2. 4 : Symmetric, Skew Symmetric And Orthogonal Matrices

A square matrix A is said to be symmetric about the principal diagonal, if the elements of 'A' on one side of the principal diagonal are the reflected images of the elements of 'A' on the side of the principal diagonal. In other words

A real square matrix $A = \left[a_{ij}\right]$ is called **symmetric** if $A^T = A \Rightarrow a_{ji} = a_{ij}$

A square matirx, A = [a_{ij}] is **skew-symmetric** if $A^T = -A \Rightarrow a_{ji} - -a_{ij}$

A square matrix A is called **orthogonal** if $A^TA = AA^T = I$ or

Orthogonal if $A^T = A^{-1}$

Properties of symmetric and skew-symmetric :

The transpose of A i.e. $A^T = \begin{cases} A \text{ when } A \text{ is symmetric} \\ -A \text{ when } A \text{ is skew symmetric} \end{cases}$

(i) If A is any square matrix then the matrix ($A + A^T$) is a symmetric matrix and the matrix ($A–A^T$) is a skew-symmetric matrix.

(ii) If A and B are two symmetric (or skew-symmetric matrices of same order, then (A + B) is also symmetric (or skew-symmetric)

The matrices $\begin{bmatrix} 2 & -3 & 7 \\ -3 & 9 & 8 \\ 7 & 8 & 11 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & -7 \\ -5 & 7 & 0 \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

are **symmetric, skew-symmetryc** and **orthogonal** matrices as you should verify.

Theorem - 1 (Eigen values of symmetric and skew-symmetic matrices)

(a) The eigen values of symmetric matrix are real

(b) The eigen values of a skew symmetric matrix are purely imaginary or zero.

Proof of this theorem will be discussed in the next section

Orthogonal Transformation and Matrices

The linear transformation Y = AX where the coefficient matrix A is an orthogonal matrix is called orthogonal transformation. If it transforms $\sum_{i=1}^{n} y_i^2$ into $\sum_{i=1}^{n} x_i^2$

The matrix of an orthogonal matrix is an orthogonal matrix $X^TX = [x_1, x_2, ..., x_n]$ $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=-1}^{n} x_i^2$ and $Y^TY = [y_1, y_2, ..., y_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=-1}^{n} y_i^2$

Y = AX where $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an example of orthogonal transformation.

By the system Y = AX we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \end{pmatrix}$$

$\Rightarrow y_1 = x_1\cos\theta + x_2\sin\theta$ and $y_2 = -x_1\sin\theta + x_2\cos\theta$

squaring and adding.

$\Rightarrow y_1^2 + y_2^2 = x_1^2 + x_2^2$

So it follows that the linear transformation Y = AX is said to be **orthogonal** if it transforms $y_1^2 + y_2^2$ into $x_1^2 + x_2^2$.

On generalising this we can say the linear transformation Y = AX where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_{11}\ a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21}\ a_{22} \ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\ a_{n2} \ldots\ldots\ldots a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$$

is said to be orthogonal if it transforms $y_1^2 + y_2^2 + y_3^2 + \ldots + y_n^2$ into $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$

Theorem - 2

The orthogonal transformation preserves the value of the inner product of vectors

Proof : Let U = Aa and V = Ab be two orthogonal transformations. We are prove that U.V. = a.b

Let U and V be two column matrices, then U.V =

$$U^T V = (Aa)^T Ab = a^T A^T Ab = a^T Ib = a^T b = (a.b)$$

Since A is an orthogonal matrix so $A^T T = A^{-1}A = I$ as $A^T = A^{-1}$ so the orthogonal transformaiton preserves the value of the inner product.

Orthonormal System

A system of column vectors $a_1, a_2, \ldots, a_n$ forms an orthonormal system

if $a_i.a_j = a_i^T a_j = \begin{cases} 0 & for\ i \neq j \\ 1 & for\ i = j \end{cases}$ ... (1)

Similarly a system of row vectors $a_1, a_2 \ldots a_n$ forms an orthonormal system if $(a_i a_j) = a_i a_j^T$

$= \begin{cases} 0\ for\ i \neq j \\ 1\ for\ i \neq j \end{cases}$

Theorem – 3 : (Orthogonality of column and row vectors)

A real square matrix is orthogonal if and only if its column vectors (and also its row vectors) from an orthonormal system i.e. $(a_i, a_j) = a_i^T a_j = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases}$ ... (1)

Necessary Part :

Let A be orthogonal matrix. We are to show that the system of column vectors $a_1, a_2, \ldots a_n$ form an orthonormal system.

Since A is an orthogonal matrix, So $A^{-1}A = A^T A = 1$

Since $a_1, a_2, a_3, \ldots a_n$ are the column vectors of A

So $A=[a_1,a_2,.....a_n]\Rightarrow A^T=\begin{bmatrix}a_1^T\\a_2^T\\.\\.\\a_n^T\end{bmatrix}$ $\quad\therefore A^{-1}A=A^TA=\begin{bmatrix}a_1^T\\a_2^T\\.\\.\\a_n^T\end{bmatrix}(a_1a_2...........a_n)$

$$=\begin{bmatrix}a^Ta_1 & a_1^Ta_2...........a_1^Ta_n\\a_2^Ta_1 & a_2^Ta_2...........a_2^Ta_n\\ \cdots & \cdots\\ \cdots & \cdots\\a_n^Ta_1 & a_n^Ta_2..........a_n^Ta_n\end{bmatrix}=I=\begin{bmatrix}1\ 0\ ....0\\0\ 1...................0\\ \cdots\\ \cdots\\0\ 0\ ....1\end{bmatrix}......(2)$$

Which shows that $a_i^Ta_j=\begin{cases}0 & for\ i\neq j\\1 & for\ i=j\end{cases}$

Hence the system of column vectos $a_1, a_2,.....,a_n$ form an orthonormal system

Sufficient Part :

If the system of column vectors $a_1, a_2,.....a_n$ of A forms an orthonormal system, then A is orthogonal. If the column vectors of A satisfy (1), then the off diagonal entries in (2) must be 0 and the diagonal entries 1. Hence $A^TA=I$ as (2) shows. Similarly $AA^T=I\Rightarrow A^T=A^{-1}$ because also $A^TA=A^{-1}A=I$ and the inverse is unique. Hence A is orthogonal.

Similarly we can show that A is orgthogonal if and only if the system of row vectors of A form a system of orthonormal system.

Theorem – 4 : (Determinant of an orthogonal matrix)

The determinant of an orthogonal matrix has the value 1 or –1.

Proof : We know by property 2 of properties of determinant that if A and B be two square matrices of same order, then det AB = det A. det B

$\Rightarrow|AB|=|A|.|B|$

and $\Rightarrow|A^T|=|A|$. we get for an orthogonal matrix.

$1=|I|=|AA^{-1}|=|AA^T|=|A|.|A^T|=|A|^2\ \Rightarrow|A|=\pm1$

Theorem – 5 : (Eigen values of an orthogonal matrix)

The eigen values of an orthogonal matrix A are real or complex conjugates in pairs and have absolute value 1. The proof of this theorem will be discussed in the next section.

Illustrative Examples

Verify the following matrices symmetric, skew - symmetric, or orthogonal ? Find their eigen values.

Example – 1: (a) $\begin{bmatrix}a & b\\-b & a\end{bmatrix}$ **(b)** $\begin{bmatrix}1 & 0 & 0\\0 & \cos\theta & -\sin\theta\\0 & \sin\theta & \cos\theta\end{bmatrix}$ **(c)** $\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$

Solution: (a) Here $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \neq A \neq -A$

Now, $A.A^T = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} = \left(a^2+b^2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Conditions : If a = 0 then A is a skew symmetric matrix.
If b = 0 then A is a symmetric matrix.
If $a^2+b^2=1$ then A is an orthogonal matrix

Let λ be some unknown scalar for which $AX = \lambda X$

$$\Rightarrow (A - \lambda I)X = 0$$

So the characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)^2 + b^2 = 0$$

$$\Rightarrow (a-\lambda)^2 = -b^2 \Rightarrow \boxed{\lambda = a \pm bi}$$

Hence the eigen values of A are $\lambda = a \pm ib$.

(b) Here A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, $A^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \neq A \neq -A$

Now $A.A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\Rightarrow \boxed{A^T = A^{-1}}$

$\Rightarrow$ A is an orthogonal matrix

Let λ be some unknown scalar for which $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

So the characteristics equation is $(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & \cos\theta-\lambda & -\sin\theta \\ 0 & \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\left\{(\cos\theta-\lambda)^2 + \sin^2\theta\right\} = 0 \Rightarrow (1-\lambda) = 0$ or $\left\{(\cos\theta-\lambda)^2 + \sin^2\theta\right\} = 0$

$\Rightarrow \lambda = 1 \quad (\cos\theta - \lambda)^2 = -\sin^2\theta$

or $(\cos\theta - \lambda) = \pm i\sin\theta$ or $\lambda = e^{\pm i\theta}$

$\therefore \lambda = 1 \; or \; e^{\pm i\theta}$

Hence the eigen values of A are $\lambda = 1 \; or \; e^{\pm i\theta}$.

(c) Here A = $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

There fore $A^T A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

and

$$AA^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{A}$$

Here A is orthogonal matrix.

Example – 2: ***Show that the main diagonal entries of a skew - symmetric matrix must be zero.***

Solution: We know that the square matrix $A = \left[a_{ij}\right]$ is said to be skew-symmetric if $a_{ij} = -a_{ij}$ for all i, j

Putting j = i we obtain $a_{ij} = -a_{ij}$

$\Rightarrow 2a_{ij} = 0 \Rightarrow \boxed{a_{ii} = 0}$

Hence the main diagonal entries of a skew symmetric matrix are zero.

Example – 3: ***Prove that eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal.*** ***[B.P.U.T. – 2016***

Proof : Let λ_i and λ_j be two different eigen values of a symmetric matrix A.

So $\lambda_i - \lambda_j \neq 0$

Let X_i and X_j be the corresponding column matrices of eigen vectors.

We are to prove that X_i and X_j are orthogonal

i.e. $X_i^T X_j = 0$ since $i \neq j$

Since X_i and X_j are nontrivial solutions of AX = λ X.

So we have $AX_i = \lambda_i X_i$ and $AX_j = \lambda_j X_j$

Since $X_i^T X_j$ is (1 x 1) element so $X_i^T X_j = \left(X_i^T X_j\right)^T = X_j^T X_i$

Again $\left(AX_i\right)^T = \left(\lambda_i X_i\right)^T \Rightarrow X_i^T A^T = \lambda_i X_i^T$

Hence $\lambda_i X_i^T X_j = X_i^T A^T X_j = X_i^T AX_j$ Since A is symmetric matrix.

$\Rightarrow \lambda_i X_i^T X_j = X_i^T \lambda_i X_j = \lambda_j X_i^T X_j \;\; \Rightarrow \lambda_i X_i^T X_j = -\lambda_j X_i^T X_j = 0$

$\Rightarrow \left(\lambda_i - \lambda_j\right)\left(X_i^T X_j\right) = 0 \qquad \Rightarrow X_i^T X_j = 0$ since $\lambda_i - \lambda_j \neq 0$

Hence X_i and X_j are orthogonal.

Example – 4 : *Show that the inverse of a skew symmetric matrix is skew symmetric.*
Proof : Let A be a square matrix which is skew symmetric

i.e. $A^T = -A$

Proof will be complete if we can show that $\left(A^{-1}\right)^T = -A^{-1}$

Since we have $A^T = -A$

$$\Rightarrow \left(A^T\right)^{-1} = (-A)^{-1} \Rightarrow \left(A^{-1}\right)^T = -A^{-1}$$

Since $\Rightarrow \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Hence A^{-1} is also skew symmetric.

2.5 : Complex Matrices : Hermitian, Skew Hermitian, Unitary

Complex matrix :

If all the elements of a matrix are real numbers, then it is called a real matrix over R. On the other hand, if atleast one element of a matrix is a complex number a + ib where a, b $\in R$ and $i = \sqrt{-1}$ then the matrix is called a **complex matrix.**

The matrix obtained by replacing the elements of a complex matrix A by the corresponding conjugate complex numbers is called the conjugate of the matrix A and it is denoted by $\overline{A}$.

Thus if $A = \begin{bmatrix} 4+5i & 5i \\ 7 & 2-3i \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 7 & 2 \end{bmatrix} + i\begin{bmatrix} 3 & 5 \\ 0 & -3 \end{bmatrix} = a + ib$

then $\overline{A} = \begin{bmatrix} 4-5i & -5i \\ 7 & 2+3i \end{bmatrix}$ and $\begin{bmatrix} 4-5i & 7 \\ -5i & 2+3i \end{bmatrix}$ are conjugate and transpose of matrices respectively

Hermitian, Skew-Hermitian, Unitary matrices

A square matrix $A = \left[a_{ij}\right]$ is called **Hermitian** if $\left(\overline{A}\right)^T = A$ *or* $\overline{a}_{ji} = a_{ij}$

Putting j = i we obtain $\overline{a}_{ji} = a_{ij} \Rightarrow a_{ij}$ is real

For example $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ is a Hermitian matrix.

Since a symmetric matrix A is always a real matrix and conjugate of a real number is the number itself.

So $A^T = A \Rightarrow \overline{A}^T = A$. Since $\overline{A} = A$

Every symmetric matrix A is also a Hermitian matrix.

A square matrix $A = \left[a_{ij}\right]$ is called Skew-Hermitian

if $\left(\overline{A}\right)^T = A$ or $\overline{a}_{ij} = -a_{ij}$

Putting $j = i$, we obtain

$\overline{a_{ij}} = -a_{ij} \Rightarrow a_{ij} + \overline{a_{ij}} = 0 \Rightarrow (\alpha + i\beta) + (\alpha - i\beta) = 0$ where $a_{ij} = \alpha + i\beta$

$\Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$

Which implies that a_{ij} is either purely imaginary or 0.

For example $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ is a skew Hermitian matrix

Since a skew - symmetric matrix A is always a real matrix and conjugate of a real number is the number itself. So it follows that if a square matrix A is skew symmetric then

$A^T = -A \Rightarrow \overline{A}^T = A^T = -A \Rightarrow A$ is skew Hermitian

Hence every skew symmetric matrix A is always a skew Hermitian matrix.

Unitary Matrix :

A square matrix $A = [a_{ij}]$ is called unitary if $\left(\overline{A}^T\right) = A^{-1}$

Since an orthogonal matrix A is always a real matrix. The conjugate of a real number is the number itself. So if a square matrix A is orthogonal then $A^T = A^{-1}$

$\Rightarrow \left(A^{-T}\right) = A^T = A^{-1} \because$ A is real

$\Rightarrow$ A is unitary matrix.

So every orthogonal matrix is always a unitary matrix.

For example $\begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$ is a unitary matrix.

Conversely if a Hermitian matrix is real, then $\overline{A}^T = A^T = A$

Hence a real Hermitian matrix is a symmetric matrix.

Similarly if a skew hermitian matrix is real then $\left(\overline{A}\right)^T = A^T = -A$. Hence a real skew Hermitian matrix is a skew symmetric matrix.

Finally if a unitary matrix is real then $\left(\overline{A}\right)^T = A^T = A^-$.

Hence a real unitary matrix is an orthogonal matrix.

This shows that Hermitian, skew-Hermitian, unitary matrices generalize symmetric, skew symmetric and orthogonal matrices respectively.

2.6 : Quadratic Form

If we take X as a column matrix with components x_1, x_2 and a coefficient matrix A of order 2 × 2 , then $Q = X^T AX$ is called a quadratic form. i.e. $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

We get $Q = X^T AX = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$

$$= a_{11}x_1^2 + a_{12}x_2 + a_{21}x_2x_1 + a_{22}x_2^2 = a_{11}x_1^2 + \left(a_{12} + a_{21}\right)x_{12} + a_{22}x_2^2$$

Similarly for general if we take X as a column matrix with n components $x_1, x_2, x_3, ..., x_n$ and the coefficient matrix A of order (n x n) then we get a quadratic form in the n variables $x_1, x_2, ...x_n$.

$$Q = X^T AX = \begin{bmatrix} x_1 & x_2 & x_3 ......x_n \end{bmatrix} \begin{bmatrix} a_{11}\ a_{12} ....a_{1n} \\ a_{21}\ a_{22} ....a_{2n} \\ .... \\ .... \\ a_{n1}\ a_{n2} ....a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

$$= a_{11}x_1^2 + \left(a_{12} + a_{21}\right)x_1x_2 + ... + \left(a_{1n} + a_{n1}\right)x_1x_n$$
$$+ a_{22}x_2^2 + ... + \left(a_{2n} + a_{n2}\right)x_2x_n + ... + a_{nn}x_n^2$$

Now let us $\dfrac{a_{ij} + a_{ji}}{2} = c_{ij}$

Clearly, $c_{ji} = c_{ij} \Rightarrow c_{ij} + c_{ji} = a_{ij} + a_{ji}$

Now $C = [C_{ij}]_{n \times m} = \dfrac{A + A^T}{2}$ is a symetric matrix

Now the quadratic form can be modified as

$$Q = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}X_iX_j$$

In short, we can write

$$Q = X^T CX \text{, where } C = \frac{A + A^T}{2} \quad \& \quad X^T = [x_1, x_2, ..., x_n]$$

A homogeneous polynomial of second degree in any number of variable is called a quadratic form e.g.

(a) $ax^2 + 2hxy + by^2 = [x\ y] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(b) $ax^2 + by^2 + cz^2 + 2\,hxy + 2\,gyz + 2\,fzx$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(c) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gxy + 2fzx + 2lxw + 2myw + 2nzw$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & f & l \\ h & b & g & m \\ f & g & c & n \\ l & m & n & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Note : (1) Every quadratic form in $X_1, X_2, \ldots. X_n$ can be expressed in form $Q = X^T CX$ where $C = \dfrac{A + A^T}{2}$

(2) If A is real, then the extreme $Q = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} X_i X_j$ is called a real quadratic form.

If the quadratic form be real, then the matrix A is a real symmetric matrix and is known as the matrix of the real quadratic form.

A real qudratic form of n variables $x_1, x_2, \ldots, x_n$ is sometimes written as $Q(x_1, x_2 \ldots, x_n)$ and is expressed as $Q = X^T AX$

So that it assumes the value 0 , when X =0 .

But, for different non-zero X, Q takes up different real values.

A real quadratic form $Q = X^T AX$ is said to be

(i) positive , definte, if Q>0 , for all $X \neq 0$;

(ii) positive, semi-definite, if $Q \geq 0$, for all X and Q = 0 for some $X \neq 0$;

(iii) negative definite, if Q < 0, for all $X \neq 0$;

(iv) negative semi-definite, if $Q \leq 0$, for all X and Q = 0 for some $X \neq 0$

(v) indefinite, if $Q \geq 0$, for some non-zero X and $Q \leq 0$, for some other non-zero X.

The associated real symetric matrix A is said to be positive definite, positive semi definite, negative definite, negative semi-definite and indefinite respectively in the corresponding cases.

Then $c_{ij} = c_{ji}$ and $c_{ij} + c_{ji} = a_{ij} + a_{ji}$ so that we may write $Q = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j$

If A is real, the efficient matrix $C = \left[c_{ij} \right]$ in this new representation is a real symmetric matrix. This shows that any real quadratic form Q in n variables $x_1, x_2, \ldots. x_n$ may be written $Q = X^T CX$ where C is a real symmetric matrix.

Hermitian Form

If X be a column vector of complex unknowns and A be a Hermitian matrix then $H = \left(\overline{X} \right)^T AX$ is called Hermitian form

Theorem – 1 : ***For every choices of the vector X, the value of a Hermitian form $H = \left(\overline{X} \right)^T AX$ where A is a Hermitian matrix, is real.***

Proof : Since Hermitian form $H = \overline{X}^T \; AX$

$$\Rightarrow \overline{H} = \left(\overline{\overline{X}^T \; AX} \right) = X^T \overline{AX} = X^T A^T \overline{X}$$

Since A is a Hermitian matrix.

$$\Rightarrow \overline{H} = \left(\overline{H}^T A^T \overline{X} \right) = \left(X^T A^T \overline{X} \right)^T = \overline{X}^T AX = H$$

Since $X^T A^T \overline{X}$ is a scalar which is (1×1) element

$\Rightarrow \overline{H} = H \;\; \Rightarrow H$ is real.

Skew Hermitian Form

If X be a column vector of complex unknowns and A be a skew hermitian matrix, then $Q = \overline{X}^T AX$ is called skew Hermitian form

Theorem – 2 : ***For every choice of the vector X the value of a skew - Hermitian form is purely imaginary or 0.***

Proof : Since the skew Hermitian form is

$Q = \overline{X}^T AX$ where A is skew Hermitian.

$\Rightarrow \overline{Q} = \left(\overline{\overline{X}^T AX}\right) = X^T \overline{A}\overline{X} = X^T\left(-A^T\right)\overline{X}$ Since $\overline{A} = -A^T$

$\Rightarrow \overline{Q} = -\left(X^T A^T \overline{X}\right) = -\left(X^T A^T \overline{X}\right)^T$ Since $X^T A^T \overline{X}$ is a scalar whichis 1 x 1 element

$\Rightarrow \overline{Q} = -\left(\overline{X}^T AX\right) = -Q$

$\Rightarrow Q + \overline{Q} = 0$ which implies that Q is purely imaginary or 0.

Theorem – 3: (Eigen Values) : Prove that eigen values of skew Hermitian matrix are purely imaginary or 0.

Proof: Let λ be the eigen value of A. Then by definition there is a vector $X \neq 0$ such that $AX = \lambda X$

From this we obtain $\overline{X}^T = AX = \overline{X}^T \lambda X = \lambda \overline{X}^T X$

Since $X \neq 0$ it follows that $\overline{X}^T X \neq 0$ and we may divide, finding $\lambda = \dfrac{\overline{X}^T AX}{\overline{X}^T X}$

The denominator $\overline{X}^T X$ is real

If A is Hermitian, then $\overline{X}^T X$ is real and for which λ is real.

If A is skew Hermitian, then the numerator $\overline{X}^T X$ is purely imaginary or 0 and for which λ is purely imaginary or 0.

Theorem – 4 : The eigen values of a unitary matrix have the absolute value - 1.

Unitary System

A system of column vectors $X_1, X_2, ...X_n$ for which

$$\overline{X}_i^T X_j = \begin{cases} 0 \text{ when } i \neq j \\ 1 \text{ when } i = j \end{cases} (i, j = 1, 2......n).....(1)$$

is called a unitary system.

Clearly if these vectors are real, then $\overline{X}_i^T X_j = X_i^T X_j$ is the innerproduct (dot product) of X_i and X_j in the elementary sense and condition (1) means that $X_1, X_2, ...X_n$ are orthogonal unit vectors. Hence a unitary system of real vectors is a system of orthogonal (mutually perpendicular) unit vectors.

Theorem – 5 (Unitary Matrix) :

The column vectors (and also the row vectors) of a unitary matrix form a unitary system.

Proof : Let A be a unitary matrix with column vectors $a_1, a_2, ...a_n$

Then $$A^{-1}A = \overline{A}^T A = \begin{bmatrix} \overline{a_1}^T \\ \overline{a_2}^T \\ . \\ . \\ \overline{a_n}^T \end{bmatrix} \begin{bmatrix} a_1 a_2 ......a_n \end{bmatrix} = \begin{bmatrix} \overline{a_1}^T a_1 & \overline{a_1}^T a_2 ....\overline{a_1}^T a_n \\ \overline{a_2}^T a_1 & \overline{a_2}^T a_2 ....\overline{a_2}^T a_n \\ .... \\ .... \\ \overline{a_n}^T a_1 & \overline{a_n}^T a_2 ....\overline{a_n}^T a_n \end{bmatrix}$$

$$\Rightarrow \overline{a_i}^T a_j = \begin{bmatrix} 0 \text{ when } i \neq j \\ 1 \text{ when } i = j \end{bmatrix}$$

This shows that the column vectors of unitary matrix form a unitary system (Proved)

Illustrative Examples

Verify the following matrices in the given problem are Hermitian, skew Hermitian or unitary and their spectrum. Find their eigen values and eigen vectors.

Example – 1: (a) $\begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1-i & 0 \end{bmatrix}$

Solutions: (a) Let $A = \begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} -i & 1-i \\ -1-i & 0 \end{bmatrix} = -\begin{bmatrix} i & -1+i \\ 1+i & 0 \end{bmatrix}$

and $A^T = \begin{bmatrix} i & -1+i \\ 1+i & 0 \end{bmatrix} = -\overline{A}$

Since $\boxed{A^T = -\overline{A}}$ so A is a skew hermitian matrix

let λ be some unknown scalar for which $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

So the characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} i-\lambda & 1+i \\ -1+i & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(i-\lambda) + 2 = 0 \Rightarrow \lambda^2 - i\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{i \pm \sqrt{-1-8}}{2} = \frac{i \pm 3i}{2} \Rightarrow \boxed{\lambda = 2i, -i}$$

Hence the eigen values of A are $\lambda = 2i, -i$

For $\lambda = 2i, \left[A - (2i)I\right]X = 0 \Rightarrow \begin{bmatrix} -i & 1+i \\ -1+i & -2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow -ix_1 + (1+i)x_2 = 0$ ..... (1) and $(-1+i)x_1 - 2ix_2 = 0$ ..... (2)

$\therefore$ equation (1) $\Rightarrow -ix_1 = -(1+i)x_2 \quad \Rightarrow \boxed{\dfrac{x_1}{1+i} = \dfrac{x_2}{i}}$

$\therefore$ For $\lambda = 2i$, the corresponding eigen vector of A is $X = \begin{bmatrix} 1+i \\ i \end{bmatrix} = \begin{bmatrix} 1+i & i \end{bmatrix}^T$

Again for $\lambda = -i, [A + iI]X = 0 \quad \Rightarrow \begin{bmatrix} 2i & 1+i \\ -1+i & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow 2ix_1 + (1+i)x_2 = 0$ ..... (3) and $(-1+i)x_1 + ix_2 = 0$ ..... (4)

$\therefore$ equation (3) $\Rightarrow 2ix_1 = -(1+i)x_2 \quad \Rightarrow \boxed{\dfrac{x_1}{-1-i} = \dfrac{x_2}{2i}}$

$\therefore$ For $\lambda = -i$ the corresponding eigen vector of A is $X = \begin{bmatrix} -1-i \\ 2i \end{bmatrix} = \begin{bmatrix} -1-i & 2i \end{bmatrix}^T$

(b) Let $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

and $\overline{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = -A^T$

Since $\boxed{A^T = -\overline{A}}$ so A is a skew - hermitian matrix.

Again $\left(\overline{A}\right)^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

Now $\left(\overline{A}\right)^T .A = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\Rightarrow \left(\overline{A}\right)^T = A^{-1} \Rightarrow$ A is an unitary matrix.

Let λ be some unknown scalar for which $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

So the characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (i-\lambda)(\lambda^2+1)=0 \Rightarrow (i-\lambda)=0$ *or* $(\lambda^2+1)=0 \Rightarrow \lambda = i$ *or* $\lambda = \pm i$

$\Rightarrow \boxed{\lambda = i, -i}$

Hence the eigen values of A are $\lambda = i, -i$

$\therefore$ For $\lambda = i$, $[A - iI]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 0.x_1 + 0.x_2 + 0.x_3 = 0$ .... (1)

$0.x_1 - ix_2 + ix_3 = 0$ .... (2)

and $\Rightarrow 0.x_1 + ix_2 - ix_3 = 0$ ...... (3)

$\therefore$ equation (2) $\Rightarrow -ix_2 = -ix_3$

$\Rightarrow \boxed{\frac{x_2}{1} = \frac{x_3}{1}}$ taking $x_1 = 0$ we get an eigen vector corresponding to $\lambda = i$ as

$$X = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = [0 \ 1 \ 1]^T$$

Again for $\lambda = -i$, $[A + iI]X = 0$

$$\Rightarrow \begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 2ix_1 + 0.x_2 + 0.x_3 = 0$ ..... (4) and $0.x_1 + i.x_2 + i.x_3 = 0$ ... (5)

$\therefore$ equation (4) $\Rightarrow \boxed{x_1 = 0}$

equation (5) = $\Rightarrow ix_2 = -ix_3 \Rightarrow x_2 = -x_3 \Rightarrow \boxed{\frac{x_2}{-1} = \frac{x_3}{1}}$

For $\lambda = -i$, the corresponding eigen vector of A is $X = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = [0 \ -1 \ 1]^T$

(c) Let $A = \begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1-i & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 1-i & 0 \\ 1+i & 0 & 1-i \\ 0 & 1+i & 0 \end{bmatrix}$

and $\Rightarrow \overline{A} = \begin{bmatrix} 0 & 1-i & 0 \\ 1+i & 0 & 1-i \\ 0 & 1+i & 0 \end{bmatrix} = A^T$

Since $\boxed{A^T = \overline{A}}$ so A is a hermitian matrix.

Let λ be some unknown scalar for which $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

So the characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -\lambda & 1+i & 0 \\ 1-i & -\lambda & 1+i \\ 0 & 1-i & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)\{\lambda^2 - 2\} + (1+i)\lambda(1-i) = 0 \Rightarrow -\lambda^2 + 4\lambda = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 4) = 0 \Rightarrow \lambda = 0 \;\; or \;\; \lambda^2 = 4$$

$$\Rightarrow \boxed{\lambda = 0, \pm 2}$$

Hence the eigen values of A are $\lambda = 0, 2, -2$

For $\lambda = 0$, $[A - 0.I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1+i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1-i & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 0.x_1 + (1+i)x_2 + 0.x_3 = 0$ ...... (1)

$(1-i)x_1 + 0.x_2 + (1+i)x_3 = 0$ .... (2)

and $0.x_1 + (1-i)x_2 + 0.x_3 = 0$ ..... (3)

$\therefore$ equation (1) $\Rightarrow \boxed{x_2 = 0}$

equation (2) $\Rightarrow (1-i)x_1 = -(1+i)x_3$

$$\Rightarrow \boxed{\frac{x_1}{-1-i} = \frac{x_3}{1-i}}$$

For $\lambda = 0$, the corresponding eigen vector of A is $X = \begin{bmatrix} -1-i \\ 0 \\ 1-i \end{bmatrix} = [-1-i \;\; 0 \;\; 1-i]^T$

Further for $\lambda = 2$, $[A - 2I]X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 1+i & 0 \\ 1-i & -2 & 1+i \\ 0 & 1-i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow -2x_1 + (1+i)x_2 + 0.x_3 = 0$ ...... (4)

$(1-i)x_1 - 2x_2 + (1+i)x_3 = 0$ .... (5)

and $0.x_1 + (1-i)x_2 - 2x_3 = 0$ .... (6)

$\therefore$ equation (4) $\Rightarrow -2x_1 = -(1+i)x_2$

$$\Rightarrow \boxed{x_1 = \left(\frac{1+i}{2}\right).x_2} \Rightarrow \frac{x_1}{\left(\frac{1+i}{2}\right)} = \frac{x_2}{1} \Rightarrow \boxed{\frac{x_1}{i} = \frac{x_2}{1+i}}$$

equation (5) $\Rightarrow (1-i)\left(\frac{1+i}{2}\right)x_2 - 2x_2 + (1+i)x_3 = 0$

$$\Rightarrow -x_2 = -(1+i)x_3 \Rightarrow \boxed{\frac{x_2}{1+i} = \frac{x_3}{1}} \quad \therefore \boxed{\frac{x_1}{i} = \frac{x_2}{1+i} = \frac{x_3}{1}}$$

For $\lambda = 2$, the corresponding eigen vector of A is $X = \begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} i & 1+i & 1 \end{bmatrix}^T$

Again for $\lambda = -2, [A+2I]X = 0$

$$\Rightarrow \begin{bmatrix} 2 & 1+i & 0 \\ 1-i & 2 & 1+i \\ 0 & 1-i & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 2x_1 + (1+i)x_2 + 0.x_3 = 0$ ..... (7)

$(1-i)x_1 + 2x_2 + (1+i)x_3 = 0$ ...... (8)

and $0.x_1 + (1-i)x_2 + 2x_3 = 0$ .... (9)

$\therefore$ equation $(7) \Rightarrow 2x_1 = -(1+i)x_2$

$$\Rightarrow \boxed{x_1 = \frac{-1(1+i)}{2}x_2} \Rightarrow \boxed{\frac{x_1}{-(1+i)} = \frac{x^2}{2}}$$

equation (8) $\Rightarrow -(1-i)\frac{(1+i)}{2}x_2 + 2x_2 + (1+i)x_3 = 0$

$$\Rightarrow x_2 = -(1+i)x_3 \Rightarrow \frac{x_2}{2} = \frac{-(1+i)x_3}{2} = \frac{x_3}{\left(\frac{-2}{1+i}\right)}$$

$$\Rightarrow \boxed{\frac{x_2}{2} = \frac{x_3}{\left(\frac{-2}{1+i}\right)}} \quad \therefore \boxed{\frac{x_1}{-(1+i)} = \frac{x_2}{2} = \frac{x_3}{\left(\frac{-2}{1+i}\right)}}$$

For $\lambda = -2$,the corresponding eigen vector of A is

$$\begin{bmatrix} -(1+i) \\ 2 \\ \left(\frac{-2}{1+i}\right) \end{bmatrix} = \left[-(1+i) \;\; 2 \left(\frac{-2}{1+i}\right) \right]^T$$

$$= \left[-(1+i) \;\; 2 \;\; -2(1-i) \right]^T = \left[(1+i) \;\; -2 \;\; 2(1-i) \right]^T$$

Example – 2 : ***Decomposition***

Show that any square matrix may be written as the sum of a Hermitian matrix and a skew Hermitian matrix. Give an example.

Solution : Let A be any square matrix.

then $A = \left(\frac{A + \overline{A}^T}{2}\right) + \left(\frac{A - \overline{A}^T}{2}\right)$

$= B + C$ where $B = \left(\frac{A + \overline{A}^T}{2}\right)$ and $C = \left(\frac{A - \overline{A}^T}{2}\right)$

Since $\overline{B}^T = \overline{\left(\frac{A + \overline{A}^T}{2}\right)}^T = \left(\frac{\overline{A} + A^T}{2}\right)^T = \frac{\overline{A}^T + \left(A^T\right)^T}{2} = \frac{\overline{A}^T + A}{2} = B$

Which shows that B is a Hermitian matrix.

Again $C = \left[\frac{A - \overline{A}^T}{2}\right]$

$\Rightarrow \overline{C}^T = \overline{\left(\frac{A - \overline{A}^T}{2}\right)}^T = \left(\frac{\overline{A} - \left(A^T\right)}{2}\right)^T = \frac{\overline{A}^T - \left(A^T\right)^T}{2} = \frac{\overline{A}^T - A}{2} = (-C)$

Which shows that C is a skew Hermitian matrix.

Hence every square matrix A can be expressed as the sum of a Hermitian and a skew Hermitian matrix.

For example A be a square matrix given by $A = \begin{bmatrix} 1+2i & 3+4i \\ 5+6i & 7+8i \end{bmatrix}$

$\therefore B = \left(\frac{A + \overline{A}^T}{2}\right) = \begin{bmatrix} 1 & 4-i \\ 4+i & 7 \end{bmatrix}$ and $C = \left(\frac{A - \overline{A}^T}{2}\right) = \begin{bmatrix} 2i & -1+5i \\ 1+5i & 8i \end{bmatrix}$

Hence $A=\left(\dfrac{A+\overline{A}^T}{2}\right)+\left(\dfrac{A-\overline{A}^T}{2}\right)$

$\Rightarrow A=C+D$

$$\Rightarrow\begin{bmatrix}1+2i & 3+4i\\ 5+6i & 7+8i\end{bmatrix}=\begin{bmatrix}1 & 4-i\\ 4+i & 7\end{bmatrix}+\begin{bmatrix}2i & -1+5i\\ 1+5i & 8i\end{bmatrix}$$

where $B=\begin{bmatrix}1 & 4-i\\ 4+i & 7\end{bmatrix}$ is a Hermitian matrix and $C=\begin{bmatrix}2i & -1+5i\\ 1+5i & 8i\end{bmatrix}$ is a skew hermitian matrix.

Example – 3 : ***Express*** $\mathbf{Q=x_1^2-4x_1x_2+7x_2^2}$ ***in quadratic form.***

Solution : The given expression $x_1^2-4x_1x_2+7x_2^2$

$$=\left[x_1x_2\right]\begin{bmatrix}1 & -4\\ 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$=x_1^2-4x_1x_2+0x_2x_1+7x_2^2$$

$$\mathrm{A}=\begin{pmatrix}1 & -4\\ 0 & 7\end{pmatrix}\Rightarrow \mathrm{A}^{\mathrm{T}}=\begin{pmatrix}1 & 0\\ -4 & 7\end{pmatrix}$$

$\because \quad \mathrm{C}=\dfrac{\mathrm{A}+\mathrm{A}^{\mathrm{T}}}{2}=\begin{pmatrix}1 & -2\\ -2 & 7\end{pmatrix}$ is the required symmetric matrix.

Or, $\quad \mathrm{Q}=x_1^2-4x_1x_2+7x_2^2=x_1^2-2x_1x_2-2x_2x_1+7x_2^2$

$$\because \quad \mathrm{C}=\begin{pmatrix}1 & -2\\ -2 & 7\end{pmatrix}$$

Example – 4 : ***Express*** $\boldsymbol{Q=x_1^2+2x_1x_2+3x_2^2+6x_2x_3+2x_3^2}$ ***in quadratic form.***

Solution : The given expression $x_1^2+2x_1x_2+3x_2^2+6x_2x_3+2x_3^2$

$$=\left[x_1x_2x_3\right]\begin{bmatrix}1 & 2 & 0\\ 0 & 3 & 6\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$=x_1^2+2x_1x_2+0x_1x_3+0x_2x_1+3x_2^2+6x_2x_3+0x_3x_1+0x_3x_2+2x_3^2$$

$$\mathrm{A}=\begin{pmatrix}1 & 2 & 0\\ 0 & 3 & 6\\ 0 & 0 & 2\end{pmatrix},\quad \mathrm{A}^{\mathrm{T}}=\begin{pmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 0 & 6 & 2\end{pmatrix}$$

$\mathrm{C}=\dfrac{\mathrm{A}+\mathrm{A}^{\mathrm{T}}}{2}=\begin{pmatrix}1 & 1 & 0\\ 1 & 3 & 3\\ 0 & 3 & 2\end{pmatrix}$ is the required symmetric matrix.

Example – 5 : ***Find the symmetric matrix C of the quadratic form $Q = X^TCX$ given by***

(a) $4x_1^2 - 8x_1x_2 + 5x_2^2$

(b) $\left(x_1 - x_2 + 4x_3\right)^2 - 4\left(x_2 - x_4\right)^2$

Solution : (a) $4x_1^2 - 8x_1x_2 + 5x_2^2$

Let $X^T AX = 4x_1^2 - 8x_1x_2 + 5x_2^2$

$$\Rightarrow X^T AX = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 4 & -8 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here $A = \begin{bmatrix} 4 & -8 \\ 0 & 5 \end{bmatrix}$

$\therefore$ The symmetric coefficient matrix $C = c_{ij}$ can be found out by $c_{ij} = c_{ji} = \dfrac{a_{ij} + a_{ji}}{2}$

So $C = \begin{bmatrix} 4 & -4 \\ -4 & 5 \end{bmatrix}$ for which $X^T AX = X^T CX$

(b) $\left(x_1 - x_2 + 4x_3\right)^2 - 4\left(x_2 - x_4\right)^2$

Let $X^T AX = \left(x_1 - x_2 + 4x_3\right)^2 - 4\left(x_2 - x_4\right)^2$

$$= \left(x_1 - x_2\right)^2 + 16x_3^2 + 8\left(x_1 - x_2\right)x_3 - 4\left(x_2^2 + x_4^2 - 2x_2x_4\right)$$

$$= x_1^2 + x_2^2 - 2x_1x_2 + 16x_3^2 + 8x_1x_3 - 8x_2x_3 - 4x_2^2 - 4x_4^2 + 8x_2x_4$$

$$\Rightarrow X^T AX = x_1^2 - 3x_2^2 - 2x_1x_2 + 16x_3^2 + 8x_1x_3 - 8x_2x_3 - 4x_4^2 + 8x_2x_4$$

$$\Rightarrow X^T AX = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 8 & 0 \\ 0 & -3 & -8 & 8 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & -2 & 8 & 0 \\ 0 & -3 & -8 & 8 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

So the symmetric coefficient matrix C is given by

$C = \begin{bmatrix} 1 & -1 & 4 & 0 \\ -1 & -3 & -4 & 4 \\ 4 & -4 & 16 & 0 \\ 0 & 4 & 0 & -4 \end{bmatrix}$ for which $X^T AX = X^T CX$

Example – 6 : ***Verify Hermitian or Skew Hermitian ? Find*** $\overline{X}^T AX$.
from the following matrices

(a) $\mathbf{A} = \begin{bmatrix} i & 1+i \\ -1+i & -2i \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 2 \\ i \end{bmatrix}$ **(b)** $\mathbf{A} = \begin{bmatrix} -i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Solution : (a) Here $A = \begin{bmatrix} i & 1+i \\ -1+i & -2i \end{bmatrix}, X = \begin{bmatrix} 2 \\ i \end{bmatrix} \Rightarrow \overline{X} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} i & -1+i \\ 1+i & -2i \end{bmatrix}$$

and $\overline{A} = \begin{bmatrix} -i & 1-i \\ -1-i & 2i \end{bmatrix} = -\begin{pmatrix} i & -1+i \\ 1+i & -2i \end{pmatrix} = -A^T$

Since $\boxed{A^T = -\overline{A}}$ so A is a skew - hermitian matrix

Again $\overline{X}^T AX = [2 \ \ -i]\begin{bmatrix} i & 1+i \\ -1+i & -2i \end{bmatrix}\begin{bmatrix} 2 \\ i \end{bmatrix}$

$$= \left[2i - i(-1+i) \quad 2(1+i) - 2\right]\begin{bmatrix} 2 \\ i \end{bmatrix} = [3i+1 \ \ 2i]\begin{pmatrix} 2 \\ i \end{pmatrix}$$

$$= \left[2(3i+1) - 2\right]$$

= [6i] which is imaginary

$$\Rightarrow \overline{X}^T \ AX = [6i]$$

(b) Here $A = \begin{bmatrix} -i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}, X = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \overline{X} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} -i & -1 & -2+i \\ 1 & 0 & 3i \\ 2+i & 3i & i \end{bmatrix}$$

and $\overline{A} = \begin{bmatrix} i & 1 & 2-i \\ -1 & 0 & -3i \\ -2-i & -3i & -i \end{bmatrix} = -\begin{bmatrix} -i & -1 & -2+i \\ 1 & 0 & 3i \\ 2+i & 3i & i \end{bmatrix} = -A^T$

Since $\boxed{A^T = -\overline{A}}$ so A is a skew hermitian matrix.

$$\text{Again } \left(\overline{X}\right)^T AX = \begin{bmatrix}0 & 1 & 2\end{bmatrix}\begin{bmatrix}-i & 1 & 2+i\\ -1 & 0 & 3i\\ -2+i & 3i & i\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$=\begin{bmatrix}-1+2(-2+i) & 6i & 5i\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}-5+2i & 6i & 5i\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$=\begin{bmatrix}6i & +10i\end{bmatrix}$$

= [16i] which is imaginary.

(Unitary Eigenbasis)

Example – 7 : ***Find a basis of eigen vectors that form a unitary system ?***

$$\textbf{(a)} \begin{bmatrix}\mathbf{0} & \mathbf{3i}\\ \mathbf{-3i} & \mathbf{0}\end{bmatrix} \qquad \textbf{(b)} \begin{bmatrix}\mathbf{4} & \mathbf{1+i}\\ \mathbf{1-i} & \mathbf{4}\end{bmatrix}$$

Solution : Here $A = \begin{bmatrix}0 & 3i\\ -3i & 0\end{bmatrix}$

So the characterisitc equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix}-\lambda & 3i\\ -3i & -\lambda\end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 9 = 0 \Rightarrow \lambda^1 = \pm 3$$

Hence the eigen values of A are $\lambda^2 = \pm 3$.

Which are distinct, so the corresponding eigen vectors form a basis of eigen vectors

$\therefore$ for $\lambda = 3$ $\quad [A - 3I]X = 0$

$$\Rightarrow \begin{bmatrix}-3 & 3i\\ -3i & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0$$

$$\Rightarrow -3x_1 - 3ix_2 = 0 \quad .....(1) \quad \text{and } -3ix_1 - 3x_2 = 0 \quad .....(2)$$

$\therefore$ Equation(1) $\Rightarrow -3x_1 = -3ix_2 \Rightarrow -ix_1 = x_2$

$$\Rightarrow \boxed{\frac{x_1}{1} = \frac{x_2}{-i}}$$

$\therefore$ An eigen vector of A corresponding to the value of $\lambda = 3$ is $X_1 = \begin{bmatrix}1\\ -i\end{bmatrix}$

$\Rightarrow$ Normalized eigen vector of A is $X_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{bmatrix}$

Again for $\lambda = -3$, $[A + 3I]X = 0$

$$\Rightarrow \begin{bmatrix} 3 & 3i \\ -3i & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow 3x_1 + 3ix_2 = 0$ .... (3) and $-3ix_1 + 3x_2 = 0$....(4)

$\therefore$ equation (3) $\Rightarrow 3x_1 = -3ix_2 \Rightarrow ix_1 = x_2$

$$\Rightarrow \boxed{\frac{x_1}{1} = \frac{x_2}{i}}$$

$\therefore$ An eigen vector of A corresponding to the value of $\lambda = -3$ is $X_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector of A is $X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$

Since $\overline{X}_1^T X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} + \frac{1}{2} = 1$

$$\overline{X}_1^T X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} + \frac{1}{2} = 0$$

$$\overline{X}_2^T X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\overline{X}_2^T X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

So a basis of eigen vectos that form a unitary system are

$$X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \textit{and } X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

So the unitary matrix $= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$

(b) Here A = $\begin{bmatrix} 4 & 1+i \\ 1-i & 4 \end{bmatrix}$

So the characteristic equation of A is

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & 1+i \\ 1-i & 4-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)^2 - 2 = 0$

$\Rightarrow (4-\lambda)^2 = 2 \Rightarrow (4-\lambda) = \pm\sqrt{2}$

$\Rightarrow \lambda = 4 \pm \sqrt{2}$

Hence the eigen values of A are $\lambda = 4 \pm \sqrt{2}$

Which are distinct, so the corresponding eigen vectors form a basis of eigen vectors

$\therefore$ for $\lambda = 4+\sqrt{2}, \left[A-\left(4+\sqrt{2}\right)I\right]X = 0$

$\Rightarrow \begin{bmatrix} -\sqrt{2} & 1+i \\ 1-i & -\sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow -\sqrt{2}x_1 + (1+i)x_2 = 0$ .... (1) and $(1-i)x_1 - \sqrt{2}x_2 = 0$ .... (2)

$\therefore$ equation (1) $\Rightarrow -\sqrt{2x_1} = -(1+i)x_2$

$\Rightarrow \boxed{\dfrac{x_1}{(1+i)} = \dfrac{x_2}{\sqrt{2}}}$

$\therefore$ an eigen vector of A corresponding to the value of $\lambda = 4+\sqrt{2}$ is $X_1 = \begin{bmatrix} 1+i \\ \sqrt{2} \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector of A is $\begin{bmatrix} \dfrac{1+i}{2} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix}$

Again for $\lambda = 4-\sqrt{2}, \left[A-\left(4-\sqrt{2}\right)I\right]X = 0$

$\Rightarrow \begin{bmatrix} \sqrt{2} & 1+i \\ 1-i & \sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \sqrt{2}x_1 + (1+i)x_2 = 0$ ....(3) and $(1-i)x_1 + \sqrt{2x_2} = 0$ ..... (4)

$\therefore$ equation (3) $\Rightarrow \sqrt{2}x_1 = -(1+i)x_2 \Rightarrow \boxed{\dfrac{x_1}{(1+i)} = \dfrac{x_2}{-\sqrt{2}}}$

$\therefore$ an eigen vector of A corresponding to the value of $\lambda = 4-\sqrt{2}$ is $X_2 = \begin{bmatrix} 1+i \\ -\sqrt{2} \end{bmatrix}$

$\Rightarrow$ Normalized elgen vector of A is $X_2 \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

Since $\overline{X_1}^T X_1 = \begin{bmatrix} \frac{1-i}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}+\frac{1}{2} = 1$, $\overline{X_1}^T X_2 = \begin{bmatrix} \frac{1-i}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}-\frac{1}{2} = 0$

$\overline{X_2}^T X_1 = \begin{bmatrix} \frac{1-i}{2} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}-\frac{1}{2} = 0$, $\overline{X_2}^T X_2 = \begin{bmatrix} \frac{1-i}{2} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}+\frac{1}{2} = 1$

So basis of eigen vectors that form a unitary system are

$X_1 = \begin{bmatrix} \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $X_2 = \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

So the unitary matrix = $\begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

2.7 : Diagonalization of Matrix

Theorem – 1 : If an n×n matrix A has a basis of eigen vetors , then D = $X^{-1}AX$.....(1) is diagonal, with the eigen values of A as the entries on the maindiagonal. Here X is the matrix with these eigen vectors as column vectors. Also $D^m = X^{-1}A^m X$ (m= 2, 3....)

Proof. We prove the result for a square matrix of order 3 × 3.

The proof can easily be extended to matrices of higher order.

Let A be a square matrix of order 3 × 3.

Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen value and

$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding eigen vectors.

Since eigen vectors are nontrival solutions of the matrix equation AX = λX.

So we have $AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, AX_3 = \lambda_3 X_3$.

Let $X = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = [X_1 \quad X_2 \quad X_3]$

Consider $AX = [AX_1 \quad AX_2 \quad AX_3] = [\lambda_1 X_1 \quad \lambda_2 X_2 \quad \lambda_3 X_3]$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

= XD where D is the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\Rightarrow AX = XD$

$\Rightarrow X^{-1}AX = X^{-1}XD = ID = D$ which proves the theorem.

It follows that $D^2 = D.D = X^{-1}AX.X^{-1}AX = X^{-1}A^2X$ since $XX^{-1} = I$

Again $D^3 = D^2.D = X^{-1}A^2X.X^{-1}AX = X^{-1}A^3X$

Hence $D^m = X^{-1}A^mX$

Orthogonal diagonalisation.

A square matrix A is said to be orthogonally diagonalisable, if there exists an orthogonal invertible matrix P such that $P^{-1}AP$ = a diagonal matrix.

In this case P is said to diagonalise A orthogonally.

Illustrative Examples

Example – 1 : Find a basis of eigen vectors and diagonalise the following matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

The characteristic polynomial is $|A - \lambda I|$ and the characteristic equation is

$|A - \lambda I| = 0$, that is , $\lambda^2(3 - \lambda) = 0$.

The eigen values of A are 3,0,0.

The eigen vector $X = [x_1, x_2, x_3]^T$ corresponding to the eigen value 3 is given by the system of equations $(A - 3.I)X = 0$, that is ,

$-2x_1 - x_2 + x_3 = 0, \quad -x_1 - 2x_2 - x_3 = 0, \quad x_1 - x_2 - 2x_3 = 0$,.

which are equivalent to $x_1 + 2x_2 + x_3 = 0$

Hence $X = k_1 [1, -1, 1]^T$, where k_1 is a non-zero real number. The eigen vectors corresponding to the eigen value 0 are $k_2 [1, 1, 0]^T$ and $k_3 [-1, 1, 2]^T$, where k_2 and k_3 are non-zero numbers.

Let $P = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, then $P^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ 3 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix}$

Now $AP = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

Now $P^{-1}AP = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ 3 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ -3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Note . If λ be a multiple eigen value of a real $n \times n$ symmetric matrix, then λ will be regular eigen value.

Example – 2 : F*ind a basis of eigen vectors and diagonalize the following matrix*

$$\begin{bmatrix} \mathbf{16} & \mathbf{0} & \mathbf{0} \\ \mathbf{48} & \mathbf{-8} & \mathbf{0} \\ \mathbf{84} & \mathbf{-24} & \mathbf{4} \end{bmatrix}$$

Solution : Here $A = \begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix}$

So the characteristic equation of A is $|A - \lambda| = 0$

$$\Rightarrow \begin{bmatrix} 16-\lambda & 0 & 0 \\ 48 & -8-\lambda & 0 \\ 84 & -24 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (16-\lambda)\ (-8-\lambda)(4-\lambda) = 0 \ \Rightarrow \lambda = 16,\ -8,\ 4\,.$$

Hence the eigen values of A are $\lambda = 16,\ -8,\ 4$

$\therefore$ For $\lambda = 16$, $\quad [A - 16.I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 48 & -24 & 0 \\ 84 & -24 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow 0.x_1 + 0.x_2 + 0.x_3 = 0 \qquad \text{..... (1)}$$

$$48.x_1 - 24.x_2 + 0.x_3 = 0 \qquad \text{..... (2)}$$

$$\text{and } 84x_1 - 24x_2 - 12x_3 = 0 \qquad \text{......(3)}$$

$\therefore$ Equation (2) $\quad \Rightarrow 48x_1 = 24\,x_2$

$$\Rightarrow \boxed{\frac{x_1}{1} = \frac{x_2}{2}}$$

$\therefore$ Equation (3) $\Rightarrow 84x_1 - 48x_2 - 12x_3 = 0 \Rightarrow 36x_1 = 12x_3 \Rightarrow 3x_1 = x_3$

$$\Rightarrow \boxed{\frac{x_1}{1} = \frac{x_3}{3}}$$

$$\therefore \boxed{\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}}$$

$\therefore$ An eigen vector of A corresponding to the eigen value $\lambda = 16$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore$ For $\lambda = -8$, $[A + 8.I]X = 0$

$$\Rightarrow \begin{bmatrix} 24 & 0 & 0 \\ 48 & 0 & 0 \\ 84 & -24 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 24x_1 + 0.x_2 + 0.x_3 = 0$ ..... (4)

$48.x_1 + 0.x_2 + 0.x_3 = 0$ ..... (5)

and $84x_1 - 24x_2 + 12x_3$ ......(6)

$\therefore$ Equation (4) $\Rightarrow \boxed{x_1 = 0}$

$\therefore$ Equation (6) $\Rightarrow 24x_2 = 12x_3$

$$\Rightarrow \boxed{\frac{x_2}{1} = \frac{x_3}{2}}$$

$\therefore$ An eigen vector of A corresponding to the eigen value $\lambda = -8$ is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Again for $\lambda = 4$, $[A - 4I]X = 0$

$$\Rightarrow \begin{bmatrix} 12 & 0 & 0 \\ 48 & -12 & 0 \\ 84 & -24 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 12x_1 + 0.x_2 + 0.x_3 = 0$ ....(7)

$48x_1 - 12x_2 + 0.x_3 = 0$ ....(8)

and $84x_1 - 24x_2 + 0 \cdot x_3 = 0$ .... (9)

$\therefore$ Equation (7) $\Rightarrow x_1 = 0$

$\therefore$ Equation (8) $\Rightarrow x_2 = 0$

$$x_3 = 1 \cdot x_3 + 0 \cdot x_2 + 0 \cdot x_1$$

$\therefore$ An eigen vector of A corresponding to the eigen value $\lambda = 4$, is $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore \; X = [X_1, X_2, X_3] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \Rightarrow |X| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 1 \neq 0$$

Since $|X| \neq 0$, so basis of eigen vectors are X_1, X_2 and X_3.

We know cofactor matrix of $X = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \text{Adj } X = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Now $X^{-1} = \dfrac{\text{Adj}X}{|X|} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

So the diagonal matrix $D = X^{-1} AX$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 & 0 \\ 16 & -8 & 0 \\ 4 & -8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Example –3 : ***Find a basis of eigenvectors and diagonalize the following matrix*** $\begin{bmatrix} \mathbf{18} & \mathbf{0} & \mathbf{0} \\ \mathbf{24} & \mathbf{-4} & \mathbf{0} \\ \mathbf{42} & \mathbf{-12} & \mathbf{2} \end{bmatrix}$

Solution : Here $A = \begin{bmatrix} 18 & 0 & 0 \\ 24 & -4 & 0 \\ 42 & -12 & 2 \end{bmatrix}$

So the characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 18-\lambda & 0 & 0 \\ 24 & -4-\lambda & 0 \\ 42 & -12 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (18-\lambda)(-4-\lambda)(2-\lambda) = 0 \;\Rightarrow \lambda = 18,\; -4, 2$

Hence the eigen values of A are λ = 18, – 4, 2

$\therefore$ For $\lambda = 18$, $[A - 18.I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 24 & -22 & 0 \\ 42 & -12 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 0.x_1 + 0.x_2 + 0.x_3 = 0$ .... (1) $\quad 24x_1 - 22x_2 + 0.x_3 = 0$ .... (2)

and $42x_1 - 12x_2 - 16x_3 = 0$ ... (3)

$\therefore$ equation (2) $\Rightarrow 12x_1 = 11x_2 \Rightarrow x_1 = {}^{11}\!/_{12}\, x_2$

Putting x_1 value is (3) $x_2 = \dfrac{32}{33} x_3$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} {}^{88}\!/_{3} \\ 32 \\ 53 \end{bmatrix} = \begin{bmatrix} 88 \\ 96 \\ 159 \end{bmatrix}$$

$\therefore$ an eigen vector of A corresponding to the eigen value $\lambda = 16$ is $X_1 = \begin{bmatrix} 88 \\ 96 \\ 159 \end{bmatrix}$

For $\lambda = -4$, $[A + 4.I]X = 0$

$$\Rightarrow \begin{bmatrix} 22 & 0 & 0 \\ 24 & 0 & 0 \\ 42 & -12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 22x_1 + 0.x_2 + 0.x_3 = 0$ .... (4) $\quad 24x_1 + 0.x_2 + 0.x_3 = 0$ ..... (5)

and $42x_1 - 12x_2 + 6x_3 = 0$ ..... (6)

$\therefore$ equation (4) $\Rightarrow \boxed{x_1 = 0}$

$-12x_2 + 6x_3 = 0 \;\Rightarrow 2x_2 = x_3 \Rightarrow x_2 = \dfrac{1}{2} x_3$

$\therefore$ an eigen vector of A corresponding to the value of $\lambda = -4$ is $X_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Again for $\lambda = 2$, $[A - 2I]X = 0$

$$\Rightarrow \begin{bmatrix} 16 & 0 & 0 \\ 24 & -6 & 0 \\ 42 & -12 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 16x_1 + 0.x_2 + 0.x_3 = 0$ ...... (7) $\quad 24x_1 - 6x_2 + 0.x_3 = 0$ ..... (8)

and $42x_1 - 12x_2 + 0.x_3 = 0$ ..... (9)

$\therefore$ equation (7) $\Rightarrow \boxed{x_1 = 0}$ equation (8) $\Rightarrow \boxed{x_2 = 0}$ $\boxed{x_3 = 1}$

$\therefore$ an eigen vector of A corresponding to the eigen value $\lambda = 2$ is $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore X = [X_1 X_2 X_3] = \begin{bmatrix} 88 & 0 & 0 \\ 96 & 1 & 0 \\ 159 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow D = X^{-1}AX = \begin{bmatrix} 18 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Exercise – 2.1

1. Find the eigen values and the eigen vectors (i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} -2 & 4 \\ 4 & 4 \end{bmatrix}$

2. Find the characteristic equation and eigen values of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

3. Find the characteristic values and the characteristic vectors $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$

4. Determine eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

5. Find the eigen values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -2 & 3 \end{bmatrix}$

6. Find the eigen values and eigen vectors of the matrices (a) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

7. Prove the following
   (a) If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also it's eigen value.
   (b) If λ is the eigen value of A and $f(A)$ is any polynomial in A, then the eigen values of $f(A)$ is $f(\lambda)$.
   (c) If λ is an eigen value of A then λ is also an eigen value of A^1.
   (d) The eigen values of a Skew-Hermitian matrix are zero or pure imaginary.
   (e) The eigen values of an unitary matrix are of magnitude 1.
8. Find the matrix A whose eigen values and the corresponding eigen vectors are given below.
   (a) (2, 2, 4) (–2, 1, 0) (–1, 0, 1) (1, 0, 1)
   (b) 1, 1, 1 (–1, 1, 1) (1, –1, 1) (1, 1, –1)
9. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$
10. Prove that characteristic roots of diagonal matrix are diagonal elements of the matrix.
11. If $\lambda_1, \lambda_2, ...\lambda_n$ are given values of a matrix A. Then A^m has the given eigen values $\lambda_1^m, \lambda_2^m........\lambda_n^m$.
12. Diagonalise the following matrices and obtain the modal matrix in each case.
   (a) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 9 & -1 & -9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$
13. Show that the matrices are diagonalizable. Find the matrix P such that $P^{-1}AP$ is a diagonal matrix.
   (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$
14. Then necessary and sufficient condition than an $n \times r$ matrix A over a field F is diagonalisable is that A has in linearly independent eigen vectors.
15. Diagonalise the matrix $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.
16. Diagonalise the matrices and obtain the modal matrix in each case.
   (a) $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Answers

2. $(1-\lambda)^2(4-\lambda) = 0, 1, 1, 4$

3. Characteristic values are 5 and –2.
Characteristic vectors are (1, –1).

4. eigen values 5, 1, 1.
eigen vectors (1, 1, 1) (2, –1, 0) & (1, 0, –1)

5. eigen values 2, 2 and 8 (4, –1) (2, 3) (1, –1)

6. (a) 3, 2, 6 (1, –1, 1) (1, 0, –1) (1, 1, 1)
(b) 0, 0, 3 (1, 0, –1) (0, 1, –1) (1, 1, 1)

8. (a) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. $\lambda^3 - \lambda^2 - 18\lambda - 40 = 0$

12. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ -1 & -1 & -13 \end{bmatrix}$

13. (a) $\lambda = 0$; (1, 0, –1) $\lambda = 1$; (–1, –1, 1) $\lambda = 2$: (1, 1, 0)

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) $\lambda = 1$: (1, –1, –1) $\lambda = 2$: (0, 1, 1) $\lambda = -2$ (8, –5, 7)

$$P = \begin{bmatrix} 1 & 0 & 8 \\ -1 & 1 & -5 \\ -1 & 1 & 7 \end{bmatrix} \quad P = \frac{1}{12}\begin{bmatrix} 12 & 8 & -8 \\ 12 & 15 & -5 \\ 0 & -1 & 1 \end{bmatrix}$$

(c) $\lambda = 0$, (3, 1, –2)
$\lambda = 2i$: (3 + i, 1 + 3i, –4) $\lambda = -2i$, [3 – i, 1 – 3i, –4]

14. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

16. (a) $\begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 4 & 2 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$