20.07.14

-: Numerical Method :-

Numerical analysis is an analysis that provides efficient methods for obtaining numerical answers to solve problems. When solving problem one usually starts with some initial data and then compute after some intermediate steps.

Classification of no.s :-

1. Exact no.
2. Approximate no.

1. Exact no.s are those which have no uncertainity or approximation associated with them.

Eg: Natural no.s, Real no.s, Rational no.s etc.

2. Approximate no.s are those in which there is no uncertainity that represents a no. to a certain degree of accuracy.

Eg: $\pi, e, \sqrt{2}, \sqrt{3}$ etc.

Significant digits :-

The digits which are used to express a no. are called significant digit or significant figure.

No.	S.D.
0.88876	5 (8, 8, 8, 7, 6)
3.1416	5 (3, 1, 4, 1, 6)
3.0259	5 (3, 0, 2, 5, 9)
9.00	3 (9, 0, 0)
7482	4 (7, 4, 8, 2)
50.05	4 (5, 0, 0, 5)

Rounding up no.s :-

Rules for rounding :-

The process of cutting all unwanted digits or superflow digits and writting the nearest representaⁿ is called rounding up no.s

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No. is rounded up to 'n' significant digits :

- Discard all the digits to the right to another digit.

- If (n+1)th digit is greater than or it is 5 followed by non-zero digit then n-th digit is increased by 1.

- If (n+1)th digit < 5, then nth digit is remain unchanged.

Eg: 1.68752 (rounding up to 4 significant digit)

Significant digits : 1, 6, 8, 7, 5, 2

Ans: 1.688

Eg: 1.68342 (up to 4 significant digit)

S.D. : - 1, 6, 8, 3, 4, 2

Ans. 1.683

- If (n+1)th digit is 5 and followed by zero or zeros, then the digit is increased by 1, if n-th digit is odd & remains unchanged (if nth digit is even)

Eg: 3.14350

S.D: 3, 1, 4, 3, 5, 0

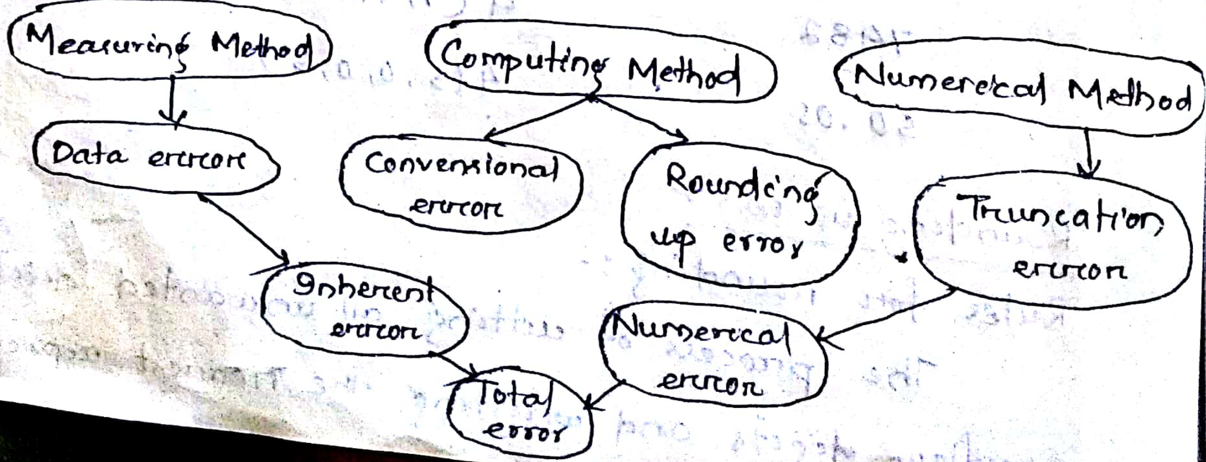
Ans. 3.144

Error :-

An error is defined as the difference between the exact value and approximation value.

$E = X - X^T$ [$E = T - A$ → Approximate value, T → True value]

Types of error :-



~~Error~~ = ~~Exact~~ Exact value - approximate error

$E_a = |x - x'| = \Delta x$

x → True/Actual/Exact value

x' → Approximate value

Relative error :-

$$\frac{\Delta x}{x} = \text{relative error } (E_r)$$

Percentage error :-

$$E_p = E_r \times 100 = \frac{\Delta x}{x} \times 100$$

Eg: Find absolute error, relative error & percentage error ?

37.46235 correspond to 4 significant values or figures ?

Ans. Let $x = 37.46235$

$x' = 37.46$

$E_a = |37.46235 - 37.46|$

$= 2.35 \times 10^{-3} = \Delta x$

$$E_r = \frac{\Delta x}{x} = \frac{2.35 \times 10^{-3}}{37.46235} = 6.2729 \times 10^{-5}$$

$E_p = 6.2729 \times 10^{-3}$

Eg: Round up the no.s correspond to 4 significant figures:-

(i) 3.26425

(ii) 36.4735

(iii) 0.70035

(iv) 0.000322167

Inherent error :-

The errors which are already present before the solution of a problem are called inherent error. Such error arises either due to the given data being approximated or due to the limitation of mathematical tables, calculators, digital computer.

It can be minimized by using better data.

For eg: Screw gauge, scall ciberes, spherometers can be located by using those elements.

Truncaⁿ error :-

This errors are caused due to approximating results or on replacing infinite process by a finite one.

Truncaⁿ error = numerical value − approximate value

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = X$$

$$\Rightarrow X' = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

NOTE :-

* If a no. is correct to 'n' decimal bases then its error is $\boxed{\frac{1}{2} \times 10^{-n}}$

* If a no. is correct to n significant digits then the maxᵐ relative error is less than equal to $\frac{1}{2} \times 10^{-n}$

* If a no. is correct to 'd' decimal base then its absolute error $\leq \frac{1}{2} \times 10^{-d}$.

* If a first significant digit of a no. is K & the no. is correct to 'n' significant figures then the relative error

is given by $E_r < \frac{1}{K \times 10^{n-1}}$

Eg: If the first significant digit of a no. 864.32 correct to 5 significant digits, then find E_r, E_a?

$K = 8$

$n = 5$

$\therefore E_r < \frac{1}{8 \times 10^4}$

$E_a = \frac{1}{2} \times 10^{-5}$

Errors in their approximation of a function :-

$y = f(x_1, x_2)$ —— ①

$\therefore$ Let y be a function of two variable x_1 & x_2.

$\therefore \delta x_1, \delta x_2$ be the errors of x_1 & x_2 respectively.

$y + \delta y = f(x_1 + \delta x_1, x_2 + \delta x_2)$ — (ii)

Expanding eq^n (ii) by using Tayler's series expanation,

$y + \delta y = f(x_1, x_2) + \left(\frac{\partial f}{\partial x_1} \delta x_1 + \frac{\partial f}{\partial x_2} \delta x_2\right)$

Neglecting the higher terms & subtracting, we get

$$\boxed{\delta y \approx \left(\frac{\partial f}{\partial x_1} \delta x_1 + \frac{\partial f}{\partial x_2} \delta x_2\right)} \quad \text{— (iii)}$$

This is the error in two variable function

$E_r = \frac{\delta y}{y}$

$E_p = E_r \times 100$

Similarly in case of 3 variable function

$y = f(x_1, x_2, x_3)$

$\delta y \approx \left(\frac{\partial f}{\partial x_1} \delta x_1 + \frac{\partial f}{\partial x_2} \delta x_2 + \frac{\partial f}{\partial x_3} \delta x_3\right)$

For 'n' variable function;

$$y = f(x_1, x_2, \ldots, n)$$

$$\delta y = \left(\frac{\partial f_1}{\partial x_1} \delta x_1 + \cdots + \frac{\partial f_n}{\partial x_n} \delta x_n \right)$$

Eg: Find the relative max^m error in u when $x = y = z = 1$ & $u = \frac{4x^2y^3}{z^4}$ & errors in x, y, z be 0.001.

Ans. Let $u = \frac{4x^2y^3}{z^4}$

$$\delta x = \delta y = \delta z = 0.001$$

$$x = y = z = 1$$

$$|\delta u| = \left| \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z \right|$$

$$= \left| \frac{8xy^3}{z^4} \times 0.001 \right| + \left| \frac{12x^2y^2}{z^4} \times 0.001 \right| + \left| \frac{16x^2y^3}{z^5} \times 0.001 \right|$$

$$|\delta u|_{max} = 0.008 + 0.012 + 0.016$$

$$= 0.036$$

$$E_r = \frac{\delta u}{u} = \frac{0.036}{4} = 0.009$$

$$E_p = 0.009 \times 100 = 0.9$$

* If $R = 10x^2y^3z^4$ & errors in x, y & z be $0.03, 0.01$ & 0.02 respectively & $x = 3, y = 1$ & $z = 2$; so calculate the max^m relative error.

→ Relative error = 0.09

& Percentage error = 9

(Ans.)

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Errors in a series of function :-

Let $f(x)$ be a function which is expanded as

$$f(x) = f(a + \overline{x-a}) - f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a)$$

Reminder term $R_n(x) = \frac{(x-a)^n}{n!}f^n(a)$

$$= \frac{(x-a)^n}{n!}f^n(\theta) \quad (a<\theta<n)$$

If the series is convergent, $R_n(x) \to 0$ as $n \to \infty$ & $f(x)$ is approximated by the first 'n' terms of the series, the max^m error will be given by the remainder term $R_n(x)$. If accuracy is required is already given it is possible to find the no. of terms so that the finite series get the desired accuracy.

Eg:- Find the no. of terms of the exponential series such that their sum gives value of e^x corrected to 8 decimal ~~bases~~ places at $x=1$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n(x)$$

$$R_n(x) = \frac{x^n}{n!}e^\theta \quad 0<\theta<x$$

$$\therefore (\text{Error})_{max} = \frac{x^n}{n!} = \frac{1}{n!} \quad (\text{at } x=1)$$

$$\frac{1}{n!} < \frac{1}{2}10^{-8}$$

$$\Rightarrow n! > 2\times 10^8$$

Hence we have 12 no. of terms in expansion in order to the sum is correspond to 8 decimal ~~bases~~ places.

Q// Find the no. of terms of the exponential series such that their sum gives value e^x correct to 6 decimal places at $x=1$

Q// Eg: ~~Expansion of 1~~ Find the no. of terms in expansion of $\tan^{-1}x$ such that their sum gives value $\tan^{-1}1$ correct to 8 significant figures.

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{n-1}}{n-1} + R_n(x)$$

Example :-

Obtain a 2nd degree polynomial approximation $f(x) = (1+x)^{\frac{1}{2}}$, $x \in [0, 0.1]$ using the taylor's series expansion about $x=0$. Use the expansion to approximate 0 to 0.05 & find bound truncation error.

$$f(x) = f(0) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{4}(1+x)^{-3/2}$

$\Rightarrow f''(0) = -1/4$

$f'''(x) = 3/8 \, (1+x)^{-5/2}$

$\Rightarrow f'''(0) = 3/8$

$$f(x) = 1 + x \cdot 1/2 - 1/4 \, x^2/2! + x^3/3! \, \frac{3}{8(1+x)^{5/2}}$$

$$R_n(x) = \frac{x^3}{3!} \frac{3}{8(1+\xi)^{5/2}}$$

$$= \frac{x^3}{16(1+\xi)^{5/2}} \qquad 0 < \xi < 0.1$$

The remainder term of Taylor series expansion is

$$R_n(x) = \frac{x^3}{3!} \cdot \frac{3}{8(1+\xi)^{5/2}}$$

$$T = (1+x)^{1/2} - \left(1 + x/2 - x^2/8\right) = \frac{x^3}{16(1+\xi)^{5/2}}$$

$$f(0.05) = 1 + \frac{0.05}{2} - \frac{(0.05)^2}{8} = 1.0247$$

Bound for truncation error

$$\frac{x^3}{16} = \frac{(0.1)^3}{16} = 6.25 \times 10^{-5}$$

Q) Calculate the sum $\sqrt{3}, \sqrt{5}$ & $\sqrt{7}$ to 4 significant figures and find its absolute and relative errors?

~~$\sqrt{3} + \sqrt{5} + \sqrt{7} = 6.613870096$~~

~~After rounding up =~~

$\sqrt{3} + \sqrt{5} + \sqrt{7} = 6.613870096$

After rounding up $= 6.614$

$\sqrt{3} = 1.732050808$

After rounding up, $\sqrt{3} = 1.732$

$\sqrt{5} = 2.236067977$

After rounding up, $\sqrt{5} = 2.236$

$\sqrt{7} = 2.645751311$

After rounding up, $\sqrt{7} = 2.646$

$\sqrt{3} + \sqrt{5} + \sqrt{7} = 6.614$

$E_a = 6.614 - 6.613870096 = 1.2990 \times 10^{-4}$

$E_R = \cancel{1.9641 \times 10^{-5}}\ 1.9641 \times 10^{-5}$

Q) 3 approximate value of a no. $1/3$, are 0.30, 0.33 & 0.34. Among the 3, which is best approximate.

$1/3 = 0.333333$

$= 0.33$ (rounding up)

$E_{a(i)} = (0.33 - 0.30) = 0.03$

$E_{a(ii)} = (0.33 - 0.33) = 0$ (Best approximation)

$E_{a(iii)} = (0.33 - 0.34) = 0.01$

27.07.16

-: Solution of Algebraic & Transidental eqn :-

Transidental eqn : mixing of all type of eqn.

Intermediate Value Theorem (IVT) :-

suppose $f(x)$ be continuous in closed interval a, b i.e $[a, b]$. $f(a)$ & $f(b)$ are of opposite sign for a shape of simplocity.

$$\text{So} \quad f(a).f(b) < 0$$

Then there is atleast one root in between a & b. This is known as IVT.

Q Find the roots of the given function?

$$x^4 - x - 10 = 0$$

Let $f(x) = x^4 - x + 10$

$f(0) = -10$

$f(1) = -10$

$f(2) = 4$

$f(1) \cdot f(2) < 0$

Then the roots are in betn 1 & 2.

Direct Method :- & Iteration Method :-

$$ax^2 + bx + c = 0$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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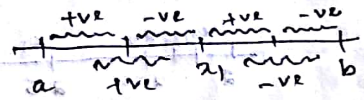
Methods:
(i) Bisection Method
(ii) Secant Method
(iii) Regular Falsi Method
(iv) Newton's Raphson's Method
(v) Muller's method

Bisection Method :-

$f(x) \longrightarrow [a,b]$

$f(a)\, f(b) < 0$



$x_1 = \frac{a+b}{2}$

If $f(x_1) = 0$, then x_1 is the root.

If $f(x_1) > 0$ then the root is in betn 'a' and 'x_1'.
If $f(x_1) < 0$ then the root is in betn x_1 and b.

It is a repeated application intermediate value theorem (IVT). we can get the initial approximation.

Bisection method consists in locating the root of the eqn $f(x)=0$ betn a & b. If $f(x)$ is continuous betn a & b and $f(a)$ & $f(b)$ are of opposite sign, then there is a root betn a & b. Then the first approximation to the root is given by $x_1 = \frac{a+b}{2}$.

If $f(x_1) = 0$, then x_1 is a root of given function. Otherwise root lies betn a to x_1 or x_1 to b, according as $f(x_1)$ is +ve or -ve. Continuing this manner until we get the roots by using our desired accuracy.

This is the condition of bisection method.

Q/ Find the root of the following bisection method correct to 2 decimal places

$x^3 - 5x + 1 = 0$ which lies betn 2 & 3.

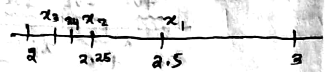
Ans :- $f(x) = x^3 - 5x + 1$, $f(2) = -1$ & $f(3) = 13$

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = 4.125 > 0$$

$$\therefore x_2 = \frac{2+2.5}{2} = 2.25$$

$$f(x_2) = 1.1406 > 0$$



$$\therefore x_3 = \frac{2+2.25}{2} = 2.125$$

$$f(x_3) = -0.0292 < 0$$

$$\therefore x_4 = \frac{2.125 + 2.25}{2} = 2.1875$$

$$f(x_4) = 0.53$$

After rounding up x_4, we get $x_4 = 2.19$ approximately.

Find the root

Q/1 $x^4 - x - 10 = 0$ correct to 2 decimal places by using Bisection method.

Q/ $x \log_{10} x = 1.2$ whose root lies betn 2 & 3. Find the root by using bisection method correct to 2 decimal places.

Q/ $xe^x = \cos x$, find the root by using bisection method correct to 4 decimal places.

Q/ $x^3 - x - 11 = 0$, find the root by using bisection method correct to 3 decimal places.

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01.08.16

Secant Method :-

The necessary formulae for secant method is given by

$$x_{k+1} = x_k - \left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right) f(x_k); \quad k = 1, 2, 3, \ldots$$

$$x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) f(x_1)$$

Q. Find the real root of $x^4 - x - 10 = 0$ correct to 3 places of decimal by using secant method or chord method ?

Ans :- $f(x) = x^4 - x - 10$

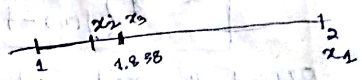
$f(0) = -10$, $f(1) = -10$

$f(2) = 4$

The roots lie betⁿ 1 & 2.

$\underline{f(1) \cdot f(2) < 0}$

$x_0 = 1$



$\Rightarrow f(x_0) = -10$

$x_1 = 2$

$\Rightarrow f(x_1) = 4$

$$x_2 = 2 - \left(\frac{2 - 1}{4 - (-10)}\right) 4$$

$$= 2 - \frac{1}{14} \times 4$$

$= 1.714$ $\qquad \Rightarrow f(x_2) = -3.083$

$$x_3 = x_2 - \left\{\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right\} f(x_2)$$

$$= 1.714 - \left\{\frac{1.714 - 2}{-3.083 - 4}\right\} (-3.083)$$

$= 1.838$

$f(x_3) = -0.425$

$$x_4 = x_3 - \left\{\frac{x_3 - x_2}{f(x_3) - f(x_2)}\right\} f(x_3)$$

$$= 1.838 - \cancel{0.425}\left(\frac{1.838 - 1.714}{-0.425 + 3.083}\right)(-0.425)$$

$$= 1.85782$$

$\therefore$ After rounding up x_4, we get $x_4 = 1.8578$ approximately.

Regular Falsi Method / False position method :-

The necessary formulae for this method is given by

$$f(x_k) \cdot f(x_{k-1}) < 0$$

$$\boxed{x_2 = x_1 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_1)}$$

Q1 Find the real root of the eq^n $x^3 - 3x + 4 = 0$ corrected to 3 places of decimal by using False position method ?

Ans:- $f(0) = 0 - 0 + 4 = 4$

$f(1) = 2$

$f(2) = 6$

$f(-1) = -1 + 4 + 4 = 7$

$f(-2) = 2$

$f(-3) = -14$

$f(-2) \cdot f(-3) < 0$

The root lies bet^n -2 and -3.

$x_0 = -3$

$f(x_0) = -14$

$x_1 = -2$

$\Rightarrow f(x_1) = 2$

In first ~~initation~~ iteration

$$x_2 = x_0 - \left\{ \frac{x_1 - x_0}{f(x_1) - f(x_0)} \right\} f(x_0)$$

$$\therefore x_2 = -3 - \left(\frac{-2+3}{2+14} \right)(-14)$$

$$= -2.125$$

$f(x_2) = 0.779 > 0$

So the root lies betn -3 to -2.125

$$\cancel{x_3 = x_1 - \left\{ \frac{x_2 - x_1}{f(x_2) - f(x_1)} \right\} f(x_1)}$$

$$\Rightarrow x_3 = -3 - \left(\frac{-2.125+3}{0.779+14} \right)(-14)$$

$$= -2.171$$

$f(x_3) = 0.2805 \simeq 0.281$

So the root lies betn -3 to -2.171

$$x_4 = -3 - \left(\frac{-2.171+3}{0.281+14} \right)(-14)$$

$$= -2.1873$$

$\therefore$ After rounding up x_4, we get $x_4 = -2.187$ approximately.

Q1 Find the real root of the eqn $xe^x = \cos x$ correct to 4 places of decimal by using False position method ?

$\therefore \quad f(x) = xe^x - \cos x, \quad f(0) = -1$

$f(1) = 1.718$

$\therefore$ The root lies betn 0 and 1

Q/ Find the 4th root of 32 using 2 places of decimal by using Regular Falsi method.

$x = \sqrt[4]{32}$

$\Rightarrow x^4 = 32$

$\Rightarrow x^4 - 32 = 0$

So $f(x) = x^4 - 32$

The root lies betn 2 and 3.

03.08.16

Newton's Raphson's Method :-

The necessary formulae for Newton's Raphson method

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$n = 0, 1, 2, \ldots$

When $n=0$, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

According to I.V.T., $f(a) \cdot f(b) < 0$

then $x_0 = \dfrac{a+b}{2}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

This method gives more accurate result.

Example:

Find the real root, $x^3 - 3x + 1 = 0$ correct to 3 decimal places by using Newton's Raphson's Method?

$\therefore f(x) = x^3 - 3x + 1$

$\Rightarrow f(0) = 1$

$\Rightarrow f(1) = -1$

$\therefore f(0) \cdot f(1) < 0$

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∴ The root lies betⁿ 0 and 1.

$$x_0 = \frac{0+1}{2} = 0.5$$

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

$$= \frac{3x_n^3 - 3x_n - x_n^3 + 3x_n - 1}{3x_n^2 - 3}$$

$$= \frac{2x_n^3 - 1}{3x_n^2 - 3}$$

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 3} = 0.333$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3} = 0.347$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 3} = 0.3472$$

∴ After rounding up x_3, we get $x_3 = 0.347$ approximately correct to 3 decimal places.

Q/ Find the real root of the eqⁿ $xe^x - 2 = 0$ correct to 3 decimal places by using Newton's Raphsons method?

$$f(x) = xe^x - 2 = 0$$

$\Rightarrow f(0) = -2$

$\Rightarrow f(1) = 0.718$

$\therefore f(0) \cdot f(1) < 0$

∴ The root lies betⁿ 0 and 1.

$$x_0 = \frac{0+1}{2} = 0.5$$

$$x_{n+1} = x_n - \frac{x_n e^{x_n} - 2}{x_n e^{x_n} + e^{x_n}}$$

$$= \frac{x_n^2 e^{x_n} + x_n e^{x_n} - x_n e^{x_n} + 2}{x_n e^{x_n} + e^{x_n}}$$

$$= \frac{x_n^2 e^{x_n} + 2}{x_n e^{x_n} + e^{x_n}}$$

$$x_1 = \frac{x_0^2 e^{x_0} + 2}{x_0 e^{x_0} + e^{x_0}}$$

$$= 0.975$$

$$x_2 = \frac{x_1^2 e^{x_1} + 2}{x_1 e^{x_1} + e^{x_1}}$$

$$= 0.863$$

$$x_3 = \frac{x_2^2 e^{x_2} + 2}{x_2 e^{x_2} + e^{x_2}} = 0.853$$

$$x_4 = \frac{x_3^2 e^{x_3} + 2}{x_3 e^{x_3} + e^{x_3}} = 0.8526$$

$\therefore$ After rounding up x_4, we get $x_4 = 0.853$ approximately correct to 3 decimal places.

Q/ Find the real root $xe^x = \cos x$ correct to 3 places of decimal by using NR method? $(0, 1)$

Q/ Find the positive root of the eq^n $x = 2\sin x$ by using NR method? $(0, \pi/2)$

Some deductions about NR method :-

(1) Iteration formula for finding $\frac{1}{N}$:-

$$x_{n+1} = x_n (2 - N x_n)$$

(2) Iteran formula to find $\sqrt{N}$:-

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

(3) Iteran formula to find $\frac{1}{\sqrt{N}}$:-

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{N x_n}\right)$$

(4) Iteran formula to find $\sqrt[K]{N}$:-

$$x_{n+1} = \frac{1}{K}\left((K-1)x_n + \frac{N}{x_n^{K-1}}\right)$$

Formula (1) :-

Let $x = \frac{1}{N}$

$\Rightarrow \frac{1}{x} = N$

$\Rightarrow \frac{1}{x} - N = 0 = f(x)$

$$x_{n+1} = x_n - \left(\frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}\right)$$

$$= x_n + \left(\frac{1}{x_n} - N\right)x_n^2$$

$$= x_n + (x_n - N x_n^2)$$

$$= 2x_n - N x_n^2$$

$$\Rightarrow x_{n+1} = x_n(2 - N x_n)$$

Q// Evaluate the following correct to 4 decimal places by using NR formula?

(a) $\frac{1}{31}$ (b) $\sqrt{28}$ (c) $\frac{1}{\sqrt{14}}$ & (d) $\sqrt[3]{24}$

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Ans:

(a) $N = 31$

$$\text{so } \frac{1}{N} = \frac{1}{31} = 0.0322$$

$$x_1 = x_0(2 - 31x_0)$$

$$= 0.0322\,(2 - 31 \times 0.0322)$$

$$= 0.0322$$

$$x_2 = x_1(2 - 31x_1)$$

$$= 0.0322$$

(b) $N = 28$

$$\Rightarrow \sqrt{N} = \sqrt{28}$$

$$\sqrt{25}, \sqrt{28}, \sqrt{36}$$

$$\uparrow 5 \qquad \uparrow 6$$

more closer

so $x_0 = 5$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{28}{x_0}\right)$$

$$= \frac{1}{2}\left(5 + \frac{28}{5}\right)$$

$$= 5.3$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{28}{x_1}\right) = 5.2915$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{28}{x_2}\right) = 5.2915$$

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(c) $N = 14$

$\frac{1}{\sqrt{9}}, \frac{1}{\sqrt{14}}, \frac{1}{\sqrt{16}}$ ← more closer

$x_0 = \frac{1}{4} = 0.25$

$\therefore x_1 = \frac{1}{2}\left(x_0 + \frac{1}{14 x_0}\right)$

$= 0.2657$

$x_2 = \frac{1}{2}\left(x_1 + \frac{1}{14x_1}\right)$

$= 0.267$

$x_3 = \frac{1}{2}\left(x_2 + \frac{1}{14 x_2}\right)$

$= 0.267$

(d) $N = 24$, $K = 3$

$(8)^{1/3}, (24)^{1/3}, (27)^{1/3} \longrightarrow$ more closer

So $x_0 = 3$

$\therefore x_1 = \frac{1}{K}\left\{(K-1)x_0 + \frac{N}{x_0^{K-1}}\right\}$

$= 2.888$

$\therefore x_2 = 2.884$

$\therefore x_3 = 2.884$

8.8.16

Methods of Iteration:-

$ax + b = 0$

$\Rightarrow x = -b/a$

$ax^2 + bx + c = 0$

$\Rightarrow ax^2 = -bx - c$

$\Rightarrow x^2 = \frac{-bx-c}{a}$

$\Rightarrow x = \sqrt{\frac{-bx-c}{a}}$

or $ax^2 + bx + c = 0$

$\Rightarrow bx = -ax^2 - c$

$\Rightarrow x = \frac{-ax^2-c}{b}$

or $ax^2 + bx + c = 0$

$\Rightarrow x(ax+b) = -c$

$\Rightarrow x = \frac{-c}{ax+b}$

suppose $f(x) = 0$ is the first degree eqn of x. Let $\phi(x)$ be an iteration funn. The convergence of an iteration depends upon the funn $\phi(x)$ by using one or more initial approximan x_0.

The necessary formulae for method of iteration is given by

$$x_{k+1} = \phi(x_k)$$

where $k = 0, 1, 2, \dots$

$\therefore f(x) = 0$ is first degree eqn of x and the new term of it is $\phi(x) = x$ & also $\phi(x)$ is continuous funn on the $[a, b]$

And $|\phi'(x)| \leq 1, \quad \forall x \in [a,b]$

Then we have to prove that 'ξ' is our desired root of the eq^n (1) and the sequence x_k is convergence to 'ξ'

Hence $x_{k+1} = \phi(x_k)$

Q) Find the root of the eq^n $f(x) = x^3 + x - 5$ by iteration?

$f(x) = x^3 + x - 5$

$f(0) = -5$

$f(1) = -3$

$f(2) = 5$

The root lies bet^n 1 and 2.

$x = 5 - x^3 = \phi_1(x)$

$\phi_1'(x) = -3x^2$

$|\phi_1'(x)| = |3x^2| > 1$

so $\phi_1(x)$ is not true.

$x^3 = 5 - x$

$\Rightarrow x = (5-x)^{1/3} = \phi_2(x)$

$\phi_2'(x) = -\frac{1}{3}(5-x)^{-2/3} = -\frac{1}{3} \cdot \frac{1}{(5-x)^{2/3}}$

$|\phi_2'(x)| = \left| \frac{1}{3(5-x)^{2/3}} \right| \leq 1$

so $\phi_2(x)$ is true.

This way of writing the eq^n is ready to take for method of iteration.

$x_{k+1} = (5 - x_k)^{1/3}$; $k = 0, 1, 2, \ldots$

$x_0 = \frac{1+2}{2} = 1.5$

$x_1 = (5-1.5)^{1/3} = 1.518$

$x_2 = (5-1.518)^{1/3} = 1.515$

$x_3 = (5-1.515)^{1/3} = 1.516$

$x_4 = (5-1.516)^{1/3} = 1.5159$

$\therefore$ $x_4 \approx 1.52$ approximately correct to 2 decimal places.

Q11 Solve the eq^n $x^3+x^2-1=0$ and find a +ve root by method of iteration.

$f(x) = x^3+x^2-1=0$

$f(0) = -1$

$f(1) = 1$

$f(0)\cdot f(1) < 0$

So the root lies between 0 and 1.

$x^2 = 1-x^3$

$\Rightarrow x = (1-x^3)^{1/2} = \phi_1(x)$

$\Rightarrow \phi_1'(x) = \frac{1}{2}(1-x^3)^{-1/2}(-3x^2)$

$= -\frac{3}{2}\frac{x^2}{(1-x^3)^{1/2}}$

$\Rightarrow \left|\phi_1'(x)\right| = \left|\frac{3}{2}\frac{x^2}{(1-x^3)^{1/2}}\right| > 1$ or < 1

so $\phi_2'(x)$ is ~~true~~ false

$\cancel{x_0 = \frac{0+1}{2} = 0.5}$

$x^3 = 1 - x^2$

$\Rightarrow x = (1-x^2)^{1/3} = \phi_1(x)$

$\phi_1'(x) = \frac{1}{3}(1-x^2)^{-2/3}(-2x)$

$|\phi_2'(x)| = \left|\frac{2}{3}\frac{x}{(1-x^2)^{2/3}}\right| > 1$ or < 1

so $\phi_2'(x)$ is false.

$x^3 + x^2 = 1$

$\Rightarrow x^2(1+x) = 1$

$\Rightarrow x^2 = \frac{1}{1+x}$

$\Rightarrow x = \sqrt{\frac{1}{1+x}} = \phi_3(x)$

$\Rightarrow \phi_3'(x) = -\frac{1}{2}\left(\frac{1}{1+x}\right)^{-1/2} \cdot \frac{1}{(1+x)^2} = -\frac{1}{2}$

$\Rightarrow |\phi_3'(x)| < 1$

so $\phi_3(x)$ is true

$x_0 = 0.5$

$x_1 = \sqrt{\frac{1}{1+x_0}} = 0.816$

$x_2 = \sqrt{\frac{1}{1+x_1}} = 0.742$

$x_3 = \sqrt{\frac{1}{1+x_2}} = 0.758$

$x_4 = \sqrt{\frac{1}{1+x_3}} = 0.754$

$$x_5 = \sqrt{\frac{1}{1+x_4}} = 0.755$$

$$x_6 = \sqrt{\frac{1}{1+x_5}} = 0.755$$

$\therefore$ so the desired root is 0.755 approximately.

H.W.

Q// Find the +ve root of $x^3+x-1=0$ by method of iteration?

Q// Find the +ve root of $xe^x = 1$ by method of iteration?

Q// Find the +ve root of $2x = \cos x + 3$ by method of iteration?

Q// Find the +ve root of $e^x - 3x = 0$ by method of iteration?

Simultaneous Linear Equation :-

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

By Crammer's rule in determinant method

$$D = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_x = \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_y = \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \Delta_3 = \begin{vmatrix} a_1 & b_1 & \Delta_1 \\ a_2 & b_2 & \Delta_2 \\ a_3 & b_3 & \Delta_3 \end{vmatrix}$$

In matrix method,

$Ax = b$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\Rightarrow \cancel{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$b = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$x = A^{-1} b$$

$$\Rightarrow x = \left(\frac{\text{adj}.A}{|A|} \right) b$$

Co. factor; $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$

7.8.16

Triangularization method :-
(Factorisation Method)

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$K = [A/b]$

Augumented matrix

$$K = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

~~Triangular~~ Diagonal matrix :

$$A = [a_{ij}]_{m \times n} \ ; \text{where if } i \neq j \text{ then } a_{ij} = 0$$

Upper triangular matrix :-

A square matrix / A diagonal matrix is said to be upper triangular matrix if it's diagonal's below entries are zeros.

$$U = [a_{ij}]_{m \times n} \ ; \text{where if } i > j \text{ then } a_{ij} = 0$$

~~Uppe~~ Lower triangular matrix :

A square matrix is said to be lower triangular matrix if it's diagonal's above entries are zeros.

$$L = [a_{ij}]_{m \times n} \ ; \text{where if } i < j \text{ then } a_{ij} = 0$$

Triangularizaⁿ method:

This method is also known as factorization method. In this method, the coefficient matrix 'A' of the system of eqⁿ can be factorized into the product of upper triangular as well as lower triangular matrix.

$$\boxed{A = LU}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Using matrix multiplicaⁿ, multiplying L & U and comparing it with the coefficient matrix, we get

step-1 :

$Ax = b$.

$\Rightarrow (LU)\, x = b$

step-2 :-

$L(Ux) = b$

$\Rightarrow Lz = b$

step-3 :

$z = Ux$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \& \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

For the shake of simplicity either L or U as a diagonally dominant matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$LU = \begin{bmatrix} u_{11} & \cancel{u_{12}} & u_{13} \\ l_{21} u_{11} & (l_{21} u_{12} + u_{22}) & l_{21} u_{12} + l_{22} u_{22} + \cancel{l_{33}} u_{33} \\ l_{31} u_{13} & l_{31} u_{12} + l_{32} u_{22} & \end{bmatrix}$$

$$LU = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} u_{11} & l_{21} u_{12} + u_{22} & l_{21} u_{13} + u_{23} \\ l_{31} u_{11} & l_{31} u_{12} + l_{32} u_{22} & l_{31} u_{13} + l_{32} u_{33} + u_{33} \end{bmatrix}$$

$u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$

$l_{21} = \frac{a_{21}}{a_{11}}$, $u_{22} = a_{22} - l_{21} \cdot a_{12}$

$l_{31} = \frac{a_{31}}{a_{11}}$, $u_{23} = a_{23} - \cancel{\frac{l_{21} \cdot a_{13}}{u_{22}}}$ $a_{23} - l_{21} \cdot a_{13}$

$u_{33} = a_{33} - l_{31} \cdot a_{13} - l_{32} \cdot u_{23}$

$$l_{32} = \frac{a_{32} - l_{31} a_{12}}{u_{22}}$$

Q// Solve by using triangularization method ?

$10x_1 + x_2 + x_3 = 12$

$x_1 + 10x_2 + x_3 = 12$

$x_1 + x_2 + 10x_3 = 12$

Here $A = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $b = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$

$$L_{21} = \frac{1}{10}, \quad l_{31} = \frac{1}{10}, \quad l_{32} = \frac{1 - \frac{1}{10}}{\frac{99}{10}} = \frac{1}{11}$$

$u_{11} = 10$

$u_{12} = 1$

$u_{13} = 1$

$$u_{22} = 10 - \frac{1}{10} = \frac{99}{10}$$

$$u_{23} = 1 - \frac{1}{10} \cdot 1 = \frac{9}{10}$$

$$u_{33} = 10 - \frac{1}{10} \cdot 1 - \frac{1}{11} \cdot \frac{9}{10}$$

$$= 10 - \frac{1}{10} - \frac{9}{110} = \frac{1100 - 11 - 9}{110} = \frac{1080}{110}$$

$$= \frac{108}{11}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{10} & 1 & 0 \\ \frac{1}{10} & \frac{1}{11} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 10 & 1 & 1 \\ 0 & \frac{99}{10} & \frac{9}{10} \\ 0 & 0 & \frac{108}{11} \end{bmatrix}$$

step : 2

$LZ = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{10} & 1 & 0 \\ \frac{1}{10} & \frac{1}{11} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} z_1 \\ \frac{1}{10} z_1 + z_2 \\ \frac{1}{10} z_1 + \frac{1}{11} z_2 + z_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$$

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$\Rightarrow z_1 = 12$

$\frac{1}{10} z_1 + z_2 = 12$

$\Rightarrow z_2 = 12 - \frac{12}{10} = 10.8$

$\frac{1}{10} z_1 + \frac{1}{11} z_2 + z_3 = 12$

$\Rightarrow z_3 = 12 - \frac{12}{10} - \frac{10.8}{11} = \frac{108}{11}$

Step : 3

$Z = UX$

$\Rightarrow UX = Z$

$$\Rightarrow \begin{bmatrix} 10 & 1 & 1 \\ 0 & \frac{99}{10} & \frac{9}{10} \\ 0 & 0 & \frac{108}{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10.8 \\ \frac{108}{11} \end{bmatrix}$$

Using backward substitution, we get

$10x_1 + x_2 + x_3 = 12$

$\frac{99}{10} x_2 + \frac{9}{10} x_3 = 10.8$

$x_3 = \frac{108}{11} \times \frac{11}{108} = 1$

$\therefore \frac{99}{10} x_2 = 10.8 - \frac{9}{10} x_3 = 10.8 - \frac{9}{10} = 9.9$

$\Rightarrow x_2 = \frac{9.9}{9.9} = 1$

$10x_1 + 1 + 1 = 12$

$\Rightarrow x_1 = 1$ So $x_1 = x_2 = x_3 = 1$ (Ans.)

Q) $x + y + z = 3$

$2x + 3y + 4z = 9$

$3x - 2y + z = 1$

Q) $3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

10.8.16 Gauss Elimination Method :-

$\cancel{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$

$\cancel{1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{2}}$

Elementary transformation :-

(i) Interchanging any row or any column, it does not impact to the matrix.

(ii) Multiplying a scalar to any row or any column, it does not impact to the matrix.

(iii) By adding any row to the corresponding as well as in case of column, it does not impact to the matrix.

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$K = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Step-1

$$\sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right]$$

Step-2

$$\sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right]$$

$a_1x + b_1y + c_1z = d_1$

$b_2'y + c_2'z = d_2'$

$c_3''z = d_3''$

Q) Solve the eq^n by using Gauss elemination method?

$x + y = 2$

$2x + 3y = 5$

Ans :

$$K = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right]$$

By taking $R_3 \rightarrow R_2 - 2R_1$

$$K = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$x + y = 2$

& $y = 1 \Rightarrow x = 1$

Q1 $x+y+z=9$

$2x-3y+4z=13$

$3x+4y+5z=40$

Ans :-

$$K = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

By taking $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$K = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

By taking $R_3 \to 5R_3 + R_2$

$$K = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

By using backward substitution method

$\therefore 12z = 60$

$\Rightarrow z = 5$

$-5y + 2z = -5$

$\Rightarrow -5y = -5 - 10$

$\Rightarrow y = 3$

$x = 9 - 3 - 5 = 1$

$\therefore x = 1, y = 3, z = 5$ (Ans.)

Q1 $9x + 4y + 3z = -1$

$5x + y + 2z = 1$

$7x + 3y + 4z = 1$

Sol^n:

$$K = \left[\begin{array}{ccc|c} 9 & 4 & 3 & -1 \\ 5 & 1 & 2 & 1 \\ 7 & 3 & 4 & 1 \end{array}\right]$$

By taking $R_2 \to 9R_2 - 5R_1$

$R_3 \to 9R_3 - 7R_1$

$$K = \left[\begin{array}{ccc|c} 9 & 4 & 3 & -1 \\ 0 & -11 & 3 & 14 \\ 0 & -1 & 15 & 16 \end{array}\right]$$

By taking $R_3 \to 11R_3 - R_2$

$$K = \left[\begin{array}{ccc|c} 9 & 4 & 3 & -1 \\ 0 & -11 & 3 & 14 \\ 0 & 0 & 162 & 162 \end{array}\right]$$

By using backward substitution method,

$162z = 162$

$\Rightarrow z = 1$

$-11y + 3z = 14$

$\Rightarrow -11y = 14 - 3 = 11 \Rightarrow y = -1$

$$9x = -1 - 4y - 3z$$
$$= -1 + 4 - 3$$
$$= 0$$
$$\Rightarrow x = 0$$

$\therefore$ Hence $x = 0$, $y = -1$ & $z = 1$ (Ans.)

* If $K = \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 7 \end{array}\right]$

The system has no solution.

If $K = \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The system has infinite no. of solution.

* $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$

$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4$

So $K = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & b_4 \end{array}\right]$

0

Inversion Method :-

(Gauss Jordan Method)

$$A^{-1} = \frac{Adj.\,A}{|A|}$$

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Q// Find the inverse by using gaussian method?

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$(A, I) = \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

By taking $R_2 \to R_2 + 5R_1$

$R_3 \to 3R_3 - 5R_1$

$$(A, I) = \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & -1 & 1 & -5 & 0 & 3 \end{array}\right]$$

By taking $R_3 \to R_2 + R_3$

$$(A, I) = \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right]$$

$3x_1 - x_2 + x_3 = 1$ | $3x_1 - x_2 + x_3 = 0$ | $3x_1 - x_2 + x_3 = 0$

$x_2 = 5$ | $x_2 = 1$ | $x_2 = 0$

$x_3 = 0$ | $x_3 = 1$ | $x_3 = 3$

$\Rightarrow x_1 = 2$ | $\Rightarrow x_1 = 0$ | $x_1 = -1$

So the inverse matrix is given by

$$A^{-1} = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

H.W

Q// $\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}$ Find the inverse of this matrix?

Q// $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ find the inverse of this matrix?

$$\text{Inverse of } A = \frac{Adj(A)}{|A|}$$

$$\text{co. factor} = C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Iteration Method.

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

Iteration method will converge at these conditions -

$|a_1| > |b_1| + |c_1|$

$|b_2| > |a_2| + |c_2|$

$|c_3| > |a_3| + |b_3|$

Gauss - Jacobi Method :-

$a_1 x = d_1 - b_1 y - c_1 z$

$$\Rightarrow x = \frac{d_1}{a_1} + \left(-\frac{b_1}{a_1}\right) y + \left(-\frac{c_1}{a_1}\right) z$$

$x = k_1 + l_1 y + m_1 z$

$y = k_2 + l_2 x + m_2 z$

$z = k_3 + l_3 x + m_3 y$

$$\left.\begin{aligned} x^{k+1} &= k_1 + l_1 y^k + m_1 z^k \\ y^{k+1} &= k_2 + l_2 x^k + m_2 z^k \\ z^{k+1} &= k_3 + l_3 x^k + m_3 y^k \end{aligned}\right\} \text{—— (i)}$$

$x^{(0)} = y^{(0)} = z^{(0)} = 0$

Initially $x^{(0)}, y^{(0)}$ & $z^{(0)}$ are zeros.

Procedure is continued till the convergence is assured correct to required decimal place to get the $(k+1)$ iterates & we use the values of the kth iterates.

In the absence of values of x, y, z we usually take $(0,0,0)$ as the initial estimate.

The iteraⁿ method will converge if the absolute values of the leading diagonal elements of the coefficient matrix[A] of the system $AX = B$ are greater than the sum of the absolute values of the other coefficient of that row.

Q1 Solve by using Gauss-Jacobi method.

$$10x + y + z = 1$$

$$x + 10y + z = 1$$

$$x + y + 10z = 1$$

Solⁿ :- $|a_1| = |10| > |1| + |1|$

$|b_2| = |10| > |1| + |1|$

$|c_3| = |10| > |1| + |1|$

Conditions are satisfied.

$$10x + y + z = 1$$

$$\to x = \frac{1}{10} + \left(-\frac{1}{10}\right)y + \left(-\frac{1}{10}\right)z$$

$$= \frac{1}{10}(1 - y - z)$$

Similarly $y = \frac{1}{10}(1 - x - z)$

& $z = \frac{1}{10}(1 - y - x)$

$$x^{k+1} = \frac{1}{10}(1 - y^k - z^k)$$

$$y^{k+1} = \frac{1}{10}(1 - x^k - z^k)$$

$$z^{k+1} = \frac{1}{10}(1 - y^k - x^k)$$

Initially put $x^{(0)} = y^{(0)} = z^{(0)} = 0$

1st iteraⁿ :-

$$x' = \frac{1}{10}, \quad y' = \frac{1}{10} \quad \& \quad z' = \frac{1}{10}$$

2nd iteraⁿ :-

$x^2 = \frac{1}{10}(1-y'-z') = \frac{1}{10}(1-\frac{1}{10}-\frac{1}{10}) = 0.08$

$y^2 = \frac{1}{10}(1-x'-z') = \frac{1}{10}(1-\frac{1}{10}-\frac{1}{10}) = 0.08$

$z^2 = \frac{1}{10}(1-y'-x') = \frac{1}{10}(1-\frac{1}{10}-\frac{1}{10}) = 0.08$

3rd iteraⁿ :-

$x^3 = \frac{1}{10}(1-y^2-z^2) = \frac{1}{10}(1-0.08-0.08) = 0.084$

$y^3 = \frac{1}{10}(1-x^2-z^2) = \frac{1}{10}(1-0.08-0.08) = 0.084$

$z^3 = \frac{1}{10}(1-x^2-y^2) = \frac{1}{10}(1-0.08-0.08) = 0.084$

4th iteraⁿ :-

$x^4 = \frac{1}{10}(1-y^3-z^3) = 0.0832$

$y^4 = \frac{1}{10}(1-x^3-z^3) = 0.0832$

$z^4 = \frac{1}{10}(1-x^3-y^3) = 0.0832$

Q/ Solve by using Gauss-Seidel method ?

$20x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

$x^{(0)} = y^{(0)} = z^{(0)} = 0$

$x^{(1)} = K_1 + l_1 y^{(0)} + m_1 z^{(0)}$

$y^{(1)} = K_2 + l_2 x^{(1)} + m_2 z^{(0)}$

$z^{(1)} = K_3 + l_3 x^{(1)} + m_3 z^{(1)}$

$x^{(2)} = k_1 + l_1 y^{(1)} + m_1 z^{(1)}$

$y^{(2)} = k_2 + l_2 x^{(2)} + m_2 z^{(1)}$

$z^{(2)} = k_3 + l_3 x^{(2)} + m_2 y^{(2)}$

<u>Sol^n</u> :-

$20x = 17 - y + 2z$

$x = \frac{17}{20} - y/20 + 1/10 x$ — (i)

$\cancel{20} y = -\frac{18}{20} - \frac{3}{20}x + \frac{z}{20}$ — (ii)

$z = \frac{25}{20} - \frac{1}{10}x + \frac{3}{10}y$ — (iii)

$x^{(0)} = y^{(0)} = z^{(0)} = 0$

<u>1st iteraⁿ :-</u>

$x' = \frac{17}{20} = 0.85$

$y' = -\frac{18}{20} + \left(-\frac{3}{20} \times \frac{17}{20}\right) = -1.0275$

$z' = \frac{25}{20} - \left(\frac{1}{10} \times -1.0275\right) + \frac{3}{10}\left(-\frac{69}{20}\right) = 1.0108$

<u>2nd iteraⁿ :-</u>

$x^2 = \frac{17}{20} - \frac{1}{20}(-1.0275) + \frac{1}{10}(1.0108)$

$= 1.0021$

$y^2 = -\frac{18}{20} - 3/20 \times (1.0021) + \frac{1.0108}{20} = -0.9998$

$z^2 = 0.9998$

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Finite Differences :

$$\underset{x_0}{+}\overset{h}{—}\underset{x_1}{+}\overset{h}{—}\underset{x_2}{+}\text{———}\underset{x_{n-1}}{+}\overset{h}{—}\underset{x_n}{+}$$

$$a \leq x_0 \leq x_1 \leq x_2 \leq \cdots\cdots \leq x_{n-1} \leq x_n \leq b$$

$$x_i = x_0 + ih \quad , \quad x_1 = x_0 + h \quad , x_0 = h$$

$$y = f(x) \qquad \cancel{x_{0i}} = x_0 + nh \qquad x_n - x_{n-1} = h$$

$$\Rightarrow x_n - x_0 = nh$$

Suppose $f(x)$ is continuous in $[a, b]$ regulated at $n+1$ equi-space nodal points, then

$$\frac{x_n - x_0}{h} = n$$

$$\frac{b-a}{h} = n$$

$$\frac{b-a}{n} = h$$

(1) Shift Difference Operator
(2) Forward " "
(3) Backward " "
(4) ~~Average~~ central " "
(5) Average " "

If 'E' is the shift operator, then

$$\left.\begin{aligned} &E(f(x_i)) = f(x_i + h) \\ &E(f(x_0)) = f(x_0 + h) = f(x_1) \\ &E(f(x_1)) = f(x_2) \\ &E(f(x_2)) = f(x_3) \\ &\boxed{E(f(x_n)) = f(x_{n+1})} \end{aligned}\right\} \text{Shift Difference Operator}$$

Higher Order Shift Operator :-

$$E^2(f(x_i)) = E(E(f(x_i)))$$
$$= E(f(x_i+h))$$
$$= f(x_i+2h)$$

$$\boxed{\begin{aligned} E^n(f(x_i)) &= f(x_i+nh) \\ E^{-n}(f(x_i)) &= f(x_i-nh) \end{aligned}}$$

Forward Difference Operator :-

Δ (Forward operator)

$$\Delta(f(x_i)) = f(x_i+h) - f(x_i)$$
$$\Delta(f(x_0)) = f(x_0+h) - f(x_0)$$
$$= f(x_1) - f(x_0)$$
$$= f_1 - f_0$$

$$\Delta(f(x_1)) = f(x_2) - f(x_1)$$
$$\boxed{\Delta(f(x_n)) = f(x_{n+1}) - f(x_n)}$$

Higher order forward difference operator :-

$$\Delta^2(f(x_i)) = \Delta(\Delta(f(x_i)))$$
$$= \Delta[f(x_i+h) - f(x_i)]$$
$$= \Delta(f(x_i+h)) - \Delta(f(x_i))$$
$$= f(x_i+2h) - f(x_i+h) - [f(x_i+h) - f(x_i)]$$
$$= f(x_i+2h) - 2f(x_i+h) + f(x_i)$$
$$= c(2\ 0)f(x_i+2h) - c(2\ 1)f(x_i+h) + c(2\ 2)f(x_i)$$

$$\boxed{c(n\ r) = {}^nc_r}$$

$\because {}^nc_0 = 1$

${}^nc_1 = n$

${}^nc_n = 1$

$$\boxed{{}^nc_r = \frac{n!}{r!(n-r)!}}$$

$$\Delta^2(f(x_i)) = \sum_{k=0}^{2} (-1)^k \, c(2\ k) f_{i+2-k}$$

$$\boxed{\Delta^n(f(x_i)) = \sum_{k=0}^{n} (-1)^k \, c(n\ k) f_{i+n-k}}$$

By using Pascal's triangle we can also simplify i.e.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

. . . .

→ The difference of a constant function is zero i.e,

$$\Delta c = 0$$

→ The operation 'Δ' is commutative i.e,

$$\Delta c(f(x)) = c\Delta(f(x))$$

→ 'Δ' is distributive i.e,

$$\Delta(f(x) + g(x)) = \Delta(f(x)) + \Delta(g(x))$$

→ If 'a' and 'b' are two constants, then

$$\Delta[af(x) + bg(x)] = a\Delta f(x) + b\Delta g(x)$$

→ Indices law holds on 'Δ'.

$$\Delta^n(\Delta^m f(x)) = \Delta^{n+m}(f(x))$$

→ $\Delta[f(x).g(x)] = f(x+h)\Delta(g(x)) + g(x)\Delta(f(x))$

Proof : $\Delta[f(x).g(x)] = f(x+h)g(x+h) - f(x)g(x)$

$= f(x+h)g(x+h) - f(x+h)g(x)$
$\quad + f(x+h)g(x) - f(x)g(x)$

$= f(x+h)[g(x+h) - g(x)]$
$\quad + g(x)[f(x+h) - f(x)]$

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$= f(x+h)\Delta g(x) + g(x)\Delta f(x)$

→ Division of two functions:-

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).\Delta[f(x)] - f(x)\Delta(g(x))}{g(x+h)g(x)}$$

Proof : $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$

$$= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{[f(x+h) - f(x)]g(x) + f(x)[g(x) - g(x+h)]}{g(x+h)g(x)}$$

$$= \frac{\Delta[f(x)]g(x) - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)}$$

$$= \frac{g(x)\Delta(f(x)) - f(x)\Delta(g(x))}{g(x+h)g(x)}$$

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→ $\Delta(e^x) = e^{x+h} - e^x = e^x(e^h - 1)$

→ $\Delta(\tan^{-1}x) = \tan^{-1}(x+h) - \tan^{-1}x$

$$= \tan^{-1}\frac{(x+h) - x}{1 + (x+h)x}$$

Eg: (i) $\Delta\left(\frac{1}{2x+3}\right) = \frac{1}{2(x+1)+3} - \frac{1}{2x+3}$

$$= \frac{1}{2x+5} - \frac{1}{2x+3}$$

(ii) $\Delta^2\left(\frac{1}{x^2 - 5x + 6}\right) = ?$

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\Rightarrow A(x-3) + B(x-2) = 1$$

$$\Rightarrow (A+B)x - 3A - 2B = 1$$

Applying $x = 3$

$$3A + 3B - 3A - 2B = 1$$

$$\boxed{B = 1}$$

Applying $x = 2$

$$\boxed{A = -1}$$

Eg:- $\Delta^2\left(\frac{1}{x^2-5x+6}\right) = \Delta\left(\Delta\left(\frac{-1}{x-2} + \frac{1}{x-3}\right)\right)$

$$= \Delta\left(\frac{-1}{(x+1)-2} - \frac{-1}{x-2} + \frac{1}{x+1-3} - \frac{1}{x-3}\right)$$

$$= \Delta\left(\left(\frac{-1}{x-1} + \frac{1}{x-2}\right) + \left(\frac{1}{x-2} - \frac{1}{x-3}\right)\right)$$

$$= \Delta\left(-\frac{1}{x-1} + \frac{1}{x-2}\right) + \Delta\left(\frac{1}{x-2} - \frac{1}{x-3}\right)$$

$$= \frac{-1}{(x+1)-1} - \frac{-1}{x-1} + \frac{1}{(x+1)-2} - \frac{1}{x-2}$$

$$+ \frac{1}{(x+1)-2} - \frac{1}{x-2} - \left(\frac{1}{x+1-3} - \frac{1}{x-3}\right)$$

$$= -\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-1} - \frac{1}{x-2} + \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x-2} + \frac{1}{x-3}$$

$$= -\frac{1}{x} + \frac{3}{x-1} - \frac{3}{x-2} + \frac{1}{x-3}$$ (Ans.)

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

Eg: Form the forward difference table for the following data ?

x	0	1	2	3	4
y	8	11	9	15	6

Ans:-

x	y	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	8				
		3			
1	11		-5		
		-2		$+13$	
2	9		8		-36
		6		-23	
3	15		-15		
		-9			
4	6				

Backward Difference Operator :
('∇')

$$\boxed{\nabla(f(x_i)) = f(x_i) - f(x_i - h)}$$; where $i = 1, 2, \ldots, n$

$\nabla(f_1) = f(x_1) - f(x_0)$

$$\boxed{\nabla f_n = f_n - f_{n-1}}$$

Higher order backward difference operator :-

$\nabla^2(f(x_i)) = \nabla(\nabla(f(x_i))) = \nabla(f(x_i) - f(x_i - h))$

$= \nabla(f(x_i)) - \nabla(f(x_i - h))$

$= f(x_i) - f(x_i - h) - [f(x_i - h) - f(x_i - 2h)]$

$= f(x_i) - 2f(x_i - h) + f(x_i - 2h)$

Scanned by CamScanner

$= c(2\ 0) f(x_i) - c(2\ 1) f(x_i - h) + c(2\ 2) f(x_i - 2h)$

$\Rightarrow \nabla^2(f(x_i)) = \sum_{k=0}^{2} (-1)^k c(2\ k) f_{i-k}$

$\therefore \boxed{\nabla^n(f(x_i)) = \sum_{k=0}^{n} (-1)^k c(n\ k) f_{i-k}}$

Backward difference table :-

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0	∇f_1			
x_1	f_1		$\nabla^2 f_2$	$\nabla^3 f_3$	
x_2	f_2	∇f_2	$\nabla^2 f_3$		$\nabla^4 f_4$
x_3	f_3	∇f_3		$\nabla^3 f_4$	
x_4	f_4	∇f_4	$\nabla^2 f_4$		

31/08/16 **Divided Differences :-**

$y = f(0) + [a\ b]$

$a \leq x_0 \leq x_1 \leq \cdots \cdots \leq x_n \leq b$

$x_i := x_0 + ih$ where $i = 1, 2, \cdots, n$

$f(x) = [x_0\ x_1]$

$f[x_0\ x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{\Delta f_0}{h}$

$f[x_0\ x_1\ x_2] = \dfrac{f[x_1\ x_2] - f[x_0 - x_1]}{x_2 - x_0} = \dfrac{\Delta f_1 - \Delta f_0}{2h}$

$= \dfrac{\Delta^2 f_0}{2h}$

$f[x_1\ x_2\ x_3] = \dfrac{f[x_2\ x_3] - f[x_1 - x_2]}{x_3 - x_1}$

$$\boxed{f[x_0\ x_1 \cdots\cdots x_n] = \frac{f[x_1 x_2 \cdots\cdots x_n] - f[x_0\ x_1 \cdots x_{n-1}]}{x_n - x_0}}$$

Divided difference table :-

x	$f(x)$	1st Divided Diff^n	2nd D. Differences	3rd D. Differences	4th D.D
x_0	$f(x_0)$				
		$f[x_0\, x_1]$			
x_1	$f(x_1)$		$f[x_0\, x_1\, x_2]$		
		$f[x_1\, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_2	$f(x_2)$		$f[x_1\, x_2\, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
		$f[x_2\, x_3]$		$f[x_1, x_2, x_3, x_4]$	
x_3	$f(x_3)$		$f[x_2\, x_3\, x_4]$		
		$f[x_3\, x_4]$			
x_4	$f(x_4)$				

Eg: Using divided difference table, calculate for the function

$$f(x) = x^2 + 2x + 2$$

whose arguments are 1, 2, 4, 7, 10

<u>Ans</u> :- $f(x) = x^2 + 2x + 2$

$f(1) = 5$

$f(2) = 10$

$f(4) = 26$

$f(7) = 65$

$f(10) = 122$

Divided difference table :-

x	$f(x)$	1st D.D	2nd D.D	3rd D.D	4th D.D
1	5	$\frac{10-5}{2-1} = 5$	$\frac{8-5}{4-1} = 1$	$\frac{1-1}{7-1} = 0$	$\frac{0-0}{10-1} = 0$
2	10	$\frac{26-10}{4-2} = 8$	$\frac{13-8}{7-2} = 1$	$\frac{1-1}{10-2} = 0$	
4	26	$\frac{65-26}{7-4} = 13$	$\frac{19-13}{10-4} = 1$		
7	65	$\frac{122-65}{10-7} = 19$			
10	122				

Interpolation :-

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n$$

A polynomial funⁿ $P(x)$ is called a interpolating polynomial of the value of $f(x)$ or its certain order derivatives coincides to the values of $f(x)$ or its n order derivatives at one or more tabular point i.e,

$$P(x_i) = f(x_i)$$

where $i = 1, 2, \cdots\cdots, n$

Linear interpolation :-

Let $f(x)$ be a contineous funⁿ on $[a\ b]$ tabulated at '$n+1$' equispaced such that the condition is given by $a \leq x_0 \leq x_1 \leq \cdots\cdots x_n \leq b$, we will find a polynomial approximation $P(x)$ of a degree less than 'n'. So we can find $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n$

$$f(x) = a_0 + a_1 x \quad \text{---(1)}$$

where a_0, a_1 are arbitory constants which to be determined.

$$f(x_0) = a_0 + a_1 x_0 = P(x_0) \quad \text{---(2)}$$

$$f(x_1) = a_0 + a_1 x_1 = P(x_1) \quad \text{---(3)}$$

$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0 \quad \text{---(4)}$$

$$\boxed{P(x)(x_0 - x_1) - x\left[f(x_0) - f(x_1)\right] + 1\left[x_1 f(x_0) - x_0 f(x_1)\right] = 0}$$

Langrange's interpolation :-

Expanding $eq^n(4)$ by column wise, we get

$$p(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$$

$$\Rightarrow p(x)(x_0 - x_1) - (x - x_1) f(x_0) + (x - x_0) f(x_1) = 0$$

$$\Rightarrow \boxed{p(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{(x_1 - x_0)}{(x_1 - x_0)} f(x_1)}$$

This is known as two point Langrange's interpolan formula.

For 3 pt.

$$p(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

[3 point Langranges interpolation formula]

For 4 pt.

$$p(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

[4 point Langranges interpolan formula]

For n pts

$$p(x) = \frac{(x - x_1)(x - x_2)(x - x_3)\cdots\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots\cdots(x_0 - x_n)} f(x_0) +$$

$$\frac{(x - x_0)(x - x_2)\cdots\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots\cdots(x_1 - x_n)} f(x_1) + \cdots\cdots$$

$$+ \frac{(x - x_0)(x - x_1)\cdots\cdots(x - x_{n-1})}{} f(x_n)$$

Newton's linear interpolation :-

Expanding 1st row

$P(x)(x_0 - x_1) - x[f(x_0) - f(x_1)] + 1[x f(x_0) - x_0 f(x_1)]$.

$\Rightarrow P(x)(x_0 - x_1) = x[f(x_0) - f(x_1)] + [x_0 f(x_1) - x f(x_0)]$

$$\Rightarrow P(x) = \frac{x[f(x_0) - f(x_1)]}{x_0 - x_1} + \frac{x_0 f(x_1) - x f(x_0)}{x_0 - x_1}$$

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0 - x_1]$$

$$f(x_0) + (x - x_0) = [x_0 \; x_1]$$

12.09.16

Q

	x_0	x_1	x_2
x	5	7	11
y	150	392	1452
	$f(x_0)$	$f(x_1)$	$f(x_2)$

Find the Langranges interpolation formula ?

Soln :-

Applying , 3 point Langranges interpolation formula, we get

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-7)(x-11)}{(5-7)(5-11)} \times 150 + \frac{(x-5)(x-11)}{(7-5)(7-11)} \times 392 + \frac{(x-5)(x-7)}{(11-5)(11-7)} \times 1452$$

Newton's forward interpolation formula :-

$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)\dots(x-x_{n-1})$

$= f(x_0) + \frac{(x-x_0)}{1!\,h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 f_0 + \dots$

$\frac{(x-x_0)(x-x_1)(x-x_2)}{3!\,h^3}\Delta^3 f_0 + \dots\dots$

$\dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{n!\,h^n}\Delta^n f_0$

$P(x_i) = f(x_i)$

(At $x = x_0$)

$P(x_0) = a_0 + a_1 \times 0 + \dots + a_n \times 0$

$\Rightarrow \boxed{P(x_0) = a_0 = f(x_0)}$

(At $x = x_1$)

$P(x_1) = a_0 + a_1(x_1 - x_0) + a_2 \times 0 + \dots + a_n \times 0$

$\Rightarrow P(x_1) = a_0 + a_1(x_1 - x_0)$

$\Rightarrow P(x_1) = f(x_0) + a_1(x_1 - x_0)$ ——— (i)

$a_1(x_1 - x_0) = \frac{f(x_1) - f(x_0)}{\cancel{(x_1 - x_0)}} \times \cancel{(x_1 - x_0)}$ $\left\{ \because a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right.$

$= f(x_1) - f(x_0)$

Putting these eq^n (i), we get

$P(x_1) = f(x_0) + f(x_1) - f(x_0)$

$\Rightarrow \boxed{P(x_1) = f(x_1)}$

Putting $(x - x_0) = uh \Rightarrow x = x_0 + uh$

$\Rightarrow x - x_1 = x_0 + uh - x_1 = uh - (x_1 - x_0) = uh - h = (u-1)h$

$\Rightarrow x - x_2 = (u-2)h$

$\Rightarrow (x - x_{n-1}) = \{u - (n-1)\}h = (u - n + 1)h$

$$\boxed{P(x_0+uh) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \dots + \frac{u(u-1)\dots(u-n+1)}{n!}\Delta^n f_0}$$

This is known as "Newton's Gregory Forward interpolan formula"

Newton's backward interpolation formula :-

$$P(x) = f_n + \frac{(x-x_n)\nabla f_n}{1!\,h} + \frac{(x-x_n)(x-x_{n-1})\nabla^2 f_n}{2!\,h^2} + \dots$$

$$\dots + \frac{(x-x_n)\dots(x-x_0)}{n!\,h^n}\nabla^n f_n$$

$x - x_n = uh$

$x - x_{n-1} = x - x_n + x_n - x_{n-1} = uh + h = (u+1)h$

$x - x_0 = (u-n+1)h$

$$\boxed{P(x_n+uh) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \dots + \frac{u(u+1)\dots(u+n-1)}{n!}\nabla^n f_n}$$

This is known as "Newton's Gregory Backward interpolan. formula."

Q) Find a cubic polynomial which takes these following values.

x	0	1	2	3
$f(x)$	1	2	1	10

Soln

$P(x) =$

PC

Q) Giv

fo

Soln

$$P(x) = f_0 + \frac{(x-x_0)}{1!\,h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 f_0 + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!\,h^3}\Delta^3 f_0$$

x	$f(x)$	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$P(x) = 1 + \frac{(x-0)}{1}1 + \frac{(x-0)(x-1)}{2!\,1^2}(-2) + \frac{(x-0)(x-1)(x-2)}{3!\,1^3}\times 12$$

$$= 1 + x + \{-x(x-1)\} + 2x(x-1)(x-2)$$

$$= 1 + x - x^2 + x + (2x^2 - 2x)(x-2)$$

$$= 1 + 2x - x^2 + 2x^3 - 4x^2 - 2x^2 + 4x$$

$$\boxed{P(x) = 2x^3 - 7x^2 + 6x + 1}$$

Q) Given that $\sin 45° = 0.7071$

$\sin 50° = 0.7660$

$\sin 55° = 0.8192$

$\sin 60° = 0.8660$

Find $\sin 52°$ by using any method of interpolation formula?

14.9.16

Q1

Soln :-

x	45	50	52	55	60
$f(x)$	0.7071	0.7660	?	0.8192	0.8660

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$$P(x_0+uh) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0$$

$$u = \frac{x-x_0}{h} = \frac{52-45}{5} = \frac{7}{5} = 1.4$$

$$P(52) = 0.7071 + 1.4 \times 0.0589 + \frac{1.4\times 0.4}{2}(-0.0057)$$

$$+ \frac{1.4\times 0.4\times(-0.6)}{2\times 3}(-0.0007)$$

$$= 0.707$$

Q1

d:	80	85	90	95	100	105
A :	5026	5674	6362	7088	7854	?

Find approximate values for the areas to the diameter of 105m using an approximate interpolation formula.

14.9.16

Q1 The following table give the population of town during last 6 census. Estimate using any suitable interpolation formula, the increase in the population during the year 1946-1948.

Year	1911	1921	1931	1941	1951	1961
Population	12	15	20	27	39	52

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911	12	3				
1921	15	5	2	0		
1931	20		2		3	-10
1941	27	7	5	3	-7	
1951	39	12	1	-4		
1961	52	13				

$$u_{1946} = \frac{x - x_0}{h} = \frac{1946-1911}{10} = 3.5$$

$$y_{1946} = y + u\Delta y + \frac{u(u-1)}{2!}\Delta^2 y + \frac{u(u-1)(u-2)}{3!}\Delta^3 y + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!} \times \Delta^5 y$$

$$\Rightarrow y_{1946} = 12 + 10.5 + 8.75 + 0 + 0.820 + 0.273$$

$$= 32.343$$

$u_{1948} = 3.7$

$$y_{1948} = 12 + 11.1 + 9.99 + 0 + 1.486 + 1.684$$

$$= 36.26$$

$\therefore y_{1948} - y_{1946} = 36.26 - 32.343 = 3.917$ (total increase in popula^n^)

(Ans.)

Q1 Find the no. of men getting wages betn Rs 10/- & Rs 15/- from the following table?

Wages (in Rs)	0–10	10–20	20–30	30–40
Frequency	9	30	35	42

Soln :-

Wages (in Rs) (x)	Frequency (y)	Δy	$\Delta^2 y$	$\Delta^3 y$
Under 10	9	30	5	2
Under 20	39	35	7	
Under 30	74	42		
Under 40	116			

$$u_{15} = \frac{15-10}{10} = 0.5 \qquad \left(\frac{x-x_0}{h} = \frac{15-10}{10}\right)$$

$$y_{(15)} = y + u\Delta y + \frac{u(u-1)}{2!}\Delta^2 y + \frac{u(u-1)(u-2)}{3!}\Delta^3 y$$

$$= 9 + 0.5 \times 30 + \frac{0.5(-0.5)}{2} \times 5 + \frac{0.5 \times (0.5) \times (-1.5)}{6} \times 2$$

$$= 9 + 15 + (-0.625) + 0.125$$

$$= 23.5 \simeq 24$$

$$y_{(10)} = 9$$

$\therefore$ No. of men getting wages betn Rs 10/- and Rs 15/-

$$= y_{15} - y_{10} = 24 - 9 = 15 \qquad \text{(Ans)}$$

Q. Estimate the value of f_{22}, f_{42} from the available data?

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	251	204

Q. Use the Newton's divided diff^n interpolation formula from the following data? Also find f_4?

x	0	2	3	6
$f(x)$	-4	2	14	158

Newton's Divided Interpolation formula :-

$$P(x) = f(x_0) + (x-x_0) f[x_0\ x_1] + (x-x_0)(x-x_1) f[x_0\ x_1\ x_2]$$
$$+ (x-x_0)(x-x_1)(x-x_2) f[x_0\ x_1\ x_2\ x_3] + \dots\dots ,$$
$$\dots + (x-x_0)(x-x_1)\dots(x-x_{n-1}) f[x_0\ x_1\ x_2 \dots x_n]$$

Ans :

x	$f(x)$	1st DD	2nd DD	3rd DD
0	-4			
		$\frac{2+4}{2-0} = 3$		
2	2		$\frac{12-3}{3-0} = 3$	
		$\frac{14-2}{3-2} = 12$		$\frac{9-3}{6-0} = 1$
3	14		$\frac{48-12}{6-2} = 9$	
		$\frac{158-14}{6-3} = 48$		
6	158			

to $P(x) = -4 + (x-0)3 + (x-0)(x-2)3 + (x-0)(x-2)(x-3)1$

$$= -4 + 3x + 3x^2 - \cancel{6x} + x^3 - 5x^2 + \cancel{6x}$$

$$= x^3 - 2x^2 + 3x - 4$$

$$f_4 = 40$$

Q) Find the Newton's divided diff. interpolaⁿ formula from the following data points?

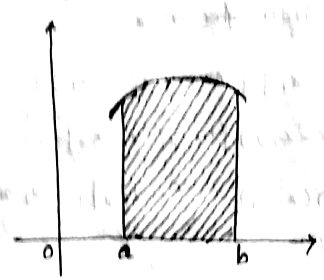
x	2	4	5	6	8	10
$f(x)$	10	96	196	350	868	1746

19.09.16

Numerical Integration :-

$f(x) \to [a\ b]$

$y = f(x)$



$$\text{Area} = \int_a^b f(x)\,dx$$

Integrand :- whom to integrate [$f(x)$].

Integral :- The sign '$\int$'.

Integration :- To find the integral of the integrand.

"Numerical Integration" is a process of computing the values of a definite integral from the tabulated values of integrand when applied to a function of single variable, the process is known as quadrature. The expression obtained is known as quadrature formula.

The accuracy of the ~~gno~~ quadrature formula depends upon following factors :

1. Size of the interval
2. Range of the integraⁿ

3. Degree of the polynomial
4. The point where it passes through

$f(x) \longrightarrow [a\ b]$

$\forall\ x \in [x_0 \ldots\ldots x_n]$

$x_i = x_0 + ih$

$\int_a^b f(x)dx$ $\quad$ $a = x_0,\ x_1,\ \ldots,\ b = x_n$

$= \int_{x_0}^{x_0+nh} f(x)\,dx$

Langrange's interpolation formula for 'n' points is given by

$$p(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

Interpolation based on integration :-

Let $f(x)$ be continuous in $[a\ b]$.

$$I = \int_a^b w(x) f(x)\,dx$$

; where $w(x) > 0$, known as weight funⁿ.

$x_i = x_0 + ih$

$$I = \int_a^b f(x)\,dx \approx \sum_{k=0}^{n} \lambda_k f_k = \lambda_0 f_0 + \lambda_1 f_1 + \cdots + \lambda_n f_n$$

Newton's Quadrature formula :-

$$R_n = \int_a^b f(x)dx - \sum_{k=0}^{n} \lambda_k f_k$$

(Reminder term)

$$\lambda_k = \frac{(-1)^{n-k}\,h}{k!\,(n-k)!} \int_0^n s(s-1)\cdots(s-n)\,ds$$

$$R_n = \frac{h^{n+2}}{(n+1)!}\int_0^n s(s-1)\cdots(s-n)\, f^{n+1}(\xi)\, ds$$

Trapezoidal Rule :-

$(n=1)$

Let $f(x) \to [x_0 \;\; x_1]$

$\forall\; x_0 < x_1$

$x_0 \xrightarrow{\quad h \quad} x_1$

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{1} \lambda_k f_k$$

$$= \lambda_0 f_0 + \lambda_1 f_1$$

As $n=1$, so $k = 0, 1$

Hence $\lambda_0 = \dfrac{(-1)^{1-0}\, h}{0!\,(1-0)!}\displaystyle\int_0^1 (s-1)\, ds$ (By taking $n=1$ & $k=0$)

$$\Rightarrow \lambda_0 = \frac{-h}{1}\left[\frac{s^2}{2} - s\right]_0^1$$

$$= -h\left(\frac{1}{2} - 1\right) = \frac{h}{2}$$

By taking $n = 1$ & $k = 1$

$$\lambda_1 = \frac{(-1)^{1-1}\, h}{1!\,(1-1)!}\int_0^1 s\, ds$$

$$= \frac{h}{1}\left[\frac{s^2}{2}\right]_0^1$$

$$= \frac{h}{2}$$

$$\int_a^b f(x)\,dx = \lambda_0 f_0 + \lambda_1 f_1$$

$$= \frac{h}{2} f(x_0) + \frac{h}{2} f(x_1)$$

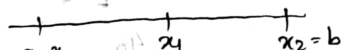
$$= \frac{h}{2}\left\{ f(x_0) + f(x_1) \right\}$$

$$\Rightarrow \boxed{\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right]}$$

$$\boxed{\begin{aligned}\int_{x_0}^{x_0+nh} f(x)\,dx &= [x_0\ x_1][x_1\ x_2]\ldots\ldots[x_{n-1}\ x_n] \\ &= \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\ldots\ldots+y_{n-1})\right]\end{aligned}}$$

Simpson's 1/3 rd rule :-

(when $n=2$)



when $n=2$, $K=0,1,2$

$x_0 < x_1 < x_2$

$x_i = x_0 + ih$

$$\int_{x_0}^{x_2} y\,dx = \sum_{K=0}^{2} \lambda_K f_K = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$$

$$\hookrightarrow \int_{x_0}^{x_0+2h} y\,dx$$

At $n=2$, $K=0$

$$\lambda_0 = \frac{(-1)^{2-0}h}{0!(2-0)!}\int_0^2 (s-1)(s-2)\,ds$$

$$= \frac{h}{2!}\left[\frac{s^3}{3} - \frac{3}{2}s^2 + 2s\right]_0^2$$

$$= \frac{h}{2}\left[\frac{8}{3} - 6 + 4\right]$$

$$= \frac{h}{2} \times \frac{2}{3}$$

$$= \frac{h}{3}$$

At $n=2, k=1$

$$\lambda_1 = \frac{(-1)^{2-1} h}{1!\,(2-1)!} \int_0^2 s(s-2)\,ds$$

$$= \frac{-h}{1!}\left[\frac{s^3}{3} - s^2\right]_0^2$$

$$= -h \times \left[\frac{8}{3} - 4\right] = -h \times \left(-\frac{4}{3}\right)$$

$$= +\frac{4h}{3}$$

At $n=2, k=2$

$$\lambda_2 = \frac{(-1)^{2-2} h}{2!\,(2-2)!} \int_0^2 s(s-1)\,ds$$

$$= \frac{h}{2}\left[\frac{s^3}{3} - \frac{s^2}{2}\right]_0^2$$

$$= \frac{h}{2}\left[\frac{8}{3} - \frac{4}{2}\right]$$

$$= \frac{h}{2} \times \frac{4}{6}$$

$$= \frac{h}{3}$$

$$\int_{x_0}^{x_0+2h} y\,dx = \frac{h}{3} f_0 + \frac{4h}{3} f_1 + \frac{h}{3} f_2$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+y_5+\cdots\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\Big]$$

$[x_0\ x_n] = [x_0\ x_1][x_1\ x_2]\cdots\cdots[x_{n-1}\ x_n]$

$[x_0\ x_n] = [x_0\ x_2][x_2\ x_4][x_4\ x_6]\cdots\cdots$

Simpson's $3/8$th rule :-

(when $n=3$)

$n=3,\ K=0,1,2,3$

$$\int_{x_0}^{x_0+3h} y\,dx = \sum_{K=0}^{3}\lambda_K f_K = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$$

At $n=3,\ K=0$

$$\lambda_0 = \frac{(-1)^{3-0}\,h}{0!\,(3-0)!}\int_0^3 (s-1)(s-2)(s-3)\,ds$$

$$= \frac{-h}{3!}\left[\frac{s^4}{4} - 2s^3 + \frac{11}{2}s^2 - 6s\right]_0^3$$

$$= \frac{-h}{6}\left[\frac{81}{4} - 54 + \frac{99}{2} - 18\right]$$

$$= -\frac{h}{6}\times\left(-\frac{9}{4}\right)$$

$$= \frac{3h}{8}$$

At $n=3,\ K=1$

$$\lambda_1 = \frac{9h}{8}$$

At $n=3,\ K=2$

$$\lambda_2 = \frac{9h}{8}$$

At $n=3,\ K=3$

$$\lambda_3 = \frac{3h}{8}$$

$$\int_{x_0}^{x_0+3h} y\,dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$$

$$= \frac{3h}{8} f(x_0) + \frac{9h}{8} f(x_1) + \frac{9h}{8} f(x_2) + \frac{3h}{8} f(x_3)$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+y_9+\cdots+y_{n-3})\right]$$

At n=5

Trapezoidal rule : $\frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$

simpson's $\frac{1}{3}$rd rule : $\frac{h}{3}\left[(y_0+y_5)+4(y_1+y_3)+2(y_2+y_4)\right]$

Simpson's $\frac{3}{8}$th rule : $\frac{3h}{8}\left[(y_0+y_5)+3(y_1+y_2+y_4) + 2(y_3)\right]$

Q// Obtain simpson's rule to find an approximate value of $\int_{-3}^{3} x^4\,dx$ by taking 7 equidistance elements or ordinates & compare it with the exact value and the value obtained by trapezoidal rule and simpson's rule and find which is the best approximaⁿ ?

Solⁿ :-

Let's divide the range of integration [-3 3] into 6 equal parts & each of with $\frac{3-(-3)}{6} = 1$

	x	$y = x^4$	
$x_6 = 3$	$x_0 = -3$	$y_0 = 81$	$y_6 = 81$
	$x_1 = -2$	$y_1 = 16$	
	$x_2 = -1$	$y_2 = 1$	
	$x_3 = 0$	$y_3 = 0$	
	$x_4 = 1$	$y_4 = 1$	
	$x_5 = 2$	$y_5 = 16$	

Trapezoidal rule :

$$\int_{x_0}^{x_0+6h} y\,dx = \int_{-3}^{3} x^4\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[162+68\right]_{-3}^{3}$$

$$= \frac{1}{2}\times 230$$

$$= 115$$

Simpson's $\frac{1}{3}$rd rule :

$$\int_{x_0}^{x_0+6h} y\,dx = \int_{-3}^{3} x^4\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[162+4\times 32+2\times 2\right]$$

$$= 98$$

Simpson's $\frac{3}{8}$th rule :

$$\int_{x_0}^{x_0+6h} y\,dx = \int_{-3}^{3} x^4\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3}{8}\left[162+3\times 34+2\times 0\right]$$

$$= \frac{3}{8}\times 264$$

$$= 99$$

Exact value :-

$$\int_{-3}^{3} x^4\,dx = \left[\frac{x^5}{5}\right]_{-3}^{3} = 97.2$$

$\therefore$ Hence, Simpson's $\frac{1}{3}$rd rule is the best approximaⁿ as it gives most neare to the exact value

Q// Compute upto 3 places of decimal, $\int_2^{10} \frac{dx}{1+x}$ in 8 equal parts ?

$$h = \frac{b-a}{n}$$

$$= \frac{10-2}{8} = 1$$

21.09.16

Q// Calculate $\int_0^6 \frac{dx}{1+x^2}$ by using trapezoidal rule and Simpson's one-third rule ?

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

Q// Calculate $\int_0^{\pi/2} e^{\sin x}\, dx$ correct to 4 decimal places ?

$$h = \frac{\pi/2 - 0}{3} = \frac{\pi}{6}$$

Q// Calculate $\int_1^2 \frac{dx}{x}$?

$$h = \frac{b-a}{n} = \frac{2-1}{6} = \frac{1}{6}$$

x	$y = 1/x$
$x_0 = 1$	$y_0 = 1$
$x_1 = \frac{7}{6}$	$y_1 = \frac{6}{7}$
$x_2 = \frac{4}{3}$	$y_2 = \frac{3}{4}$
$x_3 = \frac{3}{2}$	$y_3 = \frac{2}{3}$

Q// A river of 800 ft wide, the depth 'd' in ft at a distance x ft. from one block is given by the following ?

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

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Q) Calculate $\int_0^{\pi/2} \sqrt{\sin x}\, dx$?

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{3} = \pi/6$$

x	$y = \sqrt{\sin x}$
$x_0 = 0$	$y_0 = 0$
$x_1 = \frac{\pi}{6}$	$y_1 = \frac{1}{\sqrt{2}}$
$x_1 = \frac{\pi}{3}$	$y_2 = 0.135$

Gaussian Quadrature :-

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(x)\,dx = f(1/\sqrt{3}) + f(-1/\sqrt{3})$$

$[a \ b] \rightarrow [-1 \ 1]$

$$x = \frac{b-a}{2}u + \frac{a+b}{2}$$

$$\int_{-1}^{1} f(u)\,du = \frac{5}{9} f\left(-\sqrt{3/5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{3/5}\right)$$

* $[a \ b] \longrightarrow [0 \ 1]$, $\int_0^1 \frac{dx}{1+x}$

$$f(x) = \frac{1}{1+x}$$

$$x = \frac{b-a}{2}u + \frac{a+b}{2}$$

$$= \frac{1}{2}u + \frac{1}{2}$$

$\Rightarrow 2x = u + 1$

$\Rightarrow u = 2x - 1$

when $x = 0$, $u = -1$

$x = 1$, $u = 1$

$$dx = \frac{1}{2}du$$

$$\int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{1/2\,du}{1 + \frac{u+1}{2}} = \int_{-1}^{1} \frac{du}{u+3} \quad \therefore f(u) = \frac{1}{u+3}$$

$$\int_{-1}^{1} f(u)\,du = f(1/\sqrt{3}) + f(-1/\sqrt{3})$$

$$= \frac{1}{3+1/\sqrt{3}} + \frac{1}{3-1/\sqrt{3}}$$

$$= \frac{3-1/\sqrt{3}+3+1/\sqrt{3}}{9-\frac{1}{3}}$$

$$= \frac{6}{\frac{26}{3}} = \frac{18}{26} = 0.6923$$

$$\int_{-1}^{1} f(u)\,du = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9} \times \frac{1}{3-\sqrt{\frac{3}{5}}} + \frac{8}{9} \times \frac{1}{3} + \frac{5}{9} \times \frac{1}{3+\sqrt{\frac{3}{5}}}$$

$$= 0.6931$$

Q11 $\int_1^2 \frac{dx}{x} = ?$ & $\int_0^1 x\,dx = ?$

Solⁿ :- $f(x) = \frac{1}{x}$

$$x = \frac{b-a}{2}u + \frac{a+b}{2}$$

$$= \frac{1}{2}u + \frac{3}{2}$$

$$= \frac{u+3}{2}$$

$\Rightarrow 2x = u+3$

$\Rightarrow u = 2x-3$

when $x = 1$, $u = -1$

$x = 2$, $u = 1$

$$dx = \frac{1}{2}\,du$$

$$\int_1^2 \frac{dx}{x} = \int_{-1}^{1} \frac{\frac{1}{2}du}{\frac{u+3}{2}} = \int_{-1}^{1} \frac{du}{u+3}$$

Q. $\int_{-1}^{1} (3x^2+5x^4)\,dx$

$$\int_0^1 (3x^2+5x^4)dx = \frac{1}{2}\int_{-1}^{1} (3x^2+5x^4)\,dx$$

So $\boxed{\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\,dx}$, if $f(x)$ is an even funⁿ.

Q. $\int_1^5 \frac{dx}{x}$, $\int_0^{\pi/2} \sin x\,dx$, $\int_{-2}^{2} e^{-x/2}\,dx$, $\int_2^3 \frac{dt}{t+1}$ by using Gaussian quadrature formula ?

27.09.16

Romberg's Integration :-

$x_i = x_0 + ih \qquad i = 1, 2, \dots, n$

$h = \frac{b-a}{n}$

$I(h) \quad I(h \; h/2)$

$I(h/2) \quad I(h/2 \; h/4)$

Romberg's integraⁿ is an extrapolation formula of finding the best approximation. The method is recursive extrapolaⁿ formula which improves the values of the integrals ~~from~~ obtained from the integral.

We conclude the two successive ~~interpolation~~ integrals of the funⁿ using 1 & 2 ~~steps~~ strips and then extrapolate these integrals using 1st order strips and again extrapolate the integrals, using 1st order extrapolation, we extrapolate the integrals using 1st order extrapolation.

$\int_0^1 \frac{dx}{1+x^2}$ $\qquad a = 0, \; b = 1$

$h = \frac{b-a}{n} = 0.5$

$h = 0.5, \; 0.25, \; 0.125$

Taking $h = 0.5$ we have to calculate $\int_0^1 \frac{dx}{1+x^2}$ as per the following.

step-1

x	0	0.5	1
$y = \frac{1}{1+x^2}$	1	0.8	0.5

$$I(h) = \frac{h}{2}\left[(y_0 + y_2) + 2y_1\right]$$

$$= \frac{0.5}{2}\left[1.5 + 2 \times 0.8\right]$$

$$= \frac{0.5 \times 3.1}{2}$$

$$= 0.775$$

Taking $h = 0.25$

x	0	0.25	0.5	0.75	1
$y = \frac{1}{1+x^2}$	1	0.94	0.8	0.64	0.5

$$I(h/2) = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.25}{2}\left[1.5 + 2 \times 2.38\right]$$

$$= 0.7825 \simeq 0.783$$

Taking $h = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.98	0.94	0.88	0.8	0.72	0.64	0.56	0.5

$$I(h/4) = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{0.125}{2}\left[1.5 + 2 \times 5.52\right]$$

$$= 0.7838 \simeq 0.784$$

The table of values are as follows

0.775		
	0.008	
0.783		-0.007
	0.001	
0.784		

$$I(h\ h/2) = \frac{4I(h/2) - I(h)}{4-1}$$

$$I(h/2\ h/4) = \frac{4I(h/4) - I(h/2)}{4-1}$$

$$I(h\ h/2\ h/4) = \frac{4I(h/2\ h/4) - I(h\ h/2)}{4-1}$$

$$I(h\ h/2) = \frac{4\times 0.783 - 0.775}{3} = 0.786$$

$$I(h/2\ h/4) = \frac{4\times 0.784 - 0.783}{3} = 0.7843$$

$$I(h\ h/2\ h/4) = \frac{4\times 0.7843 - 0.786}{3} = 0.7837 \simeq 0.784$$

Q11 $\int_0^{\pi/2} \sin x\, dx = ?$

Soln: $h = \pi/4, \pi/8, \pi/16$

For $h = \pi/4$

x	0	$\pi/4$	$\pi/2$
y	0	~~0.707~~ 0.014	1

$$I(h) = \frac{\frac{\pi}{4}}{2} \times (1 + 2\times \cancel{0.707}\ 0.014)$$

$$= \cancel{0.948}\ 0.404$$

For $h = \pi/8$

x	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
y	0	0.0068	~~0.707~~ 0.014	0.0205	1

$$I(h/2) = \frac{\pi/8}{2}\left[1 + 2\times 0.0418\right]$$

$$= 0.213$$

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For $h = \frac{\pi}{16}$

x	0	$\pi/16$	$\pi/8$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$	$5\pi/16$	$3\pi/8$	$7\pi/16$	$\pi/2$
y	0	0.0034	0.0068	0.0103	0.014	0.0171	0.0205	0.024	1

$$I(h/4) = \frac{\pi/16}{2}\left[(y_0+y_8) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \frac{\pi}{32}\left[1 + 2\times 0.0725\right]$$

$$= \frac{\pi}{32}\times 1.145$$

$$= 0.1124$$

$$I(h\ h/2) = \frac{4\times 0.213 - 0.404}{3} \simeq 0.149 \simeq 0.15$$

$$I(h\ h/2)\ \cancel{h/4)} \quad \cancel{4\times 0.078}$$

$$= \frac{4\times 0.1124 - 0.213}{3} = 0.0789$$

$$I(h\ h/2\ \ h/4) = \frac{4\times 0.0789 - 0.15}{3} = 0.055 \simeq 0.06$$

Q. $\int_1^2 \frac{dx}{x}$ and $\int_0^2 \frac{dx}{x^2+4}$

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$$\frac{dy}{dx} = f(x\ y)$$

$$y(x_0) = y_0$$

$$x_i = x_0 + ih \ ; \ i = 1, 2, \ldots n$$

when $x = x_0$, $y = y_0$

Euler's method :-

If $\frac{dy}{dx} = f(x\ y)$

$y(x_0) = y_0$

$$\boxed{y_{n+1} = y_n + hf(x_n\ y_n)} \quad n = 0, 1, 2, \dots$$

Q. Use Euler's method with $h = 0.1$ to find the soln of the eqn:

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{with } y(0) = 0,\ 0 \le x \le 0.5.$$

$\frac{dy}{dx} = f(x_n\ y_n) = x_n^2 + y_n^2$

$x_0 = 0,\ y_0 = 0,\ h = 0.1$

$y_1 = y_0 + hf(x_0\ y_0)$

$= y_0 + h(x_0^2 + y_0^2)$

$= 0 + 0.1(0+0)$

$= 0.000$ at $x = 0.1$

$y_2 = y_1 + hf(x_1\ y_1)$

$= 0 + h(x_1^2 + y_1^2)$

$= 0 + 0.1(0.1^2 + 0.0^2)$

$= 0.001$ at $x = 0.2$

$y_3 = y_2 + h(x_2^2 + y_2^2)$

$= 0.001 + 0.1(0.2^2 + 0.001^2)$

$= 0.005$ at $x = 0.3$

$y_4 = y_3 + h(x_3^2 + y_3^2)$

$= 0.005 + 0.1(0.3^2 + 0.005^2)$

$= 0.014$ at $x = 0.4$

$y_5 = y_4 + h(x_4^2 + y_4^2) = 0.014 + 0.1(0.4^2 + 0.014^2)$

$= 0.03$ at $x = 0.5$

Q) Given $y' = \frac{y-x}{y+x}$, $y = 1$ when $x = 0$. Find approximate value of y for $x = 0.1$ by Euler's method using 4 steps?

$$h = \frac{0.1}{4} = 0.025$$

$$\frac{dy}{dx} = \frac{y_n - x_n}{y_n + x_n} = f(x_n, y_n)$$

$$x_0 = 0, \quad y_0 = 1$$

$$y_1 = y_0 + h\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$= 1 + 0.025\left(\frac{1-0}{1+0}\right)$$

$$= 1.025 \quad \text{at} \quad x = 0.025$$

$$y_2 = y_1 + h\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.025 + 0.025\left(\frac{1.025 - 0.025}{1.025 + 0.025}\right)$$

$$= 1.025 + 0.025\left(\frac{1}{1.05}\right) = 1.048 \quad \text{at} \quad x = 0.05$$

$$y_3 = y_2 + h\left(\frac{y_2 - x_2}{y_2 + x_2}\right)$$

$$= 1.048 + 0.025\left(\frac{1.048 - 0.05}{1.048 + 0.05}\right)$$

$$= 1.071 \quad \text{at} \quad x = 0.075$$

$$y_4 = y_3 + h\left(\frac{y_3 - x_3}{y_3 + x_3}\right) = 1.071 + 0.025\left(\frac{1.071 - 0.075}{1.071 + 0.075}\right)$$

$$= 1.0927 \quad \text{at} \quad x = 0.1$$

(Ans)

Q. Apply Euler's method to find the approximate value of 'y' corresponding to $x = 0.1$ with 4 division, given that

$$\frac{dy}{dx} = y - x^2 \quad \text{at } y = 1 \text{ when } x = 0 \text{ ?}$$

$$h = \frac{0.1}{4} = 0.025$$

Modified Euler's Method:-

The 1st appr. value of y is computed from Euler's method and then improved by the following relation :-

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_n^{(i)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(i-1)})\right]$$

$y_n^{(i)}$ denotes the iteraⁿ of y_n upto 'i' times

Q. Use modified Euler's method with 1 step to find the value of 'y' at $x = 0.1$ to ~~find~~ five significant figures where

$$\frac{dy}{dx} = x^2 + y \quad \text{and} \quad y = 0.094 \text{ with } x = 0$$

Given $h = 0.1$, $x_0 = 0$, $y_0 = 0.094$

$$\cancel{y_1} = y_0 \cancel{+h}\left(\frac{y_0 - x_0}{y_0 + x_0}\right) = \cancel{0.094 + 0.1\left(\frac{0.094}{0.094}\right)}$$

$$y_1 = y_0 + h(x_0^2 + y_0)$$

$$= 0.094 + 0.1(0^2 + 0.094)$$

$$= 0.1034 = y_1^0$$

Using modified Euler's method, for the first approximaⁿ of y_1, $n = 1$ & $i = 1$

$$y_1^1 = y_0 + \frac{0.1}{2}\left[f(x_0\ y_0) + f(x_1\ y_1^0)\right]$$

$$= 0.094 + 0.05\left(0^2 + 0.094 + 0.1^2 + 0.1034\right)$$

$$= 0.10437 \simeq 0.1044$$

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$n = 1, \; i = 2$

$$y_1^2 = y_0 + \frac{0.1}{2}\left[f(x_0\ y_0) + f(x_1\ y_1^1)\right]$$

$$= 0.094 + 0.05\left(0^2 + 0.094 + 0.1^2 + 0.10437\right)$$

$$= 0.10441$$

$$y_1^3 = 0.094 + 0.05\left(0.094 + 0.1^2 + 0.10441\right)$$

$$= 0.10442$$

$$y_1^4 = 0.094 + 0.05\left(0.094 + 0.1^2 + 0.10442\right)$$

$$= 0.10442$$

Since third and fourth approximation of y_1 is same,

so $y_1 = 0.10442$ at $x = 0.1$.

Q) Find the value of 'y' when $x = 0.2$.

$\frac{dy}{dx} = \log(x+y)$, $y = 1$ for $x = 0$ using modified Euler's method.

$h = 0.2$

$y_0 = 1, \; x_0 = 0$

$$y_1 = y_0 + hf(x_0\ y_0) = 1 + 0.2\log(x_0 + y_0)$$

$$\cancel{= 1 + \log} \qquad = 1 + 0.2\log(0+1)$$

$$= 1$$

when $n = 1, \; i = 1$

$$y_1^1 = y_0 + \frac{h}{2}\left[f(x_0\ y_0) + f(x_1\ y_1^0)\right]$$

$$= 1 + \frac{0.2}{2}\left[\log(x_0 + y_0) + \log(x_1 + y_1^0)\right]$$

$$= 1.0079$$

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when $n=1$, $i=2$

$$y_1^2 = 1 + 0.1\left(\log(0+1) + \log(0.2 + 1.0079)\right)$$

$$= 1.00820$$

when $n=1$, $i=3$

$$y_1^3 = 1 + 0.1\left(\log 1 + \log(0.2 + 1.00820)\right)$$

$$= 1.00821$$

when $n=1$, $i=4$

$$y_1^4 = 1 + 0.1\left(\log 1 + \log(0.2 + 1.00821)\right)$$

$$= 1.00821$$

Since third and fourth approximaⁿ of y_1 is same

So $y = 1.00821$ at $x = 0.2$.

iii) Q) $\frac{dy}{dx} = x + |\sqrt{y}|$ at boundary condition $y=1$, $x=0$ and $0 \leq x \leq 0.4$ in the steps of 0.2.

$h = 0.2$

$y_0 = 1$

$x_0 = 0$

$$y' = \frac{dy}{dx} = x + |\sqrt{y}|$$

$x_1 = 0.2$

$x_2 = 0.4$

Range-Kutta Method :-

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

$$x_i = x_0 + ih \; ; \; i = 1, 2, \ldots, n$$

Working rule for finding RK method :-

Let $\frac{dy}{dx} = f(x,y)$ represent any 1st order eq^n and 'h' denotes the interval between equidistance values of 'x'. If the initial values are x_0 & y_0, the first increment in 'y' can be calculated by using following formula.

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$$

$$K_3 = hf(x_0 + h, y_0 + K')$$

$$K' = hf(x_0 + h, y_0 + K_1)$$

Finally the increment, $K = \frac{1}{6}(K_1 + 4K_2 + K_3)$

$y_1 = y_0 + K$, gives the value of ordinates at the next point.

This method is known as RK Method.

4th order RK Method :-

This method is derived in the same way by using RK Method.

Let $\frac{dy}{dx} = f(x, y)$

$$y(x_0) = y_0$$

$$x_i = x_0 + ih \; ; \; i = 1, 2, \ldots, n$$

$K_1 = hf(x_0, y_0)$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y_1 = y_0 + K$

$x_1 = x_0 + h$

Q) Using RK method of order 4 find an approximate solⁿ of $\frac{dy}{dx} = x+y$. Initial condition -

$y = 1$

$x = 0$

Find y when $x = 0.2$?

<u>solⁿ</u> :-

$\frac{dy}{dx} = f(x,y) = x+y$

$x_0 = 0, y_0 = 1$

$h = 0.2$

$y_1 = y_0 + K$ —— (i)

$K_1 = hf(x_0, y_0) = h(x_0 + y_0)$

$= h(0+1) = 0.2$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$= hf(0+0.1, 1+0.1)$

$= hf(0.1, 1.1)$

$= 0.2(0.1 + 1.1)$

$= 0.24$

$K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2})$

$= 0.2 f(0.1, 1+\frac{0.24}{2}) = 0.2(0.1+1.12)$

$= 0.244$

$K_4 = hf(x_0+h, y_0+K_3)$

$= 0.2 f(0.2, 1+0.244)$

$= 0.2(0.2+1.244)$

$= 0.2888$

$K = \frac{1}{6}(K_1+2K_2+2K_3+K_4)$

$= \frac{1}{6}(0.2+2\times0.24+2\times0.244+0.288)$

$= 0.242$

$y_1 = y_0+K$

$\Rightarrow y_1 = 1+0.242 = 1.242$

Q) Apply RK method to find an approximate value of y for $x=0.2$ in steps of 0.1, if $\frac{dy}{dx} = x+y^2$ given that $y=1$ when $x=0$.

$\frac{dy}{dx} = f(x,y) = x+y^2$

$x_0 = 0, \; y_0 = 1$

$h = 0.1$

$K_1 = hf(x_0, y_0) = h(x_0+y_0^2)$

$= 0.1(0+1^2) = 0.1$

$K_2 = hf(x_0+h/2, y_0+K_1/2)$

$= hf(0+0.05, 1+0.05)$

$= 0.1\times(0.05+1.05^2)$

$= 0.1153$

$K_3 = hf(x_0 + h/2, y_0 + K_2/2)$

$= 0.1 f(0+0.05, 1+0.058)$

$= 0.1(0.05 + 1.058^2)$

$= 0.117$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1 f(0+0.1, 1+0.117)$

$= 0.1(0.1 + 1.117^2)$

$= 0.1348$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}(0.1 + 2\times 0.1153 + 2\times 0.117 + 0.1348)$

$= 0.116$

$y_1 = y_0 + K = 1 + 0.116$

$\Rightarrow y_1 = 1.116$

<u>step-2</u>

$h = 0.1$

$x_1 = x_0 + h = 0.1$

$y_1 = 1.116$

$y_2 = y_1 + K$

$K_1 = hf(x_1, y_1)$

$= 0.1(0.1 + 1.116^2) = 0.134$

$K_2 = hf(x_1 + h/2, y_1 + K_1/2)$

$= 0.1 f(0.1 + 0.05, 1.116 + 0.063)$

$= 0.1(0.15 + 1.179^2)$

$= 0.154$

$K_3 = hf(x_1 + h/2,\ y_1 + K_2/2)$

$= 0.1 f(0.1 + 0.05,\ 1.116 + 0.077)$

$= 0.157$

$K_4 = hf(x_1 + h,\ y_1 + K_3)$

$= 0.1 f(0.1 + 0.1,\ 1.116 + 0.157)$

$= 0.182$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 0.156$

$y_2 = y_1 + K$

$= 1.116 + 0.156$

$= 1.272$

Q// Solve the initial value problem –

$y' = x + y,\ y_0 = 1$ by RK method of order 4, with $h = 0.1$, $x \in (0, 0.5)$ also find the error at $x = 0.5$ if the exact sol^n is $y = 2e^x - x - 1$.

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x$$

$P = -1,\ Q = x$

$$I.F = e^{\int P dx} = e^{-\int dx} = e^{-x}$$

$$ye^{-x} = \int x e^{-x} dx + C_1$$

$$= \frac{xe^{-x}}{-1} - \int \frac{e^{-x}}{-1} dx + C$$

$$= xe^{-x} + \frac{e^{-x}}{-1} + C_1$$

$ye^{-x} = -x$

$y = -$

$\Rightarrow 1 = -$

$\Rightarrow C_1 =$

H.W.

Q// Use R

when $x =$

Q// Giv

19.10.16

Muller

P

$$y e^{-x} = -xe^{-x} - e^{-x} + c_1$$

$$y = -x - 1 + c_1 e^{x}$$

$$\Rightarrow 1 = -1 + c_1 e^{0}$$

$$\Rightarrow c_1 = 2$$

H.W.

Q. Use RK method to find an approximate value of y when $x = 1.1$ given that $y = 1.2$ when $x = 1$ and

$$\frac{dy}{dx} = 3x + y^2$$

Q. Given that $\frac{dy}{dx} = \frac{y-x}{y+x}$

$y_0 = 1$. Find $y_{0.5}$ taking $h = 0.5$?

27.02.16

Muller's Method

Let y_{i-2}, y_{i-1}, y_i be the corresponding value of $f(x)$

$P(x) = A(x - x_i)^2 + B(x - x_i) + y_i$ be a parabolic equation passing through $(x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1}), (x_i, y_i)$ then

$$y_{i-1} = A(x_{i-1} - x_i)^2 + B(x_{i-1} - x_i) + y_i = 0 \quad \text{—(ii)}$$

$$y_{i-2} = A(x_{i-2} - x_i)^2 + B(x_{i-2} - x_i) + y_i = 0 \quad \text{—(iii)}$$

Solving eq^n (ii) & (iii), we get

$$A = \frac{(x_{i-2} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i-2} - y_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)(x_{i-2} - x_i)}$$

$$B = \frac{(x_{i-2}-x_i)^2(y_{i-1}-y_i) - (x_{i-1}-x_i)^2(y_{i-2}-y_i)}{(x_{i-2}-x_{i-1})(x_{i-1}-x_i)(x_{i-2}-x_i)}$$

Putting the value of A & B in the given eqⁿ (1), the quadratic eqⁿ P(x) gives the next approximation.

$$x_{i+1} - x_i = \frac{-B \pm \sqrt{B^2 - 4Ay_i}}{2A}$$

For the direct solⁿ from P leads to loss of accuracy. Therefore the maxᵐ accuracy can be calculated by,

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

If $B > 0$, we use the positive sign to the square root of the eqⁿ and if $B < 0$ we use -ve sign to the sqrt of the eqⁿ.

Q// Find the root of the eqⁿ, $y(x) = x^3 - x^2 - x - 1 = 0$ by using Muller's method by taking initial approximaⁿ

$$x_0 = 0,\ x_1 = 1,\ x_2 = 2$$

Let $x_{i-2} = 0$

$x_{i-1} = 1$

$x_i = 2$ be the approximate root of the given eqⁿ $y(x) = x^3 - x^2 - x - 1$.

$y_{i-2} = 0^3 - 0^2 - 0 - 1$

$= -1$

$y_{i-1} = 1^3 - 1^2 - 1 - 1 = -2$

$y_i = 2^3 - 2^2 - 2 - 1 = 1$

$$A = \frac{(0-2)(-2-1) - (1-2)(-1-1)}{(1-0)(1-2)(0-2)} = \frac{4}{2} = 2$$

$$B = \frac{(0-2)^2(-2-1)-(1-2)^2(-1-1)}{(0-1)(1-2)(0-2)}$$

$$= \cancel{\frac{8+2}{1}} = \frac{-12+2}{-2} = 5$$

$$x_{i+1} - x_i = \frac{-B \pm \sqrt{B^2-4Ay_i}}{2A}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4\times 2\times 1}}{2\times 2}$$

$$= -5 \pm .$$

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2-4Ay_i}}$$

$$= 2 - \frac{2\times 1}{5 \pm \sqrt{17}} \quad \text{(Taking +ve root)}$$

$$\simeq 1.7807$$

The procedure is repeated to three approximation 1, 2 & 1.7807.

$x_{i-2} = 1$, $x_{i-1} = 2$, $x_i = 1.7807$

$y_{i-2} = -2$

$y_{i-1} = 1$

$y_i = -0.3052$

$$A = \frac{(1-1.7807)(1+0.3052) - (2-1.7807)(-2+0.3052)}{(2-1)(2-1.7807)(1-1.7807)}$$

$$= -3.78$$

$$B = \frac{(1-1.7807)^2(1+0.3052) - (2-1.7807)^2(-2+0.3052)}{(1-2)(2-1.7807)(1-1.7807)}$$

$$= -5.123$$

$$x_{i+1} = x_i - \frac{2y_i}{B \pm \sqrt{B^2 - 4Ay_i}}$$

$$= 1.7807 - \frac{2 \times (-0.3052)}{-5.123 + \sqrt{-5.123^2 - 4 \times (-3.78) \times (-0.3052)}}$$

$$= 1.8378$$

Procedure is repeated with three approximation 2, 1.7807 & 1.8378

$A = 4.619024$

$B = 5.467225$

$x_{i+1} = 1.839284$

The procedure can be repeated by using three approximation 1.7807, 1.8378, 1.839284

$y_{i-2} = -0.304808$, $y_{i-1} = -0.007757$, $y_i = -0.00095$

$A = 4.200000$, $B = 5.200000$

$x_{i+1} = 1.839287$

Q// Using Muller's method find the root of the eq^n $x^3 - 2x + 5 = 0$ which lies bet^n 2 & 3.

$x_{i-2} = 1.9$, $x_{i-1} = 2$, $x_i = 2.1$

25.10.16

Multistep Method :-

In Multistep method, evaluation of $f(x, y)$ at previous points i.e. f only at the node points not any intermediate points bet^n (x_i, y_i) & (x_{i+1}, y_{i+1}).

One-step Method :-

A one step method uses information at the current point (x_i, y_i) and the next pt. (x_{i+1}, y_{i+1}) to compute new solution value y_{i+1}.

General formula for one step method :-

$$y_{i+1} = y_i + h\left[b_0 f_{i+1} + b_1 f_i\right]$$

Two-step method :-

A two-step method uses values of y and or f at the current points (x_i, y_i) and previous points (x_{i-1}, y_{i-1}) as well as the next pts. (x_{i+1}, y_{i+1}).

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + h\left[b_0 f_{i+1} + b_1 f_i + b_2 f_{i-2}\right]$$

In addition to a no. of steps utilize multistep methods are also distinguished according to whether the coefficient of the f_{i+1} term is 0 or not.

Explicit Method :-

Multistep methods in which the coefficient of the f_{i+1} term is '0', then it is called explicity method.

Implicit method :

Multistep methods in which the coefficient of the f_{i+1} term is not '0', ~~then it is called implicity method~~. Then the unknown y_{i+1}

Three-steps method

In this case, the general form of three step method is given by

$$y_{i+1} = a_1 y_i + a_2 y_i + a_3 y_{i-2} + h(b_0 f_{i+1} + b_1 f_i + b_2 f_{i-1} + b_3 f_{i-2})$$

According to that we can achieve general multi-step method.

$$\boxed{y_{i+1} = \sum_{j=1}^{k} a_j y_{i-j+1} + h\sum_{j=0}^{k} b_j f_{i-j+1}}$$

Adam's Multistep Method :-

In this general form of writing this method upto 'K' steps

$$y_{i+1} = y_i + h\sum_{j=0}^{K} b_j f_{i-j+1}$$

Note :-

→ The explicit Adam's method is known as ~~Adam's bashforth~~ Adam's-bashforth method i.e, $\boxed{b_0 = 0}$.

→ The implicit Adam's method is known as Adam's Moulton - Moulton method i.e, $\boxed{b_0 \neq 0}$.

Adam's-bashforth method :-

The most popular explicit multistep method is known as Adam's-bashforth method.

The general Adam's-bashforth method (K-steps) has the form

$$y_{i+1} = y_i + h\sum_{j=1}^{K} b_j f_{i-j+1}$$

Note

1. Adam's method are characterised by the fact that $a_1 = 1$ and all other $a_i = 0$.
2. Adam's-bashforth method are explicit so $b_0 = 0$.
3. For Adam's-bashforth method, the no. of steps is same as the order of the method.

2nd order Adam's-bashforth method :

For this method,

(i) y_0 is given by initial condn for differential eqn.

(ii) y_1 is found from a one-step method such as RK method

(iii) Then for $i = 1, 2, \ldots, n-1$ i.e,

$$y_{i+1} = y_i + \frac{h}{2}[3f_i - f_{i-1}]$$

~~By comparing this formula~~

3rd order
Adam's - baseforth method :

For this method,

(i) y_0 is given

(ii) y_1, y_2 are found from a one-step method.

(iii) Then for $i = \cancel{2, 3, \ldots, n-1}$ $2, 3, \ldots, n-1$

$$y_{i+1} = y_i + \frac{h}{12}\left[23f_i - 16f_{i-1} + 5f_{i-2}\right]$$

Adam's - Moulton's Method :-

2nd order AM method (AMM2) :-

The 2nd order AM method is a one step method, so y_0 is given by the initial condition for the differential eq^n, then $i = 0, 1, 2, \ldots, n-1$

$$y_{i+1} = y_i + \frac{h}{2}\left[f_{i+1} + f_i\right]$$

AMM3 :-

AMM3 is a two step method

(i) y_0 is given.

(ii) y_1 is found from a one-step method then for $y = 1, 2, \ldots, n-1$

(iii) $y_{i+1} = y_i + \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right]$

Adam's Predictor - Corrector Method :-

Adam's Predictor - corrector method of 2nd order :-

ABM 2 P-C Method :-

Q// Using AB3 method solve the initial value problem

$$\frac{dy}{dx} = -2xy^2$$

$$y(0) = 1 \quad \text{in } [0, 1]$$

$$h = 0.2$$

Used formula :- $y_{i+1} = y_i + \frac{h}{12}\left[23f_i - 16f_{i-1} + 5f_{i-2}\right]$

$$\frac{dy}{dx} = -2xy^2$$

$x_0 = 0, \; y_0 = 1, \; h = 0.2$

$$y_3 = y_2 + \frac{h}{12}\left[23f_2 - 16f_1 + 5f_0\right]$$

$x_0 = 0, \; x_1 = 0.2, \; x_2 = 0.4, \; x_3 = 0.6, \; x_4 = 0.8, \; x_5 = 1$

Here to use this method, we need to use the value of y_1 & y_2. We can calculate 3 values using any single step method of 3rd order. We use Taylor's series to find this method.

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x)$$

$$y'(x) = -2xy^2$$

$$y''(x) = -2y^2 - 4xyy'$$

$$y'''(x) = -8yy' - 4x(y')^2 - 4xyy''$$

$x_0 = 0, \; y_0 = 1$

$$y'(x) = 0$$

$$y''(x) = -2$$

$$y'''(x) = 0$$

$$y_1 = y(x_1) = y(0) + hy'(0) + \frac{h^2}{2!}y''(0) + \frac{h^3}{3!}y'''(0)$$

$$= 1 + 0 + (-0.04) + 0$$

$$= 0.96$$

$$f_1 = f(x_1, y_1) = -2x_1y_1^2$$

$$= -2 \times 0.2 \times (0.96)^2$$

$$= -0.3686$$

$$y'(0.2) = -0.3686$$

$$y''(0.2) = -1.5601152$$

$$y'''(0.2) = -8\times 0.96\times(-0.3686) - 4\times 0.2\times(-0.3686)^2 - 4\times 0.2\times 0.96\times(-1.5601152)$$

$$= 3.920323706$$

$$y(x_1+h) = y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1'''$$

$$= 0.96 + 0.2\times(-0.3686) + \frac{0.2^2}{2}(-1.5601152) + \frac{0.2^3}{6}(3.920323706)$$

$$= 0.8603047943$$

$$y(0.4) = 0.8603047943$$

$$f_2 = f(x_2\ y_2) = -2x_2 y_2^2 = -2\times 0.4\times(0.8603047943)^2$$

$$= -0.5920994713$$

$$y(0.6) = y_3 = y_2 + \frac{h}{12}[23f_2 - 16f_1 + 5f_0]$$

$$= 0.8603 + \frac{0.2}{12}[23\times(-0.59209) - 16\times(-0.3686) + 5\times 0]$$

$$= 0.7316255$$

$$x_3 = 0.6$$

$$f_3 = -2x_3 y_3^2 = -0.64232$$

$$y(0.8) = y_4 = y_3 + \frac{h}{12}[23f_3 - 16f_2 + 5f_1]$$

$$= 0.6126$$

$$f_4 = -2x_4 y_4^2 = -0.60044$$

$$y(1) = y_5 = y_4 + \frac{h}{12}[23f_4 - 16f_3 + 5f_2]$$

$$= 0.5043750441$$

(Ans.)

ABM-2 P-C method

Here we use 2^{nd} Adam's-bashforth method as predictor with a 2^{nd} order Adam's Molton method as corrector.

(i) y_0 is given as initial cond^n

(ii) y_1 is found from a one-step method

(iii) $y_i = 1, 2, \ldots, n-1$

(iv) $y^*_{i+1} = y_i + \frac{h}{2}[3f_i - f_{i-1}]$

$$y_{i+1} = y_i + \frac{h}{2}\left[f^*_{i+1} + f_i\right]$$

ABM-3 P-C Method :-

Similarly y_0 is given by initial data. y_1 & y_2 are found by one-step method. Then for $y = 0, 1, \ldots, n-1$

$$y^*_{i+1} = y_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}]$$

$$y_{i+1} = y_i + \frac{h}{12}[5f^*_{i+1} + 8f_i - f_{i-1}]$$

ABM-4 P-C Method :-

y_0 is given, y_1, y_2, y_3 are found from a one-step method for $y = 3, \ldots, n-1$

$$y^*_{i+1} = y_i + \frac{h}{24}\left[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\right]$$

$$y_{i+1} = y_i + \frac{h}{24}\left[9f^*_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}\right]$$

$$y_{i+1} = y_i + \frac{h}{24}\left[55y_i - 59y_{i-1} + 37y_{i-2} - 9y_{i-3}\right]$$

R. ① Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data

x :	5	6	9	11
$f(x)$:	12	13	14	16

② From the following table estimate the number of students who obtained marks in computer programming between 75 & 80

Marks :	35-45	45-55	55-65	65-75	75-85
No of students :	18	40	64	50	28

③ Estimate the value of $f(22)$ and $f(42)$ from the following data

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

④

Milne's predictor and corrector formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right]$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}\right]$$

Knowing 4 consecutive values of y, namely