Residue :- In a Laurent series expansion the coefficient of $\frac{1}{z^n}$ is b_n is the fun^n $f(z)$ is called residue to $f(z)$ at $z=a$

Residue at Simple Poles :- Suppose $z=z_0$ be the simple pole, then residue of $f(z)$ at $z=z_0$ is given by $\boxed{\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \left[(z-z_0) f(z)\right]}$

(Or) $\boxed{\operatorname*{Res}_{z\to z_0} f(z) = \lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{P(z)}{q'(z)}}$

Q- $f(z) = \frac{z^2}{(z-1)(z+2)}$. Find the residue.

Pole: $(z-1)(z+2)=0$

$z=1$ & $z=-2$. (S.P)

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}\left[\cancel{(z-1)}\frac{z^2}{\cancel{(z-1)}(z+2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^3-z^2}{z^2+2z-z-2}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^3-z^2}{z^2+z-2}\right]$$

$$\cancel{= \lim_{z\to 1} z} \quad (6) \quad = 1/8.$$

Method II:-

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{z^2}{z+2+z-1}.$$

$$= \lim_{z\to 1} \frac{z^2}{2z} = \frac{1}{1} = 1.$$

Q- $f(z) = \dfrac{z^2}{(z-1)^2(z+3)}$

Sol^n :- Pole $(z-1)^2(z+3) = 0$

$z = 1, 1$ & $z = -3$

MP of order 2   SP

$$\operatorname*{Res}_{z\to 1} f(z) = \lim_{z\to 1}\left[\frac{z^2}{(z-1)^2(z+3)}\right]$$

Residue of Multiple Pole :-

$$\operatorname*{Res}_{z\to z_0} f(z) = \frac{1}{(m-1)!}\left\{\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

Pole, $z = 1, 1$ MP of order 2.

$$\operatorname*{Res}_{z\to 1} f(z) = \frac{1}{(2-1)!}\left\{\lim_{z\to 1}\frac{d}{dz}\cancel{(z-1)^2}\frac{z^2}{\cancel{(z-1)^2}(z+3)}\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left(\frac{z^2}{z+3}\right)\right]$$

$$= \lim_{z\to 1}\frac{2z(z+3) - z\times z^2}{(z+3)^2} = \frac{2\times 4-1}{(1+3)^2}$$

$$= \lim_{z\to 1} \frac{8-1}{16} = \frac{7}{16}.$$

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$$\boxed{\lim_{z\to 1} \frac{d}{dz}\frac{d}{dz}\left(\frac{z^2}{z+3}\right)}$$

$$Q = \frac{z^3}{(z-1)^3(z+3)}$$

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(3-1)!}\left\{\lim_{z\to 1} \frac{d^2}{dz^2}(z-1)^3 \frac{z^2}{(z-1)^3(z+3)}\right\}$$

$$= \lim_{z\to 1} \frac{d}{dz}\frac{d}{dz}\left(\frac{z^2}{z+3}\right)$$

RESIDUE Theorem :-

Let $f(z)$ be analytic in the closed path C, except at finitely many points inside & outside C. Let $z_1, z_2 \ldots z_k$ be the poles lies inside C:

$$\int_C f(z)\,dz = 2\pi i \left[\begin{matrix}\text{sum of the residue}\\ \text{of poles lies inside } C\end{matrix}\right]$$

$$= 2\pi i\left[\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \ldots + \operatorname*{Res}_{z=z_k} f(z_k)\right].$$

$$= 2\pi i \sum_{z=1}^{k} \operatorname*{Res}_{z=z_k} (f(z))$$

Q- Find the residue,

$$\int \frac{2z+3}{z^3+4z} dz \quad C:|z|=1 \text{ circle}$$

Sol^n -

Clearly, $f(z) = \dfrac{2z+3}{z^3+4z}$.

Pole: $z^3+4z=0$

$\Rightarrow z(z^2+4)=0$

$z=0$ & $z=\pm 2i$

Clearly $z=0$ be the S.P lies inside

$$\underset{z=z_0}{\text{Res}} f(z) = \lim_{z\to 0} \frac{2z+3}{3z^2+4} = \frac{3}{4},$$

$$\int \frac{2z+3}{z^3+4z} dz = 2\pi i \times \frac{3}{4} = \frac{3\pi i}{2}.$$

Clearly, $z=\pm 2i$ be the S.P. lies outside C.

Q- Evaluate $\displaystyle\int \frac{z+1}{z^4-2z^3} dz \quad C:|z|=\frac{1}{2}$

Pole: $z^4-2z^3=0$

$\Rightarrow z^3(z-2)=0$

$z=0,0,0$ & $z=2$.

M.P of order 3, S.P

$5\times4\times3\times2$

20

12

clearly, $z=0$, is a MP of order 3 lies inside c.

$$\underset{z\to 0}{\text{Res}} = \frac{1}{(3-1)!}\left\{\lim_{z\to 0}\frac{d^2}{dz^2} z^3 \frac{z+1}{z^4-2z^3}\right\}$$

Assignment

Q1- $\int_C \frac{\sin 2z}{z^6} dz \quad c: |z|=1$

$\left(\frac{z-0}{z-z_0}\right)=1$
$z\to z_0$

Q2- $\int \frac{z+23}{z^3-4z-5} dz \quad c: |z-2|=4$

Q3- $\int_C \frac{e^z+z}{z^3-2} dz \quad c: |z|=\pi/2$

(1) $z^6=0 \Rightarrow z=0$ is a pole of order 6

$$\underset{z\to 0}{\text{Res}} \frac{1}{(6-1)!}\left[\lim_{z\to 0}\frac{d^5}{dz^5}\left[\cancel{(z-0)^6}\frac{\sin 2z}{\cancel{z^6}}\right]\right]$$

$$= \frac{1}{5!}\left[\lim_{z\to 0}\frac{d^5}{dz^5}(\sin 2z)\right]$$

$$= \frac{1}{120}\left[\lim_{z\to 0} 32\cos 2z\right]$$

$$= \frac{1}{\cancel{120}} \times \cancel{32} = \frac{\cancel{8}4}{\cancel{30}} = \frac{4}{15}$$

60, 30, 16, 8, 15

(2) $\int \frac{z-23}{z^2-4z-5}\,dz$ $\quad C : |z-2| = 4$

$z - z_0 = 4$

$z_0 = 2$

z^3-4z-5

$\Rightarrow \cancel{z^3+(-5+1)z-5=0}$

$\Rightarrow \cancel{z^3-5z+z-5=0}$

$\Rightarrow \cancel{z(z^2-5)+1(z-5)=0}$

$\cancel{\Rightarrow (z+1)(z-5)(z^2-5)=0}$

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

$z^2-4z-5=0$

$\Rightarrow z^2-5z+z-5=0$

$\Rightarrow z(z-5)+1(z-5)=0$

$\Rightarrow (z-5)(z+1)=0$

$\Rightarrow z=5 \;\&\; z=-1$

$z=5$ lies inside the circle.

$\operatorname{Res}_{z\to 5} f(z) = \lim_{z\to 5} \frac{z-23}{2z-4}$

$= \frac{5-23}{2\times 5-4}$

$= \frac{-18}{6} = -3$ Ans

$C : |z| = \pi/2$ $\quad z-(2)^{1/3}$

(3) $\int_C \frac{e^z+z}{z^3-2}\,dz.$ $\quad z^3 = (2)^{1/3}$ $\quad (z-z_0) f(z)$

$z^3 = 2$

$z^3 - 2 = 0 \Rightarrow z = (2)^{1/3}$ of order 3

$\Rightarrow z = \sqrt[3]{2}, \sqrt[3]{2}, \sqrt[3]{2}$ of order 3

$\underset{z\to(2)^{1/3}}{\text{Res}} \frac{1}{2!}\left[\lim_{z\to(2)^{1/3}} \frac{d^2}{dz^2}\left[z-(2)^{1/3}\right]\frac{e^z+z}{z^3-2}\right]$

$= \frac{1}{2}\left[\lim_{z\to(2)^{1/3}} \frac{d^2}{dz^2}\, z-(2)^{1/3}\, \frac{e^z+z}{z^3-(2^{\frac{1}{3}})^3}\right].$

$= \frac{1}{2}\left[\lim_{z\to(2)^{1/3}} \frac{d^2}{dz^2} \frac{e^z+z}{z^2+z\times 2^{1/3}+2^{\frac{2}{3}}}\right].$

(3) $\int_C \frac{e^z+z}{z^3-2}\,dz \quad C : |z| = \pi/2.$

$= (z^3)-(2^{\frac{1}{3}})^3 = (z-2^{\frac{1}{3}})(z^2+2^{\frac{2}{3}}+2z^{1/3})$

$z = 2^{1/3}$ simple pole

$\underset{z\to 2^{1/3}}{\text{Res}} = \lim_{z\to 2^{1/3}} \frac{e^z+z}{3z^2}$

$2^{\frac{1}{3}}$

$= \frac{e^{2^{1/3}} + 2^{1/3}}{3\times(2^{\frac{1}{3}})^2} = \frac{3.52+1.25}{3\times 1.58}$

$= \frac{3.52+1.25}{3\times 1.58} = 1.006$

Dt:- 06/03/18

Q- $\int_C \frac{z\cosh\pi z}{z^4+13z^2+36}$ $C: |z|=\pi$

Pole:- $z^4+13z^2+36=0$

$z=\pm 2i, \pm 3i$ They are simple poles & all lies inside C.

So, $\underset{z=2i}{\text{Res}} f(z) = \lim_{z\to 2i} \frac{z\cosh\pi z}{4z^3+26z}$

$= \frac{2i\cosh 2\pi i}{4(2i)^3+26(2i)} = \frac{2i}{4\times 8(-i)+52i}$

$= \frac{2i}{-32i+52i} = 1/10 = 0.1$

$\underset{z\to -2i}{\text{Res}} f(z) = \lim_{z\to -2i} \frac{z\cosh\pi z}{4z^3+26z}$

$= \frac{-2i\cosh\pi(-2i)}{4(-2i)^3+26(-2i)}$

$= \frac{-2i\times 1}{4\times -8\times -i-52i} = \frac{-2i}{32i-52i}$

$= 1/10 = 0.1$

$\underset{z\to 3i}{\text{Res}} f(z)$, $f(z) = \lim_{z\to 3i} \frac{3i\cosh\pi(3i)}{4(3i)^3+26(3i)}$

$= \frac{3i(-1)^3}{4\times 27\times(-i)+78i} = \frac{-3i}{-108i+78i}$

$\underset{z\to 3i}{\text{Res}} f(z) = \lim_{z\to -3i} \frac{-3i\cosh\pi(-3i)}{4\times(-3i)^3+26(-3i)}$

$$= \frac{-3i \times (-1)}{4\times(-27)(-i)-78i} = \frac{3i\times 1}{108i-78i}$$

$$= \frac{3i}{30i} = 1/10 = 0.1$$

Improper Integral:-

Evaluation of Real Integrals:-

(I) $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

(II) $\int_{-\infty}^{\infty} f(x)\, dx$ (III) $\int_{-\infty}^{\infty} e^{imx} \cos mx\, dx$ or $\int_{-\infty}^{\infty} e^{imx} \sin mx\, dx$

(IV) $\int_{-\infty}^{\infty} \frac{f(x)}{x}\, dx$

(I) $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

→ In this case, we consider closed contour 'C' as a unit circle.

i.e., $|z| = 1$

$\Rightarrow z = e^{i\theta}$ $dz = ie^{i\theta} d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

Also, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$

$= \frac{z^2+1}{2z}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2zi}$

→ Check pole & residue

→ $\oint_C f(z)\,dz = 2\pi i\,[\text{sum of the residue}]$

Q) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ $\quad C: |z| = 1$

In this case, we consider closed contour 'C' as a unit circle

$|z| = 1 \quad z = e^{i\theta}$

$\Rightarrow d\theta = \dfrac{dz}{iz}$

$\cos\theta = \dfrac{z^2+1}{2z}$

$$\int \frac{\frac{dz}{iz}}{2+\frac{z^2+1}{2z}} = \int \frac{\frac{dz}{iz}}{\frac{4z+z^2+1}{2z}}$$

$$= \int \frac{dz}{i\cancel{z}} \times \frac{2\cancel{z}}{z^2+4z+1} = \int \frac{2\,dz}{i(z^2+4z+1)}$$

$$= \frac{2}{i}\int \frac{dz}{z^2+4z+1} = \frac{2}{i}\oint_c f(z)\,dz$$

$f(z) = \dfrac{1}{z^2+4z+1}$

Pole:- $z^2+4z+1 = 0$

$z = -2 \pm \sqrt{3}$

Hence, $-2+\sqrt{3}$ lies inside c.

So, $\underset{z=-2+\sqrt{3}}{\text{Res}} f(z) = \lim\limits_{z \to -2+\sqrt{3}}$

$$= \frac{1}{2(-2+\sqrt{3})+4} = \frac{1}{-4+2\sqrt{3}+4}$$

$$= 1/2\sqrt{3}$$

So, $f(z)\,dz = \dfrac{1}{\sqrt{3}}\pi i$

So, $I = \dfrac{2}{i} \times$

Q. $\displaystyle\int_0^{\pi} \frac{\cos\theta}{17-8}$

$= \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{c}{17}$

In this case

'C' as a u

$z = e^{i\theta}$

$\cos\theta = \dfrac{z}{}$

$= \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{z^2}{17}$

$= \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{z^2+}{34z}$

$= \dfrac{1}{2}\displaystyle\int_0^{2\pi}$

$= \dfrac{1}{2i}\displaystyle\int_0^{2\pi}$

$f(z) =$

So, $\oint f(z)\,dz = 2\pi i\left(\frac{1}{2\sqrt{3}}\right)$

$$= \frac{1}{\sqrt{3}}\pi i = \frac{\pi i}{\sqrt{3}}$$

So, $I = \frac{2}{\cancel{i}} \times \frac{\pi \cancel{i}}{3} = \frac{2\pi}{3}$

Q- $\int_0^{\pi} \frac{\cos\theta}{17-8\cos\theta}\,d\theta \quad |z|=1$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos\theta}{17-8\cos\theta}\,d\theta$$

In this case, we consider closed contour 'C' as a unit circle, $|z|=1$

$z = e^{i\theta} \Rightarrow d\theta = dz/iz$

$\cos\theta = \frac{z^2+1}{2z}$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\frac{z^2+1}{2z}}{17-8\left(\frac{z^2+1}{2z}\right)} \times \frac{dz}{iz}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{z^2+1/2z}{\frac{34z-8z^2-8}{2z}} \times \frac{dz}{iz}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{z^2+1}{-8z^2+34z-8} \times \frac{dz}{iz}$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{1}{\frac{(-8z^2+34z-8)z}{z^2+1}}\,dz$$

$f(z) = \frac{z^2+1}{-8z^3+34z^2-8z}$

Pole :- $8z^3 - 34z^2 + 8z = 0$

$\Rightarrow z(-8z^2 + 34z - 8) = 0$

$\Rightarrow z = 0$ & $z = 1/4, 4$

lies inside C.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-34 \pm \sqrt{(34)^2 - 4\times(-8)\times(-8)}}{-16}$$

$$= \frac{-34 \pm 30}{-16}$$

$$= \frac{-34+30}{-16}, \frac{-34-30}{-16}$$

$$= 1/4 \ \& \ 4.$$

So,

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z \to 0} \frac{z^2+1}{68z - 24z^2 - 8}$$

$$= -1/8$$

$$\operatorname*{Res}_{z=1/4} f(z) \lim_{z \to 1/4} \frac{z^2+1}{68z - 24z^2 - 8}$$

$$= \frac{\left(\frac{1}{4}\right)^2 + 1}{68 \times \frac{1}{4} - 24 \times \left(\frac{1}{4}\right)^2 - 8} = 0.141.$$

$$\operatorname*{Res}_{z=4} f(z) \lim_{z \to 4} \frac{z^2+1}{68z - 24z^2 - 8}$$

$$= \frac{16+1}{272 - 24 \times 16 - 8} = -0.141$$

$$I = \oint_C f(z)\, dz = \frac{1}{2i} \times 2\pi i \left(-\frac{1}{8} + 0.141 - 0.141\right)$$

$$= \pi/8.$$

Q- $\displaystyle\int_0^{2\pi} \frac{\cos\theta \, d\theta}{2 + \sin\theta}$

$$= \int_0^{2\pi} \frac{\frac{z^2+1}{2z}}{2 + \frac{z^2-1}{2iz}} \times \frac{dz}{iz}$$

(576)

$$= \int_0^{2\pi} \frac{\frac{z^2+1}{\cancel{2}z}}{\frac{4iz+z^2-1}{\cancel{2}iz}} \times \frac{dz}{\cancel{i}z}$$

$$= \int_0^{2\pi} \frac{z^2+1}{(z^2+4iz-1)z}\, dz$$

$$f(z) = \frac{z^2+1}{(z^2+4iz-1)z}$$

Pole:- $z^2+4iz-1=0$, $z=0$

$(z^2-1)+4iz = 0+0i$

$z^2-1=0 \Rightarrow z^2=1$

$z=\pm 1$

$4iz=0$, $z=0$

$z=0$, $z=\pm 1$ lies inside C.

$$\boxed{\begin{aligned} z &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-4i\pm\sqrt{4i^2-4\times1\times-1}}{2} \\ &= \frac{-4i\pm\sqrt{-16+4}}{2} \\ &= \frac{-4i\pm 2\sqrt{3}i}{2} \\ &= \frac{-4i+2\sqrt{3}i}{2} \;\&\; \frac{-4i-2\sqrt{3}i}{2}. \end{aligned}}$$

$$\operatorname*{Res}_{z=0} f(z) \lim_{z\to 0} \frac{z^2+1}{z(z^2+4iz-1)}$$

$$= \lim_{z\to 0} \frac{z^2+1}{3z^2+8iz-1}$$

$$= \frac{1}{-1} = -1$$

$$\operatorname*{Res}_{z=1} f(z) \lim_{z\to 1} \frac{z^2+1}{3z^2+8iz-1}$$

$$= \frac{2}{3+8i-1} = \frac{2}{8i+2}$$

$$= \frac{2}{2(1+4i)}$$

$$= \frac{1}{1+4i}$$

$$\operatorname*{Res}_{z=-1} f(z) \lim_{z\to -1} \frac{z^2+1}{(3z^2+8iz-1)}$$

$$= \frac{1+1}{3-8i-1}$$

$$= \frac{2}{2-8i} = \frac{2}{2(1-4i)}$$

$$= 1/1-4i$$

HW

Q = $\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta}\, d\theta$

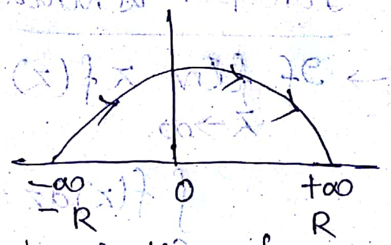
Q2- $\int_0^{2\pi} \frac{1+4\cos\theta}{17-8\cos\theta}\, d\theta$

Q3- $\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta}\, d\theta$

Date :- 08|03|18

Type A:-

$$\int_{-\infty}^{\infty} f(x)\,dx$$



Consider integral of the form

$$I = \oint_C f(z)\,dz$$

Steps:-

1) In this case the close contour C consists of upper half of the circle of radius R however large from $-R$ to $+R$.

2) Determine poles of $f(z)$ which of them lies inside C.

3) After that find the residue.

→ If the pole lies in 1st & 2nd quadrant are assumed to be pole lies inside ~~angle~~ C.

$$I = \oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

$$= 2\pi i \left[\text{sum of residue of poles lies inside}\right]$$

Taking limit $R \to \infty$

$$\lim_{R\to\infty} \int_{C_1} f(z)\,dz = 0$$ Jordan Leema

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \left[\text{sum of res. poles}\right]$$

Jordan Lemma:-

→ If $\lim_{z\to\infty} z f(z) = 0$

$$\int_{C_1} f(z)\,dz = 0$$

→ If $\lim_{z\to\infty} f(z) = 0$

$$\int_{C_1} e^{imz} f(z)\,dz = 0$$ (m is a positive real no.)

Q. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx$

Let $I = \oint_C f(z)\,dz$

Where, $f(z) = \dfrac{1}{(1+z^2)^2}$

Where C is the contour $C : |z| = 1$

Pole:- $(z^2+1)^2 = 0$

$z = \pm i, \pm i$

$z = i, i$ be the multiple pole of order 2 lies inside

$$\operatorname{Res}_{z\to i} f(z) = \frac{1}{(2-1)!}\left\{\lim_{z\to i} \frac{d}{dz} (z-i)^2 \frac{1}{(z+i)^2(z-i)^2}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{1}{(z+i)^2}\right\}$$

$$= \lim_{z\to i} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{(2i)^3} = \frac{-2}{8i^3}$$

$$= \frac{-2}{-8i} = \frac{-1}{4}i$$

$$I = \oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

$$= 2\pi i \,[\text{sum of res. of poles}]$$

Taking $R \to \infty$

$$I = \oint_C f(z)\,dz = \lim_{R\to\infty} \cancel{\int_{C_1} f(z)\,dz}^{\to 0} + \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$$

$$= \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \left[\frac{-i}{4}\right]$$

$$= \pi/2$$

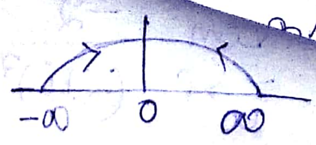
$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx = \pi/2$$

$$\rightarrow \lim_{z\to\infty} z \cdot \frac{1}{(1+z^2)^2}$$

$$= \frac{z}{z^4\left(1+\frac{1}{z^2}\right)^2}$$

$$= \frac{1}{z^3\left(1+\frac{1}{z^2}\right)^2} = 0.$$

Q- $\int_{-\infty}^{\infty} \frac{dx}{x^4+16}$



Let $I = \oint f(z)\,dz$

Where $f(z) = \frac{1}{z^4+16}$

where c is the contour $c : |z| = 1$

Pole :- $z^4+16=0 \quad z^4=-16$

$\Rightarrow z = \pm 2i, \pm 2i$

$z = 2i, 2i$ be the multiple pole of order 2 lies inside c.

$$\operatorname*{Res}_{z\to 2i} f(z) = \frac{1}{(2-1)!}\left\{\lim_{z\to 2i} \frac{d}{dz}(z-2i)\times\frac{1}{(z+2i)^2(z-2i)^2}\right\}$$

$$= \lim_{z\to 2i} \frac{-2}{(z+2i)^3}$$

$$= \frac{-2}{(2i+2i)^3} = \frac{-2}{(4i)^3}$$

$$= \frac{2}{64i} = \frac{1}{32i} = \frac{-i}{32}.$$

$$I = \oint_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

$$= 2\pi i\,[\text{sum of residue of poles lies inside } c.]$$

Taking $R \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left[\frac{-i}{32}\right].$$

$$= \pi/16.$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+16}\,dx = \pi/16.$$

Q- $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$

Q- $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)(9+x^2)}$

Q- $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+5)^2}$

⇒ Type 3:-

1) $\displaystyle\int_{-\infty}^{\infty} \cos mx\, f(x)\,dx$ or $\displaystyle\int_{-\infty}^{\infty} \sin nx\, f(x)\,dx$

$$e^{imx} = \cos mx + i\sin mx$$

$$\cos mx = \frac{e^{imx}+e^{-imx}}{2}$$

$$\sin mx = \frac{e^{imx}-e^{-imx}}{2}$$

2) Check pole & residue which lies inside C.

3) $\displaystyle I = \oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$

$= 2\pi i\,[\text{sum of res. poles}]$

4) $\displaystyle\lim_{R\to\infty} \int_{C_1} f(z)\,dz \neq \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$

$= 2\pi i\,[\text{sum of residue poles}]$

5) According to Jordan Leema

$$\lim_{R\to\infty} \int f(z)\,dz = 0$$

But $\lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$

$$= \int_{-\infty}^{\infty} e^{imx} f(x)\,dx = 2\pi i \left[\text{sum of residue poles}\right]$$

$$= \int_{-\infty}^{\infty} (\cos mx + i\sin mx) f(x)\,dx = 2\pi i [\quad].$$

$$\Rightarrow \underbrace{\int_{-\infty}^{\infty} \cos mx\, f(x)\,dx}_{u} + i \underbrace{\int_{-\infty}^{\infty} \sin mx\, f(x)\,dx}_{v}$$

Equating real & imaginary parts respectively, we get

$$\Rightarrow \int_{-\infty}^{\infty} \cos mx\, f(x)\,dx = \text{Real Part}$$

$$\& \int_{-\infty}^{\infty} \sin mx\, f(x)\,dx = \text{Imag. Part}$$

By method of contour integration

$$\rightarrow \int_{0}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a} e^{-ma}$$

$$\rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2} = \frac{\pi}{a} e^{-ma}$$

$$\rightarrow \int_{-\infty}^{\infty} \frac{\sin mx}{x^2+a^2}\,dx = 0$$

& also deduce $\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2e}$

$$\rightarrow \int_{0}^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx = \frac{\pi}{2e} e^{-ma}$$

$$\rightarrow \int_{0}^{\infty} \frac{x \sin x}{x^2+a^2} = \frac{\pi}{2e} e^{-m}$$

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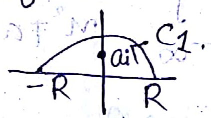
Let us consider the funⁿ $\frac{\cos mz}{z^2+a^2}$

$= \text{Real part of } \left(\frac{\cos mz + i\sin mz}{z^2+a^2}\right)$

$= \text{R.P of } \frac{e^{imz}}{z^2+a^2}$

Let $f(z) = \frac{e^{imz}}{z^2+a^2}$

Let C be the contour consisting of semi-circle C_1 of radius r which is large enough to include all poles of $f(z)$ in the upper half of the plane & also part of the real axis.



Step2:- Pole: $z^2+a^2=0$

$z = \pm ai$

$\underset{z=ai}{\text{Res}}\, f(z) = \lim_{z\to ai} \frac{e^{imz}}{2z} = \frac{e^{-ma}}{2ai}$

Step3:- $I = \int_C f(z)\,dz$

$= \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\,[\ \ ].$

$= \lim_{R\to\infty} \cancel{\int_{C_1} f(z)\,dz}^{\,0} + \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = 2\pi i \left(\frac{e^{-ma}}{2ai}\right)$

$= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{imz}}{z^2+a^2}\,dz + \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx = \frac{\pi e^{-ma}}{a}.$

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$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\cos mx + i\sin mx}{x^2+a^2}\right)dx = \frac{\pi}{a}e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}dx + i\int_{-\infty}^{\infty}\frac{\sin mx}{x^2+a^2}dx = \frac{\pi}{a}e^{-ma} + i0$$

Equating real part & imaginary part, we get

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{a}e^{-ma}$$

$$\int_{-\infty}^{\infty}\frac{\sin mx}{x^2+a^2}dx = 0 \quad \text{—— (2)}$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{2a}e^{-ma} \quad \text{—— (1)}$$

$$\Rightarrow \text{Put } a=1,\ m=1$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{\cos mx}{x^2+a^2} = \frac{\pi}{2}e$$

$$\Rightarrow = 2\int_{0}^{\infty}\frac{\cos x}{1^2+1^2} = \frac{\pi}{2}e$$

$\Rightarrow$ Differentiating eqn (1) w.r.t m

$$= \int_0^{\infty} \frac{\cancel{-}x \sin(mx)}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma} (\cancel{-}a)$$

$$= \int_0^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma}$$

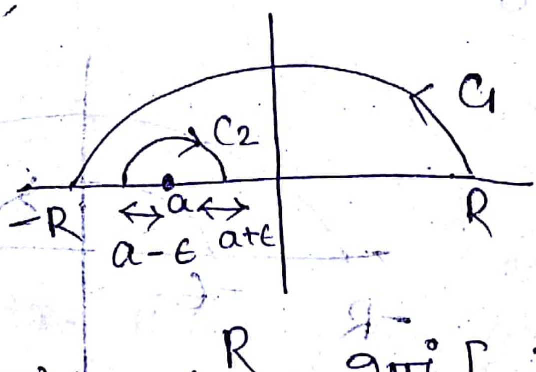
$$\boxed{\int_0^{\infty} \frac{\sin mx}{x^2+a^2}\,dx \quad \text{take img. part}}$$

Case IV :-

$$\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx$$ (simple poles on real axis.)

If $f(z)$ has a simple pole at $z=a$ on the real axis, then the principal value $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \cdot \left[\begin{matrix}\text{sum of} \\ \text{residue}\end{matrix}\right] + \pi i \cdot \left[\begin{matrix}\text{sum of} \\ \text{residue}\end{matrix}\right]$

where, the first sum extends over all poles in the upper half of the planes and the second sum extends over all poles in real axis.



$$I = \int_C f(z)\,dz =$$

$$\cancel{\int_{C_1} f(z)\,dz} \to 0 \; +$$

$$\int_{-R}^{a-\epsilon} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{a+\epsilon}^{R} = 2\pi i [\;\;]$$

For sufficiently large R & ϵ +ve 2nd contour changes its sign.

$$= \int_{\infty}^{a} f(z)\,dz - \pi i\,[\;\;] + \int_{a}^{\infty} f(z)\,dz$$

$$= 2\pi i \left[\begin{matrix}\text{residue}\\ \text{of } f(z)dz\end{matrix}\right]$$

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i\,[\;\;] + \pi i\,\{\text{r. of } R\}$$

Q.- $\int_0^{\infty} \frac{\sin x}{x}$

Consider the funⁿ $f(z) = \frac{\sin z}{z}$

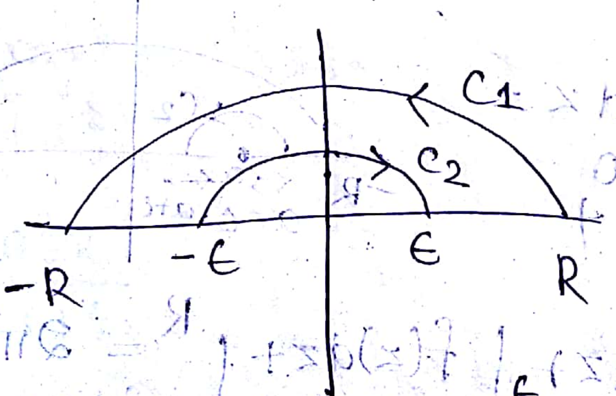
$= \text{Img. part of } \frac{e^{iz}}{z}$

or Im. part of $\left(\frac{\cos z + i\sin z}{z}\right)$

$$I = \oint_C \frac{e^{iz}}{z}\,dz$$

Pole :- $z = 0$ which lies on real axis.

$$\operatorname{Res}_{z=0} f(z) = \lim_{z \to 0} \frac{e^{iz}}{1} = 1.$$



$$\int_C f(z)\,dz = \int_{C_1} \cancel{f(z)\,dz}^{\,0} + \int_{-R}^{-\epsilon} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{\epsilon}^{R} f(z)\,dz =$$

Taking $\epsilon \to$ +ve & $R \to \infty$

$$= 0 + \int_{-\infty}^{0} f(x)\,dx - \pi i[1] + \int_{0}^{\infty} f(x)\,dx = 2\pi i[0]$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\cos x + i\sin x}{x}\right) = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = 0 + \pi i$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x} = 0$$

$$\& \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

$$\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi/2$$

Q- $\int_{-\infty}^{\infty} \frac{dx}{x^2 - ix}$ $f(z) = \frac{1}{z^2 - iz}$

$z^2 - iz = 0$

$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z \to 0} \frac{1}{2z - i} = i$ $z = 0$ & $z = i$.

$\underset{z=i}{\text{Res}}\, f(z) = \lim_{z \to i} \frac{1}{2z - i} = -i$

$2\pi i[-i] + \pi i(i) = 2\pi - \pi$
$= \pi$.

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$\int_0^{\infty} \frac{\sin x}{x} dx$. Consider the fun^n

$\frac{\sin z}{z} = \text{I.P of} \left(\frac{\cos z + i \sin z}{z}\right)$

$= \text{I.P. of } \frac{e^{iz}}{z}$

$f(z) = \frac{e^{iz}}{z}$

Pole: $z = 0$

Q- $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} = \frac{-\pi \sin 2}{e}$

Q- $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$

Q1- b) $x^2+4x+5=0$ $\quad I = \oint_C \frac{e^{iz}}{z} dz$.

Poles -- $\frac{-4+2i}{2}$, $\frac{-4-2i}{2}$

$= -2+i, \; -2-i$

$\cancel{\operatorname{Res}_{z=-2+i} f(z) = \lim_{z\to -2+i} eiz}$

$\operatorname{Res}_{z=-2+i} f(z) = \lim_{z\to -2+i} \underline{e^{iz}}$

$\int_{-\infty}^{\infty} \frac{\sin z}{z^2+4z+5}$ Imag. part of $\frac{\cos z + i\sin z}{z}$

$= I.P$ of $\frac{e^{iz}}{z^2+4z+5}$

Let $f(z) = \frac{e^{iz}}{z^2+4z+5}$.

Poles $= -2+i, -2-i$

$$\operatorname{Res}_{z=-2+i} = \lim_{z\to -2+i} \frac{e^{iz}}{2z+4}$$

$$= \frac{e^{(-2+i)i}}{2(-2+i)+4} = \frac{e^{-2i+i^2}}{-4+2i+4}$$

$$= \frac{e^{-2i-1}}{2i}$$

$$I = \int_{O}^{R} f(z)\,dz + \int f(z)\,dz$$

$$= 2\pi i \left(\frac{e^{2i-1}}{2i}\right).$$

$$= \underline{\pi(e^{-2i-1})}$$

$$\lim_{R\to\infty}\int_{\Gamma} \cancel{f(z)\,dz} + \lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz = \pi\left(e^{-2i-1}\right)$$

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi e^{(-2i-1)}$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5}\,dx = \pi e^{(-2i-1)}$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2+4x+5}\,dx = \pi e^{-1}\cdot \pi e^{-2i} = \frac{\pi}{e}(\cos 2 - i\sin 2)$$

Equating Real & Img part

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4x+5}\,dx = \frac{\pi}{e}\cos 2$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = \frac{\pi}{e}\cdot -\sin 2 = \frac{\pi(-\sin 2)}{e} = \frac{-\pi\sin 2}{e}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = \frac{-\pi\sin 2}{e}$$

Q2. $$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} \qquad x^2+a^2=0$$

$$(x^2+a^2)(x^2+b^2)=0$$

$$\Rightarrow \cancel{x^4+x^2b^2+x^2a^2}+a^2b^2=0$$

$$x^2 + a^2 = 0 \Rightarrow x^2 = -a^2$$

$$\Rightarrow x = \pm ai$$

$$\& \; x = \pm bi$$

$$f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$$

$$\underset{z=ai}{\text{Res}} f(z) = \lim_{z \to ai} \frac{e^{iz}}{(2z+a^2)(2z+b^2)}$$

$$= \frac{e^{(ai)i}}{(2ai+a^2)(2ai+b^2)}$$

$$= \frac{e^{ai^2}}{(2ai+a^2)(2ai+b^2)}$$

$$= \frac{e^{-a}}{a(2i+a)} = \frac{e^{-a}}{2ai \times 2ai}$$

$$= \frac{e^{-a}}{4a^2 \times i^2} = \frac{e^{-a}}{-4a^2}$$

$$\underset{z=bi}{\text{Res}} f(z) = \lim_{z \to bi} \frac{e^{iz}}{2z \times 2z} = \frac{e^{iz}}{4z^2}$$

$$= \frac{e^{-b}}{4b^2 i^2} = \frac{e^{-b}}{-4b^2}$$

$$I = \int_{C} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

$$= 2\pi i \left[\frac{e^{-a}}{-4a^2} + \frac{e^{-b}}{-4b^2} \right]$$

$$= 2\pi i \left[\frac{-4b^2 e^{-a} - 4a^2 e^{-b}}{-4a^2 \times -4b^2} \right]$$

$$= \frac{2\pi i}{4} \left[\frac{e^{-a}}{a^2} - \frac{e^{-b}}{b^2} \right]$$

$$= \frac{\pi i}{2} \left[\frac{e^{-a}}{a^2} - \frac{e^{-b}}{b^2} \right]$$

$$= \frac{\pi i}{2}$$