi}{2}} = \frac{5\sqrt{2}}{6}$$

4.5 : Application of Line Integrals

In this example, the integrands and the end points of the paths of integration are the same, but the values of the line integrals are different. This illustrates the important fact that.

In general, the values of a line integral of a given function depends not only on the end points but also on the geometric shape of the path of integrals are of the form

$$f(x,y,z)\frac{dx}{ds},\ f(x,y,z)\frac{dy}{ds} \text{ or } f(x,y,z)\frac{dz}{ds}$$

where $\frac{dx}{ds}, \frac{dy}{ds}$ and $\frac{dz}{ds}$ are the derivatives of the functions occuring in the parametric representation of the path of integration. Then we simply write

$$\int_C f(x,y,z)\frac{dx}{ds}ds = \int_C f(x,y,z)\,dx$$

and similar expressions in the other two cases.
For sums of these types of integrals along the same path C, we adopt the notation

$$\int_C f\ dx + \int_C g\ dy + \int_C h\ dz = \int_C (f\ dx + g\ dy + h\ dz)$$

In many cases, the functions *f*, g, h are components f_1, f_2, f_3 of a vector function

$$f = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

Then $f_1dx + f_2dy + f_3dz = \left(f_1\frac{dx}{ds} + f_2\frac{dy}{ds} + f_3\frac{dz}{ds}\right)ds$, the expression in parenthesis on the right being the dot product of the vector *f* and the unit tangent vector.

$$\frac{dr}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k},$$

where r (s) represents the path of the integration of the line integral. Therefore,

$$\int_C (f_1dx + f_2dy + f_3dz) = \int_C f \cdot \frac{dr}{ds}ds = \int_C f \cdot d\vec{r}$$

where $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.
Now if f represents a force whose point of application moves along a curve

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \quad a \le t \le b$$

from a point A to a point B in space, then $\int_C f \cdot d\vec{r}$ represents the work done by the force *f* in moving a particle from point A to point B along the curve C.

The representation in equation i.e., $\int_C f \cdot \frac{d\vec{r}}{ds}ds$ emphasises the fact that the work done by the force *f* is the value of the line integral along the curve of the tangential component of the force field *f*.

4.6 : Path Independence – Conservative Fields

Let $\vec{F}$ be a vector field with components F_1, F_2, F_3 and F_1, F_2, F_3 be continuous throughout some connected region D.
Consider two point A and B in D. Suppose that C is any piecewise smooth curve joining A and B given by $x = x(t)$, $y = y(t)$, $z = z(t)$, $t_1 \le t \le t_2$.

Region D is connected if any two points of D can be joined by a broken line of finitely many linear segments all of which belong to D.

If there exists a differentiable function f such that

$\vec{F} = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$, then along C, $f = f[x(t), y(t), z(t)]$ is a function of t and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \nabla f \cdot \left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}\right) = \nabla f \cdot \frac{dr}{dt}$$

We thus have $\vec{F} \cdot d\vec{r} = \nabla f \cdot d\vec{r} = \nabla f \cdot \frac{d\vec{r}}{dt} dt = \frac{df}{dt} dt$

Now integrating $\vec{F} \cdot d\vec{r}$ along C from A to B, we get

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \frac{df}{dt} dt \nabla\vec{f} = \vec{F} = \int_{t_1}^{t_2} d\left[f\{x(t), y(t), z(t)\}\right]$$

$$= f(x(t), y(t)z(t))\Big|_{t_1}^{t_2} = f[x(t_2), y(t_2), z(t_2)] - f[x(t_1), y(t_1), z(t_1)]$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$$

The value of the integral $[f(B) - f(A)]$ does not depend on the path C at all. This result is analogue of the First Fundamental Theorem of Integral Calculus, viz.,

$$\int_a^b f'(x) dx = f(b) - f(a)$$

The only difference is that we have $\nabla f \cdot d\vec{r}$ in place of $f'(x)\, dx$. This analogy suggests that if we define a function f by the rule.

$$\int f(x', y', z') = \int_A^{(x', y', z')} \vec{F} \cdot d\vec{r}$$

then it will also be true that $\nabla f = \vec{F}$

This result $\nabla f = \vec{F}$ is indeed true when the right-hand side of relation is path-independent. Thus a necessary and sufficient condition for the integral $\int_A^B F \cdot d\vec{r}$ to be independent of the path joining the points A and B in some connected region D is that there exists a differentiable function of such that

$$\vec{F} = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

throughout D, where components f_1, f_2, f_3 of vector field F are continuous throughout D and then

$$\int_A^B \vec{F}\cdot d\vec{r} = f(B) - f(A).$$

When F is a force such that the work-integral from A to B is the same for all paths, the field is said to be conservative. Using the above result, we can say that *:* A force field F is conservative if and only if it is a gradient field, *i.e.,* $F = \nabla f$, for some differentiable function f.

A function $f(x, y, z)$ that has the property that it's gradient gives the force vector F is called a potential function. Sometimes a minus sign is introduced, e.g., the electric intensity of a field is the negative of the potential gradient in the field.

Let us consider the following example.

Exact Differential Form :

Any expression $M(x, y, z)\,dx + N(x, y, z)dy + P(x, y, z)\,dz$ is a differential form. A differential form is exact if $Mdx + Ndy + Pdz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$ for some scalar function f through out D.

Component Test for Exactness of Mdx + Ndy + Pdz :

The differential form $Mdx + Ndy + Pdz$ is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

This is equivalent to saying that the field $F = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative.

Illustrative Examples

Example – 1 : Find the potential for conservative Fields and component test for conservative fields. Let $\boldsymbol{F = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}}$ ***be a field whose component functions have continuous first partial derivatives. Then*** $\vec{F}$ ***is conservative if and only if***

$$\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}} = \frac{\partial \boldsymbol{N}}{\partial \boldsymbol{z}}, \frac{\partial \boldsymbol{M}}{\partial \boldsymbol{z}} = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}} \textit{ and } \frac{\partial \boldsymbol{N}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{M}}{\partial \boldsymbol{y}}.$$

Solution : Given $\vec{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial z}$$

The others equation (2) are proved similarly.

Once we know that F is conservative, we want to find a potential function for F. i.e. $\nabla\vec{f} = \vec{F}$.

$$\frac{\partial f}{\partial x} = M, = \frac{\partial f}{\partial y} = N, = \frac{\partial f}{\partial z} = P.$$

Example – 2 : Show that $2xydx + (x^2 - z^2)\, dy - 2yzdz$ is exact and evaluate the integral

$$\int_{(0,0,0)}^{(1,2,3)} 2xydx + \left(x^2 - z^2\right)dy - 2yzdz\,..$$

Solution : We let $M = 2xy$, $N = x^2 - z^2$, $P = -2yz$ and apply the test for exactness :

$$\frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$$

The equalities tells us $2xydx + (x^2 - z^2)dy - 2yz\,dz$ is exact so

$$2xydx + (x^2 - z^2)\,dy - 2yz\,dz = df$$

for some function f, and the integral's value is $f(1, 2, 3) - f(0, 0, 0)$

$$M = \frac{\partial f}{\partial x} = 2xy \qquad ...(1)$$

$$N = \frac{\partial f}{\partial y} = \left(x^2 - z^2\right) \qquad ...(2)$$

$$P = \frac{\partial f}{\partial z} = -2yz \qquad ...(3)$$

From the first equation (1) and integrate w.r.t. x we get

$$f(x, y, z) = x^2y + g(y, z) \qquad ...(4)$$

Again differentiating (4) *w.r.t.* y partially and equalising (2)

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2$$

$$\Rightarrow \frac{\partial g}{\partial y} = -z^2 \Rightarrow g(y) = -z^2y + h(z) \qquad ...(5)$$

Hence y is a function of z

$$f(x, y, z) = x^2y - z^2y + h(z) \qquad ...(6)$$

Again differentiating (6) partially *w.r.t.* z. and equalising with (3)

$$\frac{\partial f}{\partial z} = -2yz + \frac{\partial h}{\partial z} = -2yz$$

$$\frac{\partial h}{\partial z} = 0 \Rightarrow h(z) = c$$

So h must be a constant

Therefore, $f(x, y, z) = x^2y - yz^2 + c$

The value of the integral

$$f(1, 2, 3) - f(0, 0, 0) = 2 - 2(9) + c - c$$
$$= 2 - 18 = -16.$$

Example – 3 : Finding a potential function & show that

$$\boldsymbol{F} = \left(e^x \cos y + yz\right)\hat{i} + \left(xz - e^x \sin y\right)\hat{j} + \left(xy + z\right)\hat{k} \text{ is conservative.}$$

Solution : $M = e^x\cos y + yz$, $N = xz - e^x\sin y$, $P = xy + z$.

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \ \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y} \qquad ....(1)$$

Once we know to find a conservative we usually want to find a potential function for F on solving $\nabla\vec{f} = \vec{F}$.

$$M = \frac{\partial f}{\partial x} = e^x \cos y + yz, \ N = \frac{\partial f}{\partial y} = xz - e^x \sin y, \ P = \frac{\partial f}{\partial z} = xy + z \qquad ....(2)$$

We integrate (1) with respect to x, treating y, z constant.

$f(x, y, z) = e^x \cos y + xyz + g(y, z)$ ....(3)

Again differentiating partially (3) w.r.t. y and compare with (2).

$$\frac{\partial f}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y}$$

i.e., $-e^x \sin y + xz + \dfrac{\partial g}{\partial y} = xz - e^x \sin y \Rightarrow \dfrac{\partial g}{\partial y} = 0$

So we get $\dfrac{\partial g}{\partial y} = 0$. Therefore, g is a function of z alone.

$f(x, y, z) = e^x \cos y + xyz + h(z)$ ....(4)

Now again differentiating (4) partially w.r.t. z, we get and compare with (2)

$$\frac{\partial f}{\partial z} = xy + \frac{\partial h}{\partial z} = xy + z \Rightarrow \frac{\partial h}{\partial z} = z.$$

So $h(z) = \dfrac{z^2}{2} + C$.

Hence $f(x,y,z) = e^x \cos y + xyz + \dfrac{z^2}{2} + c$

Example – 4 : Show that $\vec{F} = (2xy - z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 - 2zx)\hat{k}$ ***is conservative and find a potential function for it.***

Solution : First we have to show that F is conservative or not clearly $M = 2xy - y^2$, $N = x^2 + 2yz$, $P = y^2 - 2zx$ and calculate $\dfrac{\partial P}{\partial y} = 2y = \dfrac{\partial N}{\partial z}, \dfrac{\partial M}{\partial z} = 0 \neq \dfrac{\partial P}{\partial x}$

$$\frac{\partial N}{\partial x} = 2x \neq \frac{\partial M}{\partial y}.$$

The two are unequal. So F is not conservative.

We find f by integrating the equations

$$M = \frac{\partial f}{\partial x} = 2xy - z^2 \qquad ... (1)$$

$$N = \frac{\partial f}{\partial y} = x^2 + 2y \qquad ... (2)$$

$$P = \frac{\partial f}{\partial z} = y^2 - 2zx \qquad ... (3)$$

We integrate first equation with respect to x holding y & z are constant

$$f(x, y, z) = x^2y - z^2x + g(y, z)$$

Again differentiating partially w.r.t. y and equalising with (3) we get

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + 2y$$

i.e., $\frac{\partial g}{\partial y} = 2y \Rightarrow g(y) = y^2 + c$

$$f(x, y, z) = x^2y - z^2x + y^2 + h(z)$$

Again differentiating w.r.t.z and equalising with (3) we get

$$\frac{\partial f}{\partial z} = -2zx + \frac{\partial h}{\partial z} = y^2 - 2zx \Rightarrow \frac{\partial h}{\partial z} = y^2$$

$\Rightarrow h(z) = y^2z + c$

Hence $f(x, y, z) = x^2y - z^2x + y^2 + y^2z + c$

Example – 5 : Find a potential function f for the field F. If $F = 2x\hat{i} + 3y\hat{j} + 4z\hat{k}$.

Solution : Here $M = \frac{\partial f}{\partial x} = 2x$ ... (1)

$$N = \frac{\partial f}{\partial y} = 3y \quad \text{... (2)}$$

$$P = \frac{\partial f}{\partial z} = 4z \quad \text{... (3)}$$

First integrate M with respect to x, i.e., y & z are constant.

$$f(x,y,z) = x^2 + g(y,z)$$

Again differentiating partially w.r.t. y and equalising with (2)

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow \frac{\partial g}{\partial y} = 3y$$

$\Rightarrow g(y) = \frac{3y^2}{2} + h(z)$ (Intergrating w.r.t.y)

$$f(x,y,z) = x^2 + \frac{3y^2}{2} + h(z)$$

Again differentiating partially w.r.t.z and equalising with (3) we get

$$\frac{\partial f}{\partial z} = 0 + \frac{\partial h}{\partial z} = 4z \Rightarrow \frac{\partial h}{\partial z} = 4z$$

$$h(z) = 2z^2 + c$$

Hence $f(x, y, z) = x^2 + \frac{3y^2}{2} + 2z^2 + c$

Example – 6 : Find the circulation of $\vec{F}$ round the curve c where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and c is the circle $x^2 + y^2 = 1, z = 0$.

Solution : The required circulation is given by

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \left(y\hat{i} + z\hat{j} + x\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = \int_C ydx + zdy + xdz = \int_C ydx \text{ as } z = 0 \text{ on C.}$$

The parameteric equation of the curve i is $x = \cos t$, $y = \sin t$, and $0 \le t \le 2\pi$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_C ydx = \int_0^{2\pi} \sin t(-\sin t)dt$$

$$= \frac{1}{2}\int_0^{2\pi}(\cos 2t - 1)dt = \frac{1}{2}\left[\frac{\sin 2t}{2} - t\right]_0^{2\pi} = (-2\pi)\cdot\frac{1}{2} = -\pi$$

Example – 7 : Find the work done when a foce $F = (x^2 - y^2 + x)\ \hat{i}\ - (2xy + y)\ \hat{j}$ moves a particle in the xy – plane from (0, 0) to (1, 1) along the probabola $y^2 = x$. Is the work done different when the path is the straight line $y = x$?

Solution : The parabola $y^2 = x$ has a parametric representation $y = t$, $x = t^2$.

From (0, 0) to (1, 1), variation of t is $0 \le t \le 1$.

$\therefore$ Work done along the parabola

$$= \int_C \vec{F}\cdot d\vec{r} = \int_C \left[\left(x^2 - y^2 + x\right)\hat{i} - (2xy + y)\hat{j}\right]\cdot\left(dx\hat{i} + dy\hat{j}\right) = \int_C \left[\left(x^2 - y^2 + x\right)dx - (2xy + y)dy\right]$$

$$= \int_0^1 \left[\left(t^4 - t^2 + t^2\right)\cdot 2t\,dt - \left(2\cdot t^2\cdot t + t\right)dt\right] \quad = \left[\frac{1}{3}t^6 - \frac{1}{2}t^4 - \frac{1}{2}t^2\right]_0^1 = -\frac{2}{3}$$

We can similarily find the work done when the particle move from (0, 0) to (1,1) along $y = x$. In this case also we find that the work done is

$$= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2xy^2 - \frac{1}{2}y^2\right]_{(0,0)}^{(1,1)} = \frac{1}{3} + \frac{1}{2} - 2 - \frac{1}{2} = -\frac{5}{3}$$

as obtained earlier. This is because, we notice that

$$F = grad\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2xy^2 - \frac{1}{2}y^2\right)$$

Example – 8 : If $\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$, find work done by $\vec{F}$ along the upper half of the circle passing through the points (–1, 0) and (1, 0).

Solution : $$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \left(\frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}\right)\left(dx\hat{i} + dy\hat{j}\right) = \int_C \frac{-ydx + xdy}{x^2 + y^2}$$

Required workdone

$$= 2\left[\int_{\Gamma_1} \frac{-ydx + xdy}{x^2 + y^2} + \int_{\Gamma_2} \frac{-ydx + xdy}{x^2 + y^2}\right]$$

$$= 2\left[\int_{\Gamma_1} \frac{dy}{1 + y^2} + \int_{\Gamma_2} \frac{-dy}{1 + y^2} = \int_1^0 \frac{dy}{1 + y^2} - \int_{-1}^0 \frac{dy}{1 + y^2}\right]$$

$$= 2\left[\left[\tan^{-1} y\right]_1^0 - \left[\tan^{-1} y\right]_{-1}^0\right]$$

$$= 2\left[\left(\tan^{-1} 0 - \tan^{-1} 1\right) - \left[\tan^{-1} 0 - \tan^{-1}(-1)\right]\right]$$

$$= 2\left[0 - \frac{\pi}{4} - \left(0 - \frac{3\pi}{4}\right)\right] = 2\left[\frac{-\pi}{4} + \frac{3\pi}{4}\right] = 2\frac{2\pi}{4} = \pi$$

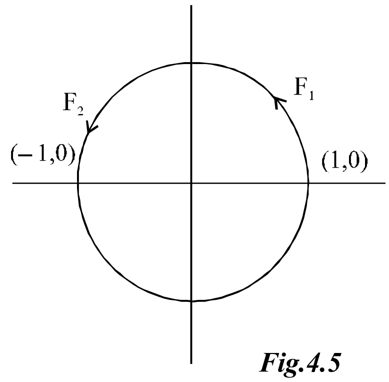

Fig.4.5

Example – 9 : If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve in xy plane $y = 2x^2$ from (0, 0) to (1, 2).

Solution : Since the integration is performed in the xy-plane (z = 0)

we can take $\vec{r} = x\hat{i} + y\hat{j}, \quad d\vec{r} = dx\hat{i} + dy\hat{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left(3xy\hat{i} - y^2\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right) = \int_C 3xydx - y^2dy = \int_C 3xydx - \int_C y^2dy$$

To integrate; change 1st integrand to function of x only with the help of $y = 2x^2$.

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 3x\cdot 2x^2dx - \int_0^2 y^2dy \Rightarrow x \text{ varies from 0 to 1 and y varies from 0 to 2.}$$

$$= \left[6\cdot\frac{x^4}{4}\right]_0^1 - \left[\frac{y^3}{3}\right]_0^2 = \frac{6}{4} - \frac{8}{3} = \frac{3}{2} - \frac{8}{3} = \frac{-7}{6}.$$

Or Second Method. Take parametric equation of the parabola $y = 2x^2$ which is $x = t$, $y = 2t^2$

$\therefore$ when $x = t$: $dx = dt$, when $y = 2t^2$; $dy = 4t\, dt$

At the pt (0, 0), $x = 0$, $\therefore$ $t = 0$

At the pt (1, 2), $x = 1$, $\therefore$ $t = 1$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C 3xydx - y^2dy = \left(3xy\frac{dx}{dt} - y^2\frac{dy}{dt}\right)dt = \int_0^1 3\cdot t\cdot 2t^2 - 4t^4\cdot 4t\, dt = \int_0^1 \left(6t^3 - 16t^5\right)dt$$

$$= \left[6\cdot\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1 = \frac{6}{4} - \frac{8}{3} = -\frac{7}{6}$$

Note that if the curve is traversed in the opposite sense i.e. from (1, 2) to (0, 0) the value of the integral would $\frac{7}{6}$ instead of $-\frac{7}{6}$.

Example – 10 : Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along.

(i) the straight line from (0, 0, 0) to (2, 1, 3).

(ii) the curve defined $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Solution : Work done $\int \vec{F}\cdot d\bar{r} = \int_C \left(3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}\right)\cdot\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$

$$= \int_C 3x^2 \cdot dx + (2xz - y)dy + zdz$$

(i) The equations of the line through (0, 0, 0) to (2, 1, 3) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \text{ or } \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$\therefore x = 2t,\ y = t,\ z = 3t$

$dx = 2dt,\ dy = dt,\ dz = 3\ dt$

When $x = 0$, $t = 0$, When $x = 2$, $t = 1$

$\therefore$ t varies from 0 to 1.

$$\text{Work done } = \int_{t=0}^{1} 3\left(4t^2\right)(2dt) + \left(12t^2 - t\right)dt + 9tdt$$

$$= \int_0^1 \left(24t^2 + 12t^2 - t + 9t\right)dt = \int_0^1 \left(36t^2 + 8t\right)dt = \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1 = 12 + 4 = 16$$

(ii) Along the curve $x^2 = 4y$, $3x^3 = 8z$

For parametric equation of the curve, we take $x = t \therefore y = \frac{t^2}{4}, z = \frac{3t^3}{8}$

$$\therefore dx = dt,\ dy = \frac{tdt}{2},\ dz = \frac{9t^2}{8}dt.$$

When $x = 0$, $t = 0$ $\quad\therefore t$ varies from 0 to 2.

When $x = 2$, $t = 2$

$$\therefore \text{ Work done } = \int_0^2 \left(3t^2\right)dt + \left(\frac{3}{4}t^4 - \frac{t^2}{4}\right)\frac{t}{2}dt + \left(\frac{3}{8}t^3\right)\left(\frac{9}{8}t^2\right)dt$$

$$= \int_0^2 \left(3t^2 + \frac{3}{8}t^5 - \frac{t^3}{8} + \frac{27}{64}t^5\right)dt = \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right)dt$$

$$= \left[t^3 - \frac{t^4}{32} + \frac{51}{64}\cdot\frac{t^6}{6}\right]_0^2 = 8 - \frac{1}{2} + \frac{51}{6} = 8 - \frac{1}{2} + \frac{17}{2} = 8 + \frac{16}{2} = 16$$