Date:- 13/01/18.

COMPLEX ANALYSIS.

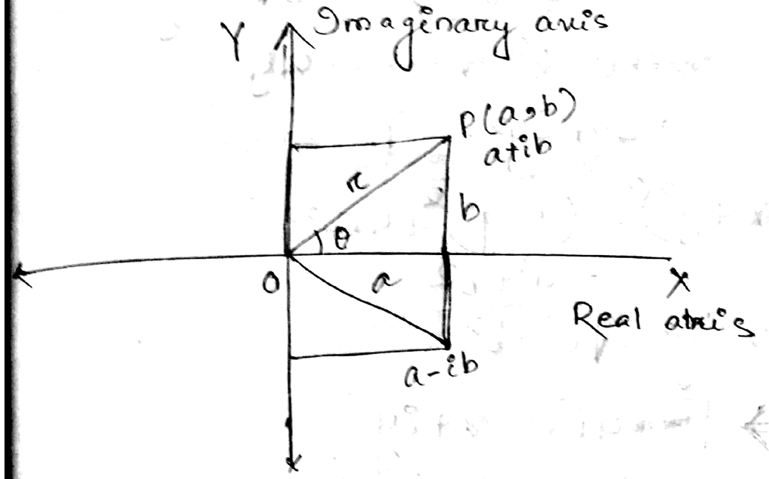
$z = a + ib$

$a \rightarrow Re\,z \quad b = Im\,z$

$z_1 \cdot z_2 = (a+ib)(c+id)$

$= (ac-bd) + i(bc+ad)$

$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$



~~argon plane~~ angle by argon plane – principal argument

$z = a + ib$

$|z| = \sqrt{a^2+b^2} = r.$

$\cos\theta = \frac{a}{r} \quad \sin\theta = \frac{b}{r}$

$\tan\theta = \frac{b}{a} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$

$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta) = re^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$.

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\boxed{\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}}$ $\boxed{\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}}$

$\Rightarrow \sqrt{a+ib} = x+iy$.

$\cancel{a^2 - b^2 + i2ab}$ $\Rightarrow a+ib = x^2 - y^2 + i2xy$

$\cancel{x = a^2 - b^2, \quad y = 2ab}$.

$a = x^2 - y^2 \qquad b = 2xy$.

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$.

$= a^2 + b^2$

$x^2 + y^2 = \sqrt{a^2+b^2}$.

$\Rightarrow \sqrt{3+4i} = x+iy$

$3+4i = x^2 - y^2 + i2xy$

$x^2 - y^2 = 3 \qquad 2xy = 4$.

$\cancel{xy = 2}$

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$.

$= 3^2 + (2xy)^2$

$= 9 + 16 \qquad x = 2$

$\Rightarrow x^2 + y^2 = 5 \qquad y = 1$

$$3+4i = 3+2\cdot 2i = 4-1+2\cdot 2\cdot i$$
$$= 2^2+i^2+2\cdot 2\cdot i$$
$$= (2+i)^2$$

$$\boxed{\begin{aligned}&(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)\\&= \cos(\alpha+\beta)+i\sin(\alpha+\beta)\end{aligned}}$$

$\Rightarrow \sqrt[3]{1} = x$

$x^3 = 1$

$x^3-1 = 0$

$(x-1)(x^2+x+1)=0$

$x=1\ ,\ x^2+x+1=0$

$$x = \frac{-1\pm\sqrt{3}i}{2},$$

$$\boxed{\begin{aligned}1 &= \cos 0+i\sin 0\\ -1 &= \cos\pi+i\sin\pi\\ i &= \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\\ -i &= \cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\end{aligned}}$$

$$\omega = \frac{-1+\sqrt{3}i}{2} \qquad \frac{-1-\sqrt{3}i}{2} = \omega^2$$

$1+\omega+\omega^2=0, \quad \omega^3=1.$

$\Rightarrow x^n = 1$

$(\cos\alpha+i\sin\alpha)^n = \cos\alpha n+i\sin\alpha n$

$x^n = 1 = \cos 0+i\sin 0$

$= \cos 2k\pi+i\sin 2k\pi$

$x = (\cos 2k\pi+i\sin 2k\pi)^{1/n}$

$, k=0,1,2\cdots n-1$

$$\boxed{x = \cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}}$$

Example:- $x^3=1 \quad n=3,\ k=0,1,2$

$x = \cos\frac{2k\pi}{3}+i\sin\frac{2k\pi}{3}$

$k=0,\ x=1$

$K = 1,$

$$x = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$= \frac{-1}{2} + \frac{i\sqrt{3}}{2}$$

$K = 2$

$$x = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

$$= \frac{-1}{2} - \frac{i\sqrt{3}}{2}.$$

$$\Rightarrow x^n = \alpha = a + ib$$

$$= r\cos\theta + ir\sin\theta$$

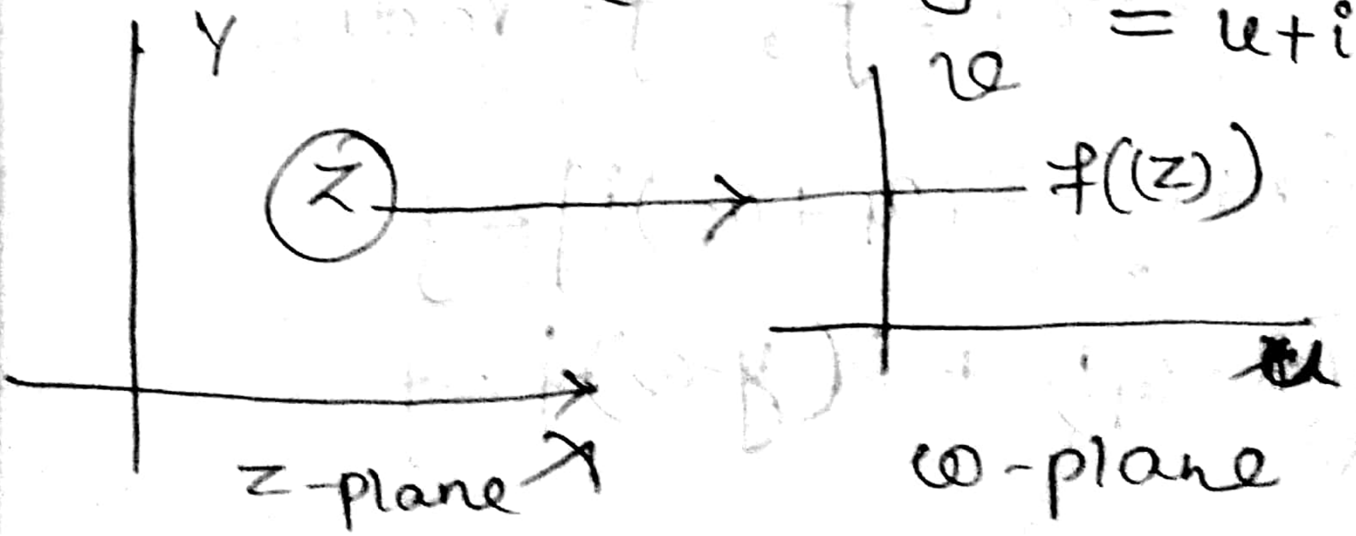
$$x^n = r[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)].$$

$$\Rightarrow \boxed{x = r^{\frac{1}{n}}\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)}$$

$$\omega = f(z) = \frac{z}{1+z} = \frac{x+iy}{1+x+iy}$$

$$= \frac{x+iy}{(1+x)+iy} = \frac{(x+iy)(1+x)-iy}{[(1+x)+iy][(1+x)-iy]}$$

$$= u+iv.$$



$$f(z) = \frac{1}{z} = \frac{1}{x+iy}$$

$$= \frac{1}{(x+iy)(x-iy)} \times (x-iy)$$

$$= \frac{(x-iy)}{(x+iy)(x-iy)}$$

$$= \frac{x}{x^2+y^2} + \frac{(-iy)}{x^2+y^2}$$

$$= u + iv$$

$u+iv$

$x^2 + i2y^2$

$x^2 - y^2$

★ **Region in the Complex Plane:-**

$$(x-h)^2 + (y-k)^2 = r^2$$

$$w = f(z) = |z| = 1$$

$$|z| = \sqrt{x^2+y^2}$$

$$x^2+y^2 = 1$$

$$(x-0)^2 + (y-0)^2 = 1$$

$$|z-a| = \rho, \quad \rho = \text{radius}$$

$$|(x+iy) - (a_1+a_2)i| = \rho$$

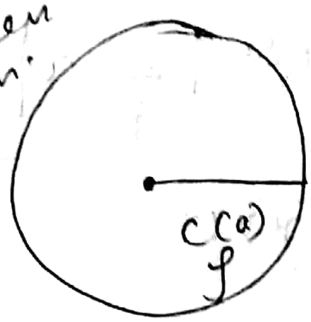
$$\Rightarrow |(x-a_1) + i(y-a_2)| = \rho$$

a, any pt. on complex region

$$\Rightarrow \sqrt{(x-a_1)^2+(y-a_2)^2}=\rho$$

$$\Rightarrow (x-a_1)^2+(y-a_2)^2=\rho^2$$

a, any pt. complex on region.



$x>0 \rightarrow$ general circle

$x<0 \rightarrow$ imaginary circle

$x=0 \rightarrow$ point.

<u>Disk</u>:- General disk $|z-a|<\rho$

Unit disk ~~$|z|<1$~~

$|z|<1 \rightarrow$ open disk, boundary not included.

$|z-a|\leq\rho \rightarrow$ closed disk, boundary included.

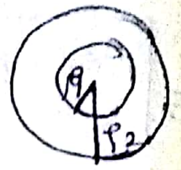
<u>Region</u>— A non-empty open connected subset of C (complex value func) is called a region.

<u>Boundary</u> — A set S is called bounded if we find a constant 'm' (+ve no. (constant) i.e., $|z|<m$, for z belongs to S.

<u>Annulus</u> — both internal & external radius

A set of all z whose distance $|z-a|$ from a is greater than ρ_1, $\rho_1<|z-a|<\rho_2$, is

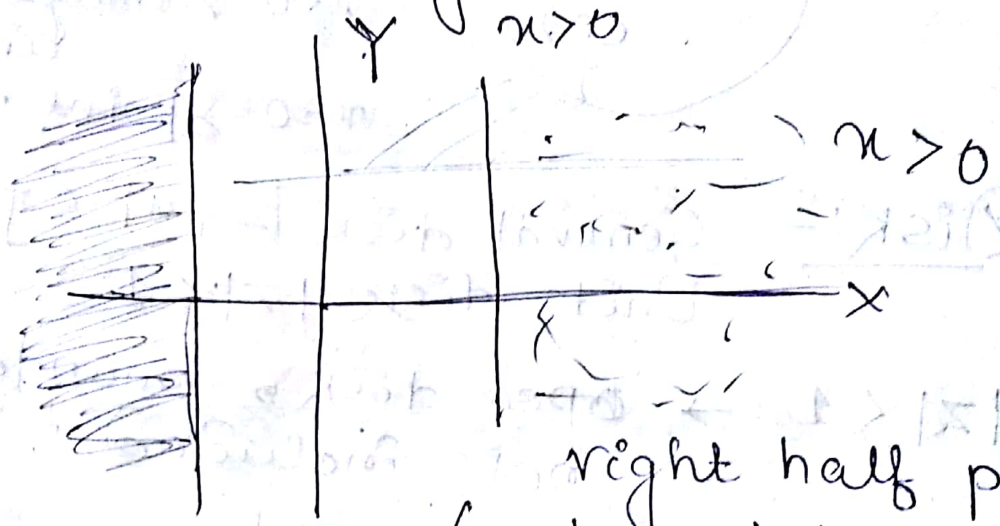
Called open annulus.



$$\boxed{\rho_1 \leq |z-a| \leq \rho_2.}$$ → close annulus.

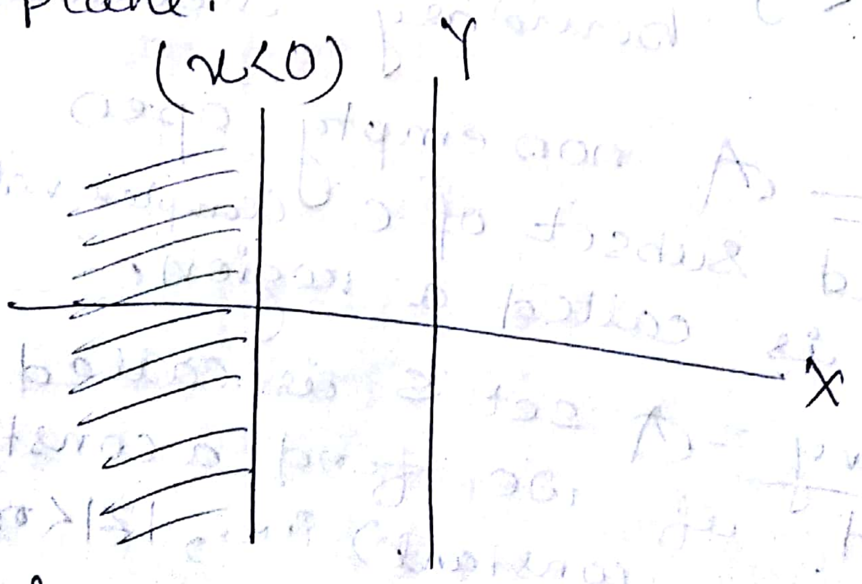
Exterior of Disk:- $|z-a| > \rho$

$z = x + iy$.

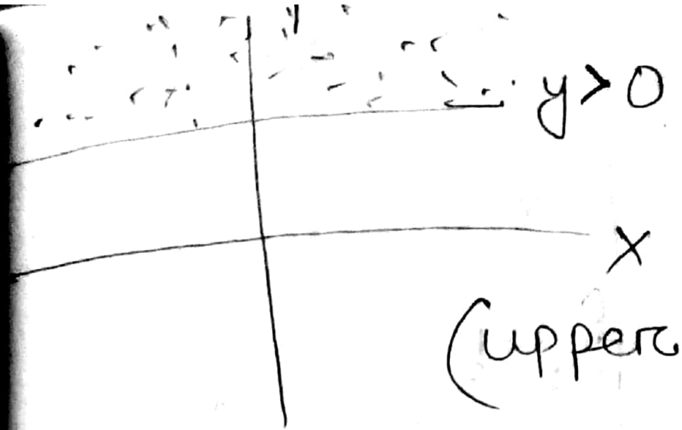


right half plane (unbounded region)

left half plane.

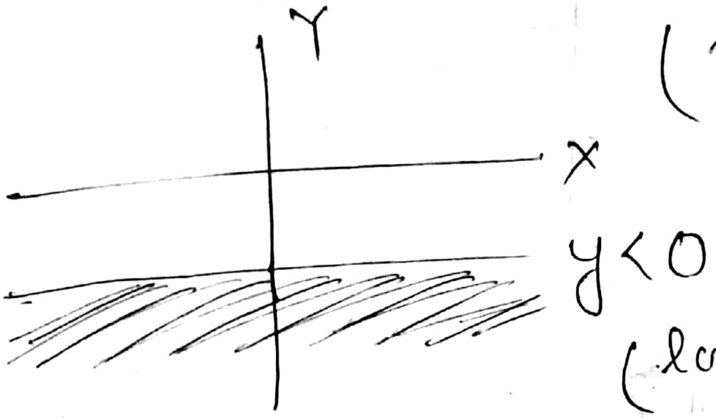


left half plane (unbounded region)



unbounded region.

(upper half plane)



(unbounded region)

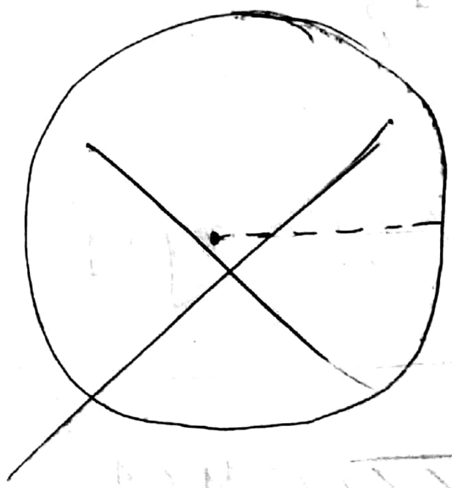
(lower half plane)

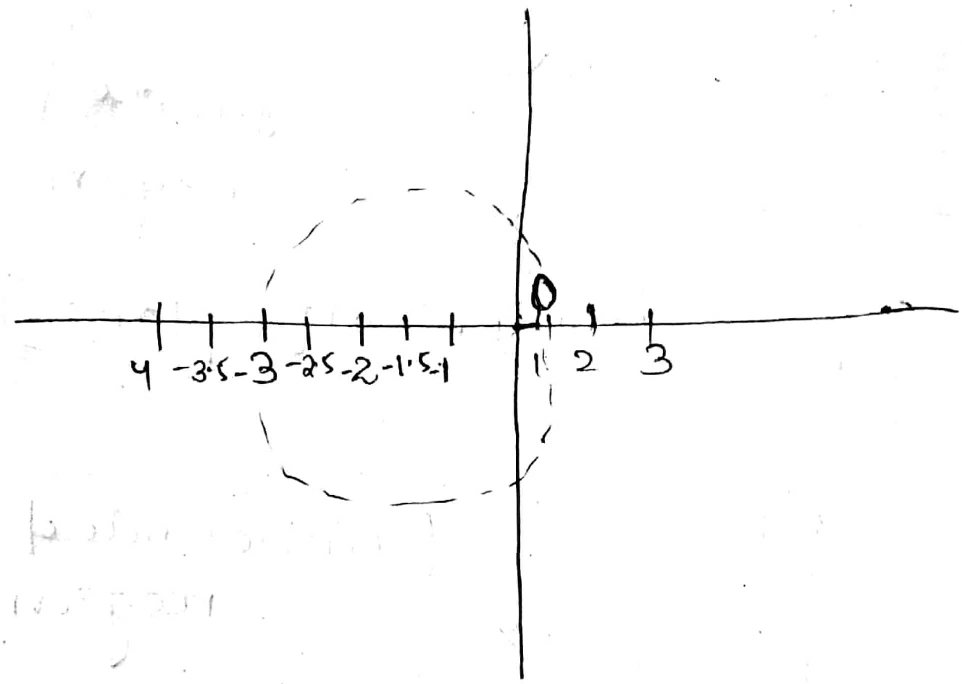
Q- Sketch the following set whether it is a region or not.

$$D = \{ z : |(2z+3)| > 4 \}$$

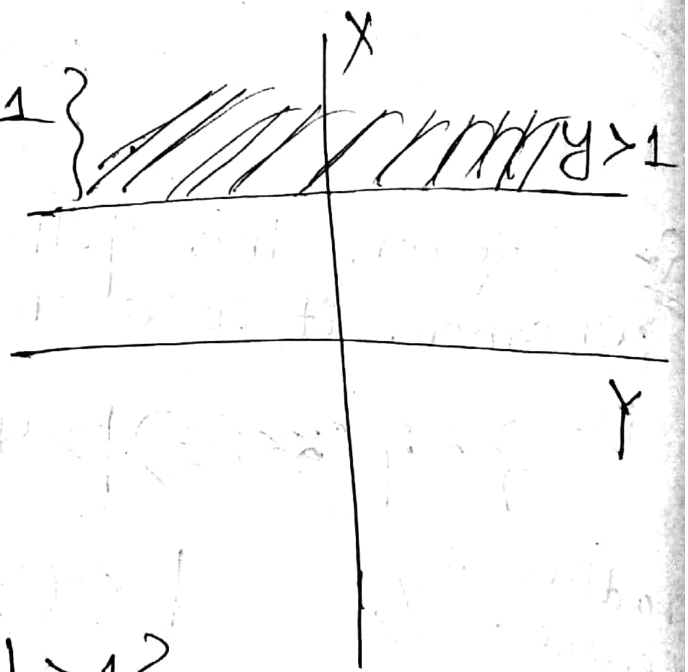
radius = 2
centre = $-3/2$

$$\left| z - \left(\frac{-3}{2}\right) \right| > 2$$



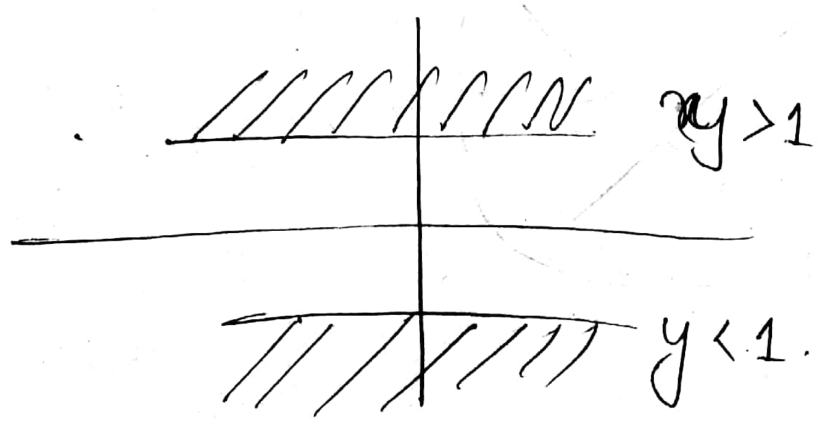


$z : \{ Im(z) > 1 \}$



or

$D = \{ z : |Im(z)| > 1 \}$



Date:- 18/01/18.

Let s be a complex number & z is a point on the 's'. F is a maping on 's'. A complex no. ω is called the value of f on 'z'.

$Z = x+iy$ | $w = f(z) = u+iv$ —

$u(x y) + iv(x y)$

$$w = f(z) = \frac{1}{z} = \frac{\cancel{x+iy}}{x+iy}$$

$$= \cancel{(x+iy)} \cdot \frac{(x-iy)}{(x+iy)(x-iy)}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$w = f(Z) = \frac{z}{1+z}$$

$$= \frac{x+iy}{(1+x+iy)} = \frac{(x+iy)(1+x) - iy}{[(1+x)+iy][1+x-iy]}$$

$$= \frac{x + x^2 \cancel{+iy} + xiy \cancel{-iy}}{[1+x+iy]\,[1+x-iy]}$$

$$= \frac{x+x^2+xiy}{[1+x+iy]\,[1+x-iy]} = \cancel{(1+x+1)}+i$$

$$= \frac{x+x^2+xiy}{[1+x+iy]^2} \quad : (x)+(iy)$$

$u = x + x^2/1+x^2+y^2$

$$= \frac{x+x^2+xiy}{(1+x)^2+(y)^2} \quad v = xy/1+x^2+y^2$$

Scanned by CamScanner

$$\boxed{\underset{x\to a}{l}\ f(x)=l}$$

$0<|x-a|<\delta.$

limit → approaching

$$\boxed{\lim_{x\to a} f(x)=l}$$

$\lim_{z\to a} f(z)=l \qquad 0<|z-a|<\delta$

$$\lim_{z\to 0}\frac{Re(z)}{1+z}$$

Re 9 img iy

Put $y=mx$

$(x+iy)\to 0$ i.e, $x\to 0,\ y\to 0$

$\cancel{\to} 0+iy$ $\quad y=mx\to y$

$x\to 0$ i.e; $y=0$.

$$\lim_{x+iy\to 0}\frac{x}{1+(x+iy)}$$

$$\lim_{x\to 0}\frac{x}{1+(x+imx)} \qquad y=mx\ x\to 0,\ x=my,\ y\to 0$$

$$\lim_{x\to 0}\frac{x}{1+x(1+mi)}=\frac{0}{1}=0.$$

(2) $$\lim_{z\to 0}\frac{Im z}{Re z}=\frac{iy}{x}$$

(3) $\lim_{z\to 0}$

$\Rightarrow \lim_{x+iy\to 0} =$

$\Rightarrow \lim_{x+iy\to 0} =$

$\Rightarrow \lim_{x\to 0} =$

$=$

★ Contin

Examine

$f(z)=$

clearly

$f(0)=0$

$\lim_{z\to 0}$

$\lim_{x\to 0}\frac{x}{|1+}$

(3) $\lim_{z \to 0} \frac{1+z^2}{\operatorname{Re} z} = \frac{1+z^2}{x}$

$\Rightarrow \lim_{x+iy \to 0} = \frac{1+z^2}{x}$

$\Rightarrow \lim_{x+iy \to 0} = \frac{1+(x+iy)^2}{x}$ $\quad x^2-y^2+2xy$

$\Rightarrow \lim_{x \cancel{+iy} \to 0} = \frac{1+x^2(1+im^2)}{x}$

$=$ doesn't exist.

☆ Continuity :-

Examine the continuity:- $\quad$ Analytic differentiable CR

$$f(z) = \begin{cases} \frac{\operatorname{Re} z}{|1+z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

clearly,

$f(0) = 0$

$\lim_{z \to 0} \frac{\operatorname{Re} z}{|1+z|} = \frac{x}{|1+z|} = \frac{x}{|1+x+iy|}$

$\lim_{x \to 0} \frac{x}{|1+x+iy|} = \frac{x}{1+\sqrt{x^2+y^2}} = \frac{x}{1+x\sqrt{1+m^2}} = 0$

Date :- 25/01/18.

$$\frac{dy}{dx} = \lim_{\Delta x \neq 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$z \to x + iy.$

$\Delta z = \Delta x + i\Delta y$

Analytic Function:-

$$\left(f(z) \pm g(z)\right)' = f'(z) \pm g'(z)$$

$$\left[f(z) - g(z)\right]' = f(z) \cdot g'(z) + g(z) \cdot f'(z)$$

$$f(z) = \frac{2z+3}{z-1}$$

$$f'(z)\Big]_{z=1} = \frac{(z-1)\cdot 2 - (2z+3)\cdot 1}{(z-1)^2}.$$

Proof:- Every differential fun is continuous but converse of the theorem is not necessarily true.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

Now, $$[f(z) - f(z_0)] = (z - z_0) \frac{f(z) - f(z_0)}{(z - z_0)}$$

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} (z - z_0) \cdot \lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)}$$

$$= 0 \cdot f'(z_0) = 0.$$

$$\lim_{z \to z_0} [f(z) - f(z_0)] = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0).$$

But converse of the theorem is not necessarily true.

$f(z) = \operatorname{Re} z$ not differentiable.

$$f(z) = \operatorname{Re} z = x$$

$$f(z + \Delta z) = x + \Delta x$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

$$= \lim_{\Delta x + i\Delta y \to 0} \frac{x + \Delta x - x}{\Delta x + i\Delta y}.$$

<u>Case I</u> – let $\Delta z \to 0$, then purely real values, i.e., imaginary part $\Delta y = 0$

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

<u>Case II</u> – let $\Delta z \to 0$, though purely imaginary values i.e, real part $\Delta x = 0$

$$\lim_{i\Delta y \to 0} \frac{0}{i\Delta y} = 0.$$

So, limit $[f'(z)]$ does not exists.

So, $f(z) = Re(z)$ is nowhere defferenteable.

Verify $f(z) = \bar{z}$ is no,where differentiable.

$$f(z + \Delta z) = \overline{z + \Delta z} = \bar{z} + \overline{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \neq 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta x + i\Delta y \neq 0} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x + i\Delta y \neq 0} \frac{\overline{\Delta x} - i\overline{\Delta y}}{\Delta x + i\Delta y}.$$

Case I - Let $\Delta z \to 0$, then purely real values i.e., imaginary part $\Delta y = 0$

$$\lim_{\Delta x \to 0} \frac{\Delta \overline{y}}{\Delta y} = 1.$$

Case II - Let $\Delta z \to 0$, through purely imaginary values i.e., real part $\Delta x = 0$

$$\lim_{i\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

$f(z) = \overline{z}$ is no-where differentiable.

→ $f(z) = |z|$ is no-where differentiable.

Analytic Function :-

Cauchy Riemann Equation (CReqn)

$$w = f(z) = u(z) + iv(z)$$
$$= u(x,y) + iv(x,y)$$

$Re\, w = u(x,y)$

$Im\, w = v(x,y)$

$$f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f(y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

A complex valued funⁿ is said to be analytic in domain 'D' if it is defined & differentiable at every point in its domain. Otherwise, a complex valued funⁿ $f(z)$ is said to be analytic at point $z = z_0$ if it is defined & differentiable at z_0 including sum of its neighbourhood point.

Differentiability → Analyticity ⇒ C.R Eqn

CR Eqn – ~~Pf of~~

Let $f(z) = u(z) + iv(z)$ be a complex valued funn defined & differentiable in a domain D, continuous first & 2nd order partial derivative of u & v exists, $f(z)$ is analytic iff it is satisfying CR equation.

⇒ $\boxed{U_x = V_y \;\; \& \;\; U_y = -V_x}$ Necessary point

Analytic ⇒ CR eqn.

Let $f(z) = u(z) + iv(z)$ be a complex valued funn defined & differentiable in a closed domain D. So we have to show that it is satisfying the CR eqn.

So, $f(z) = u(x,y) + iv(x,y)$ is analytic.

Of course its 1st & 2nd

Order ~~diff~~ partial derivative of u & v must exist & finite.

$$f'(z) = \text{exist \& finite.}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$z = x + iy$$

$$\Delta z = \Delta x + i\Delta y.$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\big(u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) + i\,v(x,y)\big)}{\Delta z}$$

$$\lim_{\Delta x + i\Delta y \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\left[\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}\right]$$

Case I :- Let $\Delta z \to 0$ though purely real values i.e, imaginary part $\Delta y = 0$.

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right].$$

$$= u_x + i v_x \text{ ———— } (*)$$

Case II :- Let $\Delta z \to 0$ though purely imaginary values i.e, $\Delta x = 0$

$$\lim_{i\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\left[\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}\right].$$

$$= \frac{1}{i} u_y + v_y$$

$$= -i u_y + v_y \text{ ———— } (**)$$

So, $f'(z)$ exists & is unique
(*) & (**) ~~exists~~ are equal

$u_x + iv_x = -iu_y + v_y$

$\Rightarrow$ $\boxed{u_x = v_y \quad \& \quad u_y = -v_x.}$

Important aspect of CR eqn

Moder name

Date:- 27/07/18.

CR eqn $\rightarrow$ analytic. (Hollomorphe)

Now, we have to prove that $f(z)$ is differentiable.

(I) $f(z) = Re(z) = x + i0$
$= u + iv$

$u = x$ & $v = 0$ * one of the condn of CR funn is satisfied

$u_x = 1$, $v_x = 0$

$u_y = 0$ $\quad u_x \neq v_y$.

$\Rightarrow$ The funn is not differential so, it is not analytic.

(II) $f(z) = \bar{z} = x - iy = u + iv$

$u = x$ & $v = -y$

$u_x = 1$ & $v_x = 0$

$u_y = 0$ & $v_y = -1$

$u_x \neq v_y$

→ CR eqn not satisfied.

→ Funn is not analytic.

so, $f(z) = \bar{z}$ is nowhere differentiable.

i^2y^2
$= -1y^2$

(iii) $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$= (x^2 - y^2) + 2xyi$

$u = x^2 - y^2$ & $v = 2xy$.

$u_x = 2x$ $\quad v_x = 2y$

$u_y = -2y$ $\quad v_y = 2x$.

$\boxed{u_x = v_y}$, $\boxed{-v_x = u_y}$

→ CR eqn satisfied

so, funn is analytic.

so, $f(z) = z^2$ is differentiable.

(iv) $f(z) = z^3 = (x+iy)^3$.

$= (x^2 - y^2 + 2xyi)(x - iy)$.

$= x^3 - xiy - xy^2 - iy^3 + 2x^2yi - 2xy^2i$

$= x^3 - iy^3 + 2x^2yi - 2xy^2i - xiy - xy^2$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$

$\cancel{u_x = 3x^2 - 3y^2}$

$u = x^3 - 3xy^2 \qquad v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2 \qquad v_x = 6y$

$u_y = -6x \qquad v_y = -3y^2 + 3x^2$

$\boxed{u_x = v_y} \qquad \boxed{v_x = -u_y}$

→ CR eqn is satisfied so, the funn is analytic & hence, it is differentiable.

★ POLAR FORM OF CR Eqn :-

$z = x + iy$

$= r\cos\theta + ir\sin\theta$

$= u + iv$

$u = r\cos\theta \qquad v = r\sin\theta.$

$u_r = \cos\theta \qquad v_r = \sin\theta$

$u_\theta = -r\sin\theta \qquad v_\theta = r\cos\theta.$

(i) $f(z) = (x+iy)^2 = z^2$

$= (r\cos\theta + i\sin\theta)^2$

$= r^2\cos 2\theta + ir^2\sin 2\theta.$

$= u + iv.$

$u = r^2\cos 2\theta \quad v = r^2\sin 2\theta.$

$U_r = 2r\cos\theta 2\theta \quad V_r = 2r\sin 2\theta.$

$U_\theta = -r^2\sin 2\theta\cdot 2 \quad V_\theta = r^2\cos\theta 2\theta\cdot 2.$

$\boxed{U_r = \frac{1}{r}V_\theta.} \quad \boxed{V_r = -\frac{1}{r}U_\theta.}$

→ CR eqn is satisfied
So, the funn is analytic &
hence it is differentiable.

(ii) $f(z) = z^4$ is analytic or not.

$f(z) = r^4\cos 4\theta + ir^4\sin 4\theta.$

$U = r^4\cos 4\theta \quad V = r^4\sin 4\theta.$

$U_r = 4r^3\cos 4\theta \quad V_r = 4r^3\sin 4\theta$

$U_\theta = -4r^4\sin 4\theta \quad V_\theta = 4r^4\cos 4\theta$

$\boxed{U_r = \frac{1}{r}V_\theta} \quad \boxed{V_r = \frac{-1}{r}U_\theta}$

→ CR eqn is satisfied
so, the funn is analytic &
hence it is differentiable.

(iii) $f(z) = z^n$ is analytic or not

$u = r^n \cos n\theta \quad v = r^n \sin n\theta$

$U_r = nr^{n-1}\cos n\theta \quad V_r = nr^{n-1}\sin n\theta$

$u_\theta = nr^n(-\sin n\theta) \quad V_\theta = nr^n \cos n\theta$

$\boxed{U_r = \frac{1}{r} V_\theta} \quad \boxed{V_r = \frac{-1}{r} U_\theta}$

→ CR eqn is satisfied, so it is analytic & hence differentiable.

Date:- 06/08/18

HARMONIC EQUATION:-

Imp Any funn satisfying laplac ~~transf~~ eqn is known as harmonic function.

$U(x, y)$

$\nabla^2 U = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$U_{xx} + U_{yy} = 0$

$f(z) = u + iv$

$\nabla^2 V = V_{xx} + V_{yy} = 0$

→ Real & imaginary part of a analytic funn is harmonic i.e., if $f(z) = u + iv$

$\nabla^2 u = u_{xx} + u_{yy} = 0$

$\nabla^2 v = v_{xx} + v_{yy} = 0$.

Q. $f(z) = x^2$ whether it is a real part of harmonic or not.

here, $v_x = 0$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial^2 u}{\partial x^2} = 2$ ∴ as $u_{xx} \neq 0$

So, u is not a real part of harmonic funcn.

ii) $f(z) = z^2 = (x+iy)^2$

$= x^2 + i^2y^2 + 2xyi$

$= x^2 - y^2 + 2xyi$

$u = x^2 - y^2$ $\quad v = 2xy$

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial^2 u}{\partial x^2} = 2$ $\quad \frac{\partial v}{\partial y} = 2\cancel{y}$ $\cancel{\frac{\partial^2 v}{\partial y^2} = 2}$

$\boxed{\cancel{\nabla} \cancel{u_{xx} = u_{yy}} = 2}$

$\cancel{u_{xx} + u_{yy} = 0}$

$\frac{\partial^2 v}{\partial y^2} = 0$ $\quad \nabla^2 u = \nabla^2 v = 0$

So, u is called the real part v is called imaginary part

(III) $f(z) = x^3 + y^3$

$\nabla^2 v$ $\dfrac{\partial v}{\partial x} = 3x^2 \qquad \dfrac{\partial^2 v}{\partial x^2} = 6x$

$\dfrac{\partial v}{\partial y} = 3y^2 \qquad \dfrac{\partial^2 v}{\partial y^2} = 6x.$

$V_{xx} + V_{yy} \neq 0 \rightarrow$ Not harmonic

★ CONJUGATE OF HARMONIC FUNCTION:-

imp.

$f(z) = u + iv$ is an analytic funⁿ, then v is called harmonic conjugate of u.

Entire Funⁿ :- A funⁿ $f(z)$ is said to be entire funⁿ if it is analytic everywhere.

Q $f(z) = e^z$ is analytic or not.

$f(z) = e^z = e^{(x+iy)}$

$= e^x . e^{iy}$

$= e^x(\cos y + i\sin y)$

$= \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}.$

Scanned by CamScanner

$u = e^x \cos y \qquad v = e^x \sin y$

$U_x = e^x \cos y \qquad V_x = e^x \sin y$

$U_y = -e^x \sin y \qquad V_y = e^x \cos y.$

$\boxed{U_x = V_y} \qquad \boxed{V_x = -U_y}$

→ CR eqn satisfied
→ It is analytic

Q1- Polynomial funn is also analytic.

Q2- Find the harmonic conjugate of u & find most general analytic funn of $f(z) = xy$.

$f(z) = xy$

$u = xy \qquad v = 0$

$\frac{\partial U}{\partial x} = \cancel{y}1 \qquad U_{yy} = 0$

$\frac{\partial^2 U}{\partial x^2} = 0 \qquad U_{xx} + U_{yy} = 0$

→ so funn is harmonic.

$f(z) = u(z) + iv(z)$

$\boxed{U_x = V_y} \qquad \boxed{U_y = -V_x.}$

→ Satisfies CR eqn
& hence it is analytic.

$U_x = y \qquad y = V_y$

$\frac{\partial v}{\partial y} = y \Rightarrow V = \int y + c$

$\Rightarrow V = \frac{y^2}{2} + h(x)$ —— ①

Differentating ① wrt to x

$V = 0 + h'(x)$

$\Rightarrow -U_y = h'(x)$

$\Rightarrow \boxed{h'(x) = -x}$

$\Rightarrow h(x) = \frac{-x^2}{2} + C$

$\Rightarrow V = \frac{y^2}{2} - \frac{x^2}{2} + C$

$\Rightarrow \boxed{V = \left(\frac{y^2 - x^2}{2}\right) + C}$

So, $f(z) = xy + \left[\left(\frac{y^2 - x^2}{2}\right) + C\right] i.$

(OR) $f(z) = u(x\ y) + iV(x\ y)$ (not)

$dV = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ (Exact cond?)

$= V_x dx + V_y dy$

$= -U_y dx + U_x dy$

$dy = -x dx + y dy$

$V = -\frac{x^2}{2} + \frac{y^2}{2} + C$

Q3- $u = ax^3 + by^3$. Find the value of a & b where u is harmonic & also find the harmonic conjugate.

$U_x = 3ax^2 \qquad U_y = 3by^2$

$U_{xx} = 6ax \qquad U_{yy} = 6by$.

$U_{xx} + U_{yy} = 0$

$\Rightarrow 6ax + 6by = 0$

$\Rightarrow ax + by = 0$ provided $a = 0$ & $b = 0$.

Let $V(xy)$ be the harmonic conjugate of $u(xy)$

So, by exact differential.

$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$

$dV = -U_y dx + U_x dy$.

$\cancel{= V_x dx + V_y dy}$

$dV = -3by^2 dx + 3ax^2 dy$

$dV = 0 + C \qquad V = C$

$\boxed{f(z) = ax^3 + by^3 + iC}$ Ans.

Date:- 07/02/18 $\int_a^b f(x)\,dx \quad \int_0^b f(x)\,dx$

$\int \rightarrow$ integral

whom to integrate $\rightarrow$ integrand

process of finding integral $\rightarrow$ Integration.

$|z| < r \rightarrow$ interior of circle

$|z| > r \rightarrow$ ext. of circle.

$\int_C f(z)\,dz$

$C \rightarrow$ contour or path. (closed)

$K \int_C f(z)\,dz$

$\int_C (f \pm g)\,dz = \int_C f\,dz \pm \int_C g\,dz.$

$\int_C (af \pm bg)\,dz = a\int_C f\,dz \pm b\int_C g\,dz$

$f(z) = u + iv$

$z = x + iy$

$dz = dx + i\,dy$

$\int_C f(z)\,dz$

$= \int_C (u + iv)(dx + i\,dy)$

$\Rightarrow \int_C (u\,dx - v\,dy) + i\,(v\,dx + u\,dy)$.

Q-i) Evaluate $\int_C z^2 dz$ along the straight line $z=0$ to $z=3+i$.

ii) along the curve made up of two line segment $z=0$ to $z=3$ from $z=3$ to $z=3+i$.

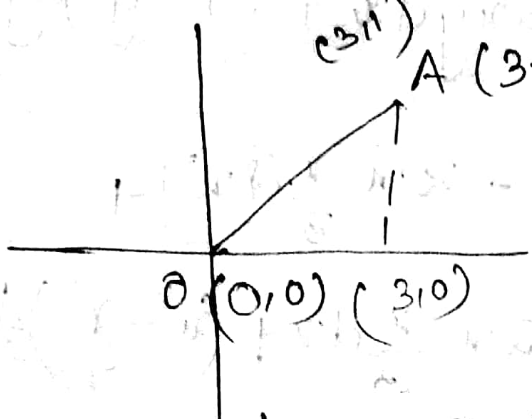
Soln.

$$\int_{z=0}^{z=3+i} z^2 dz$$

$$= \int_{z=0}^{z=3+i} \frac{z^3}{3} + C = \frac{1}{3}\left[z^3\right]_0^{3+i}$$

$$= \frac{1}{3}\left[(3+i)^3 - 0\right] = \frac{1}{3}\left[27 - 1 + 27i - 9\right]$$

$$= \frac{1}{3}\left[27 + 27i - 10\right] = \frac{1}{3}\left[17 + 27i\right].$$



z varies from 0 to $3+i$ means z varies from $(0,0)$ to $(3,1)$. The line joining betn this is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow y - 0 = \frac{1 - 0}{3 - 0}(x - 0)$$

$$\Rightarrow y = \frac{1}{3} x \Rightarrow \boxed{y = \frac{x}{3}}$$

$$\Rightarrow x = 3y$$

$$\Rightarrow dy = \frac{1}{3} dx$$

$$\int_{(0,0)}^{(3,1)} z^2 \, dz = \int_{(0,0)}^{(3,1)} (x + iy)^2 (dx + i\,dy)$$

$$= \int_{(0,0)}^{(3,1)} \left[(x^2 - y^2) + i 2xy \right] [dx + i\,dy]$$

$$= \int_{(0,0)}^{(3,1)} \left[(x^2 - y^2) dx - 2xy\,dy \right] + i \left[2xy\,dx + (x^2 - y^2) dy \right]$$

$$= \int_0^3 \left[\left(x^2 - \frac{x^2}{9} \right) dx - 2x \frac{x}{3} dx \right] + i \left[\int_0^3 2x \frac{x}{3} dx + \left(x^2 - \frac{x^2}{9} \right) \frac{1}{3} dx \right]$$

$$= \int_0^3 \left[\left(x^2 - \frac{x^2}{9}\right)dx - 2\frac{x^2}{3}dx\right] + $$
$$+ i \int_0^3 \left[2\frac{x^2}{3}dx + \left(x^2 - \frac{x^2}{9}\right)\frac{1}{3}dx\right]$$

$$= \left[\int_0^3 x^2 dx - \frac{x^2}{9}dx - \frac{2}{3}\int_0^3 x^2 dx\right] +$$
$$i\ \frac{2}{3}\int_0^3 x^2 dx + \frac{1}{3}\int_0^3 x^2 dx - \frac{1}{9}\int_0^3 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^3 - \frac{1}{9}\left[\frac{x^3}{3}\right]_0^3 - \frac{2}{3}\left[\frac{x^3}{3}\right]_0^3 +$$
$$i\left[\frac{2}{3}\left[\frac{x^3}{3}\right]_0^3 + \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 - \frac{1}{9}\left[\frac{x^3}{3}\right]_0^3\right]$$

$$= \frac{1}{3}27 - \frac{1}{9\times 3}27 - \frac{2}{9}\times 27 + i\left[\frac{2}{9}\times 27 + \right.$$
$$\left.\frac{1}{9}\times 27 - \frac{1}{27}\ 27\right]$$

$$= \frac{27}{3} - \frac{27}{27} - \frac{\cancel{54}^{6}}{\cancel{9}} + i\left[\frac{54}{9} + \frac{27}{9} - \frac{27}{27}\right]$$

$$= \frac{27}{3} - 1 - 6 + i\left[6 + 3 - 1\right].$$

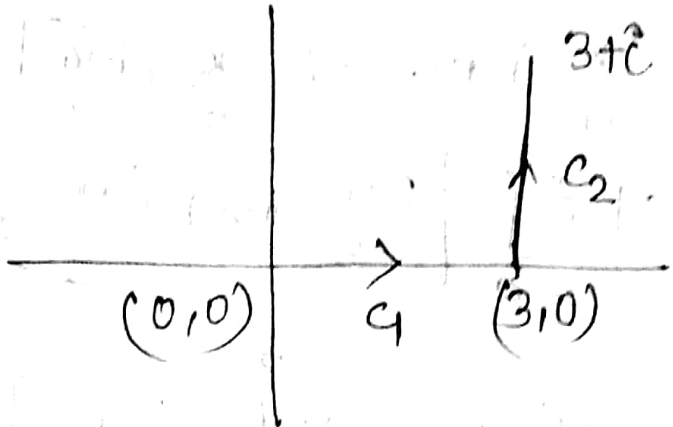
$$= \frac{27}{3} - 7 + i[8]$$

$$= \frac{27-21}{3} + i8 = \frac{26}{3} + i8.$$

Now put $x = 3y$

(11)

3+i (3,1)



$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz$$

C_1 : $\int_{C_1} z^2 dz =$ Here $x=3$ So, $dx=0$

$$= \int_0^3 (x+iy)^2 (dx + i\,dy) \qquad y=0,\ dy=0$$

$$\cancel{= \int_0^3 3^2\, dx =} \text{ [illegible]}$$

$$\cancel{= \int_0^3 3^2\, dx = 9[x]_0^3 = 9\times 3 = 27}$$

$$= \int_0^3 x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}\,27 = 9.$$

C_2 : $x=3$, $dx=0$.

$$= \int_0^1 (3+iy)^2\, i\,dy\,.$$

$$= \int_0^1 (9 - y^2 + 6iy)\, i\,dy$$

$$= \int_0^1 (9 - y^2 + 6iy)\, i\, dy$$

$$= \int_0^1 [9i\,dy - iy^2 dy - 6y\,dy]$$

$$= -i\left[\frac{y^3}{3}\right]_0^1 - 6\left[\frac{y^2}{2}\right]_0^1 + 9i$$

$$\cancel{= \frac{1}{2} - 3} = -\frac{i}{3} - 3 + 9i$$

$$= 9i - \frac{i}{3} - 3 = -3 + \frac{26i}{3}$$

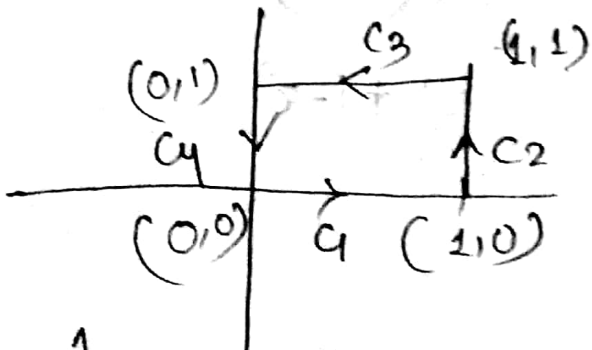
$$C = C_1 + C_2$$

$$= 9 - 3 + \frac{26i}{3} = 6 + \frac{26i}{3}$$

HW

Q- Evaluate C is a sq. with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$

Evaluate $\int_C z^2 dz$.



Here $y=0$
so, $dy=0$.

$C_1 \to \int_0^1 (x+iy)^2 (dx + i\,dy)$

$$= \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$C_2 : \int_0^1 (1+iy)^2 \, i \, dy$ $\quad x=1$, $dx=0$

$= (1-y^2+2iy) i \, dy$

$= [i]_0^1 dy - \int_0^1 y^2 i \, dy - \int_0^1 2y \, dy$ $\quad \frac{-1}{3}-1 = \frac{-1-3}{3}$

$= i - \left[\frac{y^3}{3}\right]_0^1 i - 2\left[\frac{y^2}{2}\right]_0^1$

$= i - \frac{i}{3} - 1 = -1 + \frac{2}{3} i$

$C_3 :- \int_1^0 (x+i)^2 dx$ $\quad y=1$, $dy=0$

$= \int_1^0 (x+i)^2 dx$

$= \int_1^0 (x^2 - 1 + 2ix) dx$

$= \left[\frac{x^3}{3}\right]_1^0 - [x]_1^0 + 2\left[\frac{x^2}{2}\right]_1^0$

$= \frac{1}{3}(0-1) - (0-1) + (0-1) i$

$= \frac{-1}{3} + 1 - 1i = \frac{-1}{3} + 1 - i$

$= \frac{2}{3} - i$

$C_4: \int_1^0 (-y^2)\, i\, dy \qquad x = 0, \; dx = 0$

$= \int_1^0 -iy^2\, dy = -i\left[\frac{y^3}{3}\right]_1^0$

$= \frac{i}{3}$

$C = C_1 + C_2 + C_3 + C_4$

$= \frac{1}{3} - 1 + \frac{2}{3}i + \frac{2}{3} - i + \frac{i}{3}$

$= +\frac{2}{3}i - i + \frac{i}{3} + \frac{1}{3} - 1 + \frac{2}{3}$

$= \frac{+2i - 3i + i}{3} + \frac{1 - 3 + 2}{3}$

$= \cancel{\frac{-5i + i}{3} + 0}$

$= \cancel{\frac{-4i}{3}} = 0$

Dt:- 08/02/18

Parametric Representation

→ Parametric Rep. of St. line from

$$\boxed{z(t) = a + t(b-a)}$$

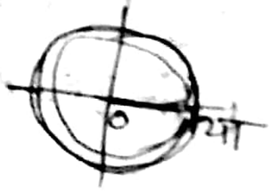
where, $\boxed{\alpha \le t \le \beta}$

⇒ Parametric Rep. of a circle.

$$|z| = r.$$

$$\Rightarrow \boxed{z(t) = re^{it}}$$

$$dz = rie^{it}$$



$$\boxed{0 \le t \le 2\pi}$$

$$\int_c f(z)\,dz = \int_c f(re^{it}) \cdot ire^{it}\,dt.$$

~~→ Parametric~~

$$|z-a| = r$$

$$z(t) = a + re^{it}$$

$$dz = rie^{it}\,dt.$$

→ Parametric Rep. of a curve —

$y = f(x)$ from a to b.

$$z = x(t) + iy(t)$$

when $x = t$ $y = f(t)$

$$\boxed{z(t) = t + if(t)}$$

St. line from $0 \to 4-7i$.

$z(t) = a + (b-a)t$

$= 0 + (4-7i-0)t$

$= (4-7i)t$

$= (4t, -7t)$ $\quad 0 \to 3+i$.

$0 \le 4t \le 4$

$\boxed{0 \le t \le 1.} \to$ Real

$0 \le -7t < -7$

$\boxed{0 \le t \le 1} \to$ Imaginary

Q $y = x^3$ from $(-2,-8)$ to $(3,27)$.

$y = f(x) \quad f(x) = x^3$.

$z = x(t) + i\,y(t) \qquad x = t$

$= t + i f(t). \qquad y = t^3$.

$\Rightarrow z(t) = t + i f(t)$

$z = t + it^3$

$\boxed{-2 \le t \le 3}$

$-8 \le t^3 \le 27 \qquad 2^3 = 8, \; 3^3 = 27$

$\Rightarrow \boxed{-2 < t \le 3}$

Q. Evaluate $\int_C \bar{z}\,dz$ C: from origin along the parabola $y=x^2$ to $1+i$.

$z = x+iy.$

Let $y = x^2.$

Let $x = t$

$y = t^2$

$z(t) = t + it^2.$

$dz = (1+2it)\,d\cancel{t}$

$\bar{z}(t) = t - it^2$

$d\bar{z}(t) = (1-2it)\,dt.$

$$\int_C \bar{z}\,dz = \int_0^1 (t-it^2)(1+2it)\,dt$$

$$= \int_0^1 t + 2it^2 - it^2 + 2it^3\,dt$$

$$= [t]_0^1 + 2i\left[\frac{t^3}{3}\right]_0^1 - i\left[\frac{t^3}{3}\right]_0^1 + 2i\left[\frac{t^3}{3}\right]_0^1$$

$$= 1 + \frac{2}{3}i - \frac{i}{3}1 + \frac{2i}{3}1.$$

$$= 1 + \frac{i}{3}.$$

Q:- Evaluate $\int_C \sin^2 z\,dz \quad C: |z| = \pi$

$z(t) = \pi e^{it}. \quad \Rightarrow z = x + iy$

$0 \le t \le 2\pi$

$\int_0^{2\pi} \sin^2(\pi e^{it})\,\pi i e^{it}\,dt \quad dz = \pi i e^{it}\,dt$

$= \int_0^{2\pi} \sin^2(\pi e^{it})\,\pi i e^{it}\,dt$

$= \pi i \int_0^{2\pi} \frac{1 - \cos 2\pi e^{it}}{2!}\,e^{it}\,dt$

$= \frac{\pi i}{2i} \int_0^{2\pi} \left(1 - \cos 2\pi e^{it}\right) e^{it}\,dt$ ILATE

$= \frac{\pi i}{2i}\left[\int_0^{2\pi} e^{it}\,dt - \int_0^{2\pi} \cos 2\pi e^{it}\,dt\right]$ e^{it}

$= \frac{\pi i}{2i}\left[e^{2\pi i} - \int_0^{2\pi} \cos 2\pi e^{it}\,e^{it}\,dt\right]$ (I) (II)

$= \frac{\pi}{2}\left[e^{2\pi i} - \right.$

$$= \frac{2}{3} i \left[27 - 9 - i + 27i \right] - \frac{2}{3} \left[27 - 9 - i + 27i \right]$$

$$= \frac{2}{3} i \left[18 + 26 i \right] - \frac{2}{3} \left[18 + 26 i \right]$$

Q. $z(t) = a + t(b-a)$ $\quad$ 0 to $3+i$

$0 \leq 3t \leq 3$

$0 \leq t \leq 1$

$$z(t) = 0 + t(3+i-0)$$
$$= 0 + t(3+i)$$
$$= t(3+i)$$
$$= 3t + it$$

$$z(t) = 3t + it$$

$$dz = (3+i)\,dt$$

$$\Rightarrow z^2 = (3t+it)^2 dt$$
$$= 6t^2 - t^2 + 6it^2$$
$$= 5t^2 + 6it^2$$

$$= \int_0^1 (5t^2 + 6it^2)(3+i)\,dt$$

$$= \int_0^1 \left[15t^2 + 5t^2 i + 18it^2 + 6i^2 t^2 \right] dt$$

$$= \int_0^1 15t^2 + 5t^2 i - 6t^2 + 18it^2\,dt$$

$$= 15\left[\frac{t^3}{3}\right]_0^1 + 5i\left[\frac{t^3}{3}\right]_0^1 - 6\left[\frac{t^3}{3}\right]_0^1 - 18i\left[\frac{t^3}{3}\right]_0^1$$

$$= \frac{15}{3} \times 1 + \frac{5}{3} \times 1i + \frac{6}{3} \times 1 - \frac{18}{3} i \times 1$$

$= 5 + \frac{5}{3}i \cancel{+} -2 - 6i$

$= 3 + \frac{5}{3}i - 6i = 3 + \frac{5i - 18i}{3}$

$= 3 + \frac{(-13i)}{3} = 3 - \frac{13i}{3}$

Date:- 10/02/18

Q- $\int_C \operatorname{Re} z \, dz$ $\quad C: (1+i)$ to $(3+2i)$ straight line given by

Soln- $z(t) = a + (b-a)t$

$= (1+i) + (3+2i-1-i)t$

$= (1+2t) + i(1+t)$.

$1 \leq (1+2t) \leq 3$. $\qquad 1 \leq (1+t) \leq 2$

$0 \leq 2t \leq 2 \qquad 0 \leq t \leq 1$

$0 \leq t \leq 1$.

$\int_C \operatorname{Re}(z) \, dz = \int_0^1 (1+2t)(2+i) \, dt$.

$= \int_0^1 2 + i + 4t + 2it \, dt$

$= 4\left[\frac{t^2}{2}\right]_0^1 + 2i\left[\frac{t^2}{2}\right]_0^1 + 2 = 2 + i + 2 + i$

$= \cancel{\frac{1}{2} \times 1 + 1 = \frac{1}{2} + 1 = \frac{3}{2}} +$

$= 4 + 2i$ Ans.

Qs. $\cancel{\int_C \bar{z}\,dz}$, $\int \text{Re}\,z^2$, c is the unit circle clockwise where $|z| = 1$.

$\boxed{\text{Ans} = 0}$

Sol^n $\Rightarrow$ $z(t) = e^{it}$

$= \cos t + i \sin t.$

$z^2 = \cos 2t + i \sin 2t.$

$\text{Re}\,z^2 = \cos 2t$

$z = \cos t + i \sin t.$

$\cancel{dz = \sin 2t + i \cos t}$

$dz = (\sin t + i \cos t)\,dt.$

$= \int_0^{2\pi} \cos 2t\,(-\sin t + i \cos t)\,dt$

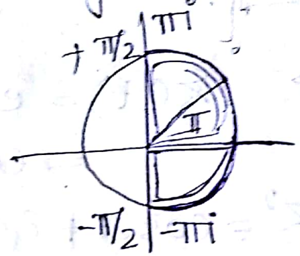
$= \int_0^{2\pi} \left(-\sin t \cos 2t + \cos 2t \cdot i \cos t\right) dt$

$= -\int_0^{2\pi} \sin t \cos 2t\,dt + \int_0^{2\pi} \cos 2t \cdot i \cos t\,dt.$

$= -\int_0^{2\pi} \sin t\,(1 - \sin^2 t)\,dt + \int$

Q. $\int_C \sin^2 z\, dz$; $C: -\pi i$ along to πi

$|z| = \pi$ in the right half plane



Soln: $z = \pi e^{it}$.

$\int_C \sin^2 \pi e^{it} \, dz.$

$dz = i\pi e^{it}$ $\quad z(t) = \pi\cos t + i\pi \sin t$

$\int_C \sin^2 \pi e^{it} \, i\pi e^{it}.$ $\quad \pi \sin t$

$-\pi < \pi \sin t \leq \pi$

$-1 < \sin t < 1$

$-\pi/2 < \sin t < \pi/2$

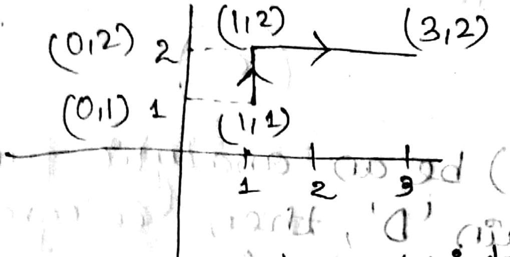
$= \int_{-\pi/2}^{\pi/2} \sin^2(\pi e^{it}) i\pi e^{it} dt$

$= \int_{-\pi/2}^{\pi/2} (1 - \cos^2 \pi e^{it}) \, i\pi e^{it} \, dt$

$= \int_{-\pi/2}^{\pi/2} i\pi e^{it} \, dt - \int_{-\pi/2}^{\pi/2} \cos^2 \pi e^{it} \, i\pi e^{it} dt$

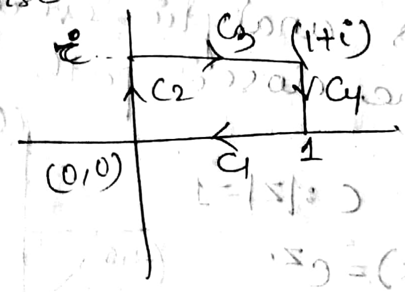
Q. $\int_C \mathrm{Re}\, z\, dz$ C: Vertically from $1+i$ to $1+2i$, then horizontally (1,2) to $3+2i$.

Soln.



C_1: Line joining from $1+i$ to $1+2i$

Q. $\int_C \mathrm{Re}\, z^2 dz$ C: boundary of the square with vertices (0, i, 1+i, 1) Clockwise



Cauchy's Theorem :-

Simple Path :- A closed path that does not intersect or touch itself is called a simple closed path.

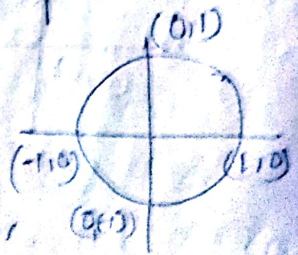
Let $f(z)$ be an analytic fun^n in a domain 'D', then integral over a path 'C' is zero.

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \oint_C f(z)\,dz = 0$$

If $f(z)$ is analytic everywhere inside the closed path.

Q- $\int_C e^z\,dz \quad C : |z| = 1$



$f(z) = e^z$

Sol^n- e^z is analytic,

so $\int_C e^z\,dz = 0 \qquad 0<\theta<2\pi$

Q- $\int \frac{dz}{z-2} \quad c : |z| = 1$

$\int_C (z-2)^{-1}\,dz \quad C : |z| = 1.$

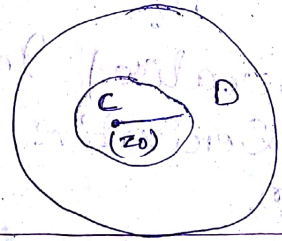
$\Rightarrow f(z) = (z-2)^{-1} dz.$

$f(z)$ Analytic everywhere, except at $z=2$. This implies that $f(z)$ is analytic inside that closed path. So, by Cauchy's theorem, so,

$f(z)\,dz = 0$

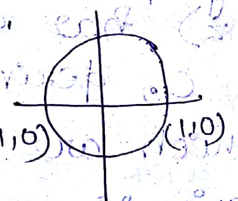
Cauchy's Integral Formula:-

If $f(z)$ is analytic in a simple connected domain D, then for any pt. at $z=z_0$ inside closed path (C in D) that encloses z_0.



$$\boxed{\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\,(f(z_0))}$$

Q. $\int \frac{e^z dz}{z}$ $C: |z|=1$.



Clearly $f(z) = e^z$ is analytic by CR eqn, so, clearly $z_0=0$ is a point that encloses to z. so by C.I formula

$\int \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

$f(z)_{z=0} = e^z = 1 \quad f(0) = 1$

Q- $\int_C \frac{dz}{z-3i}$ $\quad C : |z| = \pi$

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$$\int \frac{f(z)\,dz}{(z-z_0)^2} \qquad \int \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Cauchy's Generalised Integral Formula or Derivative of Analytic fun^n:-

If a fun^n $f(z)$ is analytic in a region C, then $f(z)$ has at any point $z = z_0$ of C, then $f(z)$ has at any point $z = z_0$ of C, derivative of all order which are analytic fun^n of C. Derivative of any point is given by $f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

$\Rightarrow \oint_C \frac{f(z)}{(z-z_0)^{n+1}}$

Working Proce

Case I:- We have $\frac{f(z)}{z-z_0}$ or

given closed we have to point $z = z_0$ the given point lies to use C. the form

$\int_C \frac{f(z)}{z-z_0}$

or $\int_C \frac{f(z)}{(z-z_0)}$

Case II:- If then we

$\int \frac{f(z)}{(z-z_0)^{n+1}}$

given

$$\Rightarrow \oint_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\left(f^n(z_0)\right)$$

Working Procedure:-

Case I:- We have to evaluate $\frac{f(z)}{z-z_0}$ or $\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ over a given closed curve C, whether we have to find out the given point $z = z_0$ lies inside or outside the given curve C. If the given point lies inside C, then we have to use C.I formula, which is of the form

$$\int_c \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

or

$$\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\left[f^n(z_0)\right]$$

Case II:- If $z = z_0$ lies outside C, then we use $g(z) = \int \frac{f(z)}{z-z_0}$ & $\int \frac{f(z)}{(z-z_0)^{n+1}}$ is analytic on the given curve C.

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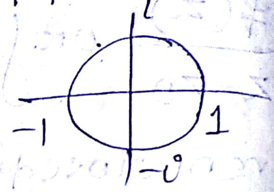
$\int_c g(z)\,dz = 0$, otherwise $z = z_0$ lies outside c, the value of the integral is also zero

Q- $\int \frac{\sin 2z}{z^4}\,dz \quad c: |z| = 1$

$$\int \frac{\sin 2z}{(z-0)^{3+1}}\,dz$$



Clearly, $f(z) = \sin 2z \quad n = 3, z_0 = 0$

$$= \frac{2\pi i}{3!} f'''(z_0)$$

$f(z) = \sin 2z$

$f'(z) = 2\cos 2z$

$f''(z) = -4\sin 2z$

$f'''(z) = -8\cos 2z.$

$f'''(z_0) = -8\cos(0)$

$= -8.$

$$\int \frac{\sin 2z}{z^4}\,dz = \frac{-8\cos z \times 2\pi i}{6}$$

$$= -\frac{8\pi i \cos 2z}{3}$$

$$= -\frac{8\pi i}{3}$$

Q. $\int \frac{e^{-z}\sin z}{z^2}\,dz$

$= \int \frac{e^{-z}\sin z}{(z-0)^{1+1}}\,dz$

clearly, $f(z) = e^{-z}\sin z$, $n=1$, $z_0=0$

$f(z) = e^{-z}\sin z$

$f'(z) = -e^{-z}\cos z$

$f(z_0) = -e^{-z}\cos(0) = -e^{-z}$

$\frac{2\pi i}{1!}(e^{-z}) = 2\pi i e^{-z}$

Q. $\int \frac{z^2\,dz}{z^4-1}$ $\quad c : |z+1| = 1$

$z-(-1) = 1.$

z^4-1

$\Rightarrow (z^2-1)(z^2+1) = 0$

$\Rightarrow (z+1)(z-1)(z+i)(z-i)$

$z = \pm 1 \quad z = \pm i$

$\int \frac{z^2}{(z+1)(z-1)(z+i)(z-i)}\,dz,$

$\int_c \frac{dz}{z+1}$, where, $g(z) = \frac{z^2}{(z-1)(z^2+1)}$

$\Rightarrow \frac{dz}{z-(-1)} = 2\pi i f(z_0)$

$z - z_0$ $\quad = 2\pi i g(z_0)$

$$g(z) = \frac{z^2}{(z^2+1)(z-1)} \quad z_0 = -1$$

$$g(z_0) = \frac{(-1)^2}{\cancel{2}((-1)^2+1)(-1-1)}$$

$$= \frac{1}{1+1i \times -2} = \frac{1}{-2 \times -2i}$$

$$2\pi i \times \frac{-1}{4i} = \frac{-1}{2}\pi$$

Q- $\displaystyle\int \frac{z^3 dz}{2z-i} \quad C: |z|=1$

$$\int \frac{z^3}{2(z-i/2)} dz \qquad z_0 = i/2$$

$$f(z) = \frac{z^3}{2} dz \qquad f(z_0) = \left(\frac{i}{2}\right)^3$$

$$\Rightarrow \frac{1}{2}\int_C \frac{z^3}{z-i/2} dz \qquad = \frac{i^3}{8}$$

$$= \frac{-1}{8} i$$

$$= 2\pi i \times \frac{1}{2} \times \frac{-1}{8} i$$

$$= \frac{-\pi i^2}{8}$$

$$= \pi/8$$

Q- $\int \frac{\cosh 2z}{3z}$ $C : |z| = 1$.

$\cancel{3(z-z_0)}$

$g(z_0) = \frac{\cosh 2z}{3}$

$g(z_0) = \frac{1}{3}$ $2\pi i \times \frac{1}{3} = \frac{2\pi i}{3}$

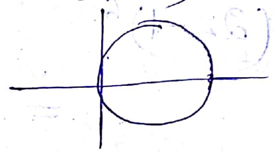
Q- $\int \frac{dz}{z^2+4}$ $C : 4x^2 + (y-2)^2 = 4$

$x^2 + \left(\frac{y-2}{2}\right)^2 = 1$

$z^2+4=0$ $\Rightarrow (x-0)^2 + \frac{(y-2)^2}{2^2} = 1$

$z = \pm 2i$ centre $(0,2)$

$\int \frac{dz}{(z+2i)(z-2i)}$ $z_0 = 2i$



$g(z) = \frac{1}{z+2i}$

$= \frac{1}{2i+2i} = \frac{1}{4i}$.

$2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$.

Q- $\frac{z^6}{(2z-1)^6} dz$ $C : |z| = 1$

$f(z) = z^6$

$\Rightarrow \frac{z^6}{2^6\left(z - \frac{1}{2^6}\right)^6}$ $\Rightarrow \frac{z^6}{2^6\left(z - \frac{1}{2^6}\right)^{5+1}}$.

$z_0 = \frac{1}{2^6} = \frac{1}{2^{5+1}} \quad n=5$

$f(z) = z^6$

$f'(z) = 6\cancel{z}5$

$f''(z) = 30z^4$

$f'''(z) = 120z^3$

$f''''(z) = 360z^2$

$f'''''(z) = 720z$

$$\frac{z^6}{(2z-1)^6} dz = \frac{1}{2^6} \times \frac{2\pi i}{5 \times 4 \times 6} \times 720 \times \frac{1}{26}$$

$$= \frac{45}{4} \times \frac{1}{64} \times \frac{2\pi i}{120}$$

$$= \frac{3\pi i}{1024}$$

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$$\int \frac{dz\, f(z)}{z-z_0} = 2\pi i f(z_0)$$

$$= \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$$

Sequence & Series :-

$1, 2, 3, \ldots\ldots n \rightarrow$ Sequence

$1+2+3+\ldots\ldots+n \rightarrow$ Series.

$$z_1+z_2+\ldots\ldots+z_n = \sum_{n=1}^{\infty} z_n.$$

Convergent $\rightarrow$ tends to particular pt.

$\lim_{n\to\infty} \{z_n\} = K$ $K =$ finite no.

divergent $\rightarrow$ tends to infinite pt.

$\{n\}_{n=1}^{\infty} = \lim_{n\to\infty} |n| = \infty \rightarrow$ divergent.

$1, \frac{1}{2}, \frac{1}{3} \ldots\ldots \frac{1}{n} = \{\frac{1}{n}\}^{\infty}$

$\lim_{n\to\infty} \left|\frac{1}{n}\right|_1^{\infty} = 0$ (finite no.) $\rightarrow$ convergent.

$1, -1, 1, -1, 1, -1, 1, -1 \ldots = \{(-1)^n\}_{n=0}^{\infty}$

$\hookrightarrow$ oscillating sequence.

even $\rightarrow 1$

odd $\rightarrow (-1)$

Series:- Sum of sequence.

$$z_1+z_2+\ldots z_n = \sum_{n=1}^{\infty} z_n$$

This series will be convergent if $\boxed{\lim_{n\to\infty} z_n < 1}$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots + a_n(z-a)^n + \cdots$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad |z| < 1$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}$$

$$\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + \cdots$$

$$\frac{1}{1+z} = (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$$

$$\frac{1}{(1+z)^2} = (1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 \cdots$$

$$\frac{1}{(1-z)^2} = (1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3$$

where, $|z| < 1$, for all cases.

$$f(z) = a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \dots = \sum_{n=1}^{\infty} a_n (z-a)^n.$$

$\frac{1}{R} \to$ be the radius of convergence.

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{C_{n+1}}{C_n} \right|$$

$$C_n = a_n z^n$$

$$C_{n+1} = a_{n+1} z^{n+1}$$

For convergency, $\left| \frac{C_{n+1}}{C_n} \right| < 1$

So, we can say $\boxed{|z-a| < R}$

$a =$ Centre.
$R =$ Radius.

In general, $\sum_{n=1}^{\infty} a_n (z-a)^n$ converges *
at $|z-a| < R$ or $|z-a| > R$
When, $|z-a| = R$, no conclusion,
which may or may not converge at the given condition.

Q. $\sum n(z+i\sqrt{2})^n$. Find the radius of convergence.

Sol^n:- $\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| < 1$

$$C_n = n\,(z+i\sqrt{2})^n$$

$$C_{n+1} = n+1\,(z+i\sqrt{2})^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{(n+1)(z+i\sqrt{2})^{n+1}}{n\,(z+i\sqrt{2})^n}\right| \cancel{<} 1$$

$n+1-n$

$$\lim_{n\to\infty}\left|\frac{n+1}{n}(z+i\sqrt{2})^1\right| < 1.$$

$$\lim_{n\to\infty}\left|1+\frac{1}{n}\right|\left|z+i\sqrt{2}\right| < 1$$

$z+\frac{1}{n}i\sqrt{2},$

$z+$

$$= \left|z-(-i\sqrt{2})\right| < 1$$

$$\boxed{\begin{matrix} a = -i\sqrt{2} \\ R = 1. \end{matrix}}$$

Taylor Series:-

* If $f(z)$ is analytic, at all points inside a circle $C: |z-a| = R$, then for all z inside C, $f(z)$ can be expanded in terms of a series i.e.,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a)$$

$$= \sum_{k=0}^{\infty} \frac{(z-a)^k}{k!} f^k(a)$$

$a =$ centre

If we put $a=0$, then the series becomes, $f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots \frac{z^n}{n!} f^n(0) + \cdots$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} f^n(0)} \longrightarrow \text{Maclaurian Series}$$

Q. Expand $f(z) = \cos z$ in terms of Taylor Series about $\pi/4$.

$a = \pi/4$

$f(z) = \cos z, \quad a = \pi/4.$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}$

$f(z) = \cos z = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right)$

$f(\pi/4) = \frac{1}{\sqrt{2}} \qquad f'(z) = -\sin z,$

$f'(\pi/4) = \frac{-1}{\sqrt{2}}.$

$f''(z) = -\cos z \qquad f'''(z) = \sin z$

$f''(\pi/4) = \frac{-1}{\sqrt{2}} \qquad f'''(\pi/4) = \frac{1}{\sqrt{2}}.$

$$\cancel{= f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{-1}{\sqrt{2}} + \frac{\left(z - \frac{\pi}{4}\right)^2}{2} \times \frac{-1}{\sqrt{2}}$$

$$+ \frac{(z - \pi/4)^3}{6} \times \frac{1}{\sqrt{2}}$$

Q- Expand $\log(1+z)$ in ascending powers of z & hence deduce $\log\sqrt{\left(\frac{1+z}{1-z}\right)}$.

Soln:- $f(z) = \log(1+z)$,

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0)$$

$f(z) = \log(1+z)$, $f(0) = 0$

$f'(z) = \frac{1}{1+z}$ $\quad f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ $\quad f''(0) = -1$

$f'''(z) = \frac{z}{(1+z)^3}$ $\quad f'''(0) = 2$

$$f(z) = 0 + z \cdot 1 + \frac{z^2}{2}(-1) + \frac{z^3}{3 \times 2} \times 2 + \cdots$$

$$\log(1+z) = 0 + z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

$$\log\sqrt{\frac{1+z}{1-z}} = \frac{1}{2}\left[\log(1+z) - \log(1-z)\right].$$

$$= \frac{1}{2}\left[z - \frac{z^2}{2} + \frac{z^3}{3} + \dots - \left(-z + \frac{z^2}{2} - \frac{z^3}{3}\right)\right.$$

$$= \frac{1}{2}\left[z - \cancel{\frac{z^2}{2}} + \frac{z^3}{3} + z - \cancel{\frac{z^2}{2}} + \frac{z^3}{3}\right.$$

$$= \frac{1}{2}\left[2z + 2\frac{z^3}{3}\right].$$

$$= \frac{1}{2}\left[2z + \frac{2z^3}{3}\right].$$

$\boxed{\text{Q. Expand } f(z) = \frac{z+1}{z+2} \text{ in Taylor series at cond}^n \text{ (i) } z=0 \text{ (ii) } z=1}$

$\underset{\text{Mac. series.}}{z=0}$

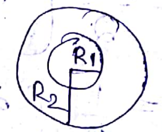
(i) when $z=0$.

$$f(z) = f(0) + z\,f'(0)$$

~~Imp~~ II Laurent's Series :-

If $f(z)$ is analytic inside & on the boundary of annulus is region R, bounded by two concentric circles C_1 & C_2 centred at ~~a~~ A & radii are R_1 & R_2 respectively.

$\boxed{R_1 < R_2}$,



then for every z in R,

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}}$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}$$

Taylor's expansion of $f(z)$ contains only +ve powers of $(z-a)$ whereas Laurent's series of $f(z)$ contains both +ve & -ve powers of $(z-a)$.

So, a_n & b_n can be calculated by using Cauchy's Integral Formula

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}}.$$

$$b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(w)}{(w-a)^{n+1}}$$

Working Procedure of Laurent Series Valid in a Given Region:-

1) Given $f(z)$ which is usually an algebraic funⁿ $\left(\frac{\phi(z)}{\psi(z)}\right)$ is resolved into partial fraction,

2) Each of the resulting term is put in the form $(1 \pm z)^{-1}$ or $(1 \pm z)^{-2}$ or so on. So as to satisfy a valid reason $z(x) = z$. We use suitable binomial expansion of 'z', simplify the like terms whose results in desired Laurent's series comprising with +ve & -ve powers of z.

$(1 \pm z)^{-1}$ or $(1 \pm z)^{-2}$.

Dt:- 16/02/18

Q. Expand $f(z) = \frac{z+1}{(z+2)(z+3)}$ in a terms of Laurent's series in valid regions (a) $|z|>3$ (b) $2<|z|<3$

Soln- $\frac{z+1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\Rightarrow A(z+3) + B(z+2) = (z+2)(z+3)$

$\Rightarrow Az + 3A + zB + 2B = z^2+3z+2z+6$

$\Rightarrow Az + zB + 3A + 2B = z^2 + 5z + 6$

$z:$ $A+B=5$ $\quad$ $3A+2B=6$

$\Rightarrow A = 5-B$ $\quad$ $3(5-B)+2B=6$

$\boxed{A=-1}$ $\quad$ $\Rightarrow 15-3B+2B=6$

$\boxed{B=2}$ $\quad$ $\Rightarrow 15-B=6$

$\Rightarrow -B = 6-15$

$\frac{-1}{z+2} + \frac{2}{z+3}$ $\quad$ $\Rightarrow -B = -9$

$\cancel{\Rightarrow B = 11}$

(a) $|z|>3$

$|z|>2$ & $|z|>3$

$\frac{|z|}{2} > 1$ $\quad$ $\frac{|z|}{3} > 1$

$\frac{2}{|z|} < 1$ $\quad$ $\frac{3}{|z|} < 1$.

$= \frac{-1}{2(1+\frac{2}{z}}$

$= \frac{-1}{z}(1+$

$= \frac{-1}{z}[1$

$\frac{2}{z}$

$= \frac{-1}{z}[1$

$\frac{2}{z}$

$= \frac{-1}{z} +$

$\frac{18}{z}$

Conditi

(b) $2<|z|<$

$$= \frac{-1}{2\left(1+\frac{2}{z}\right)} + 2\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= -\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{2}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{-1}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^{2}-\left(\frac{2}{z}\right)^{3}+\cdots\right]+$$

$$\frac{2}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^{2}-\left(\frac{3}{z}\right)^{3}+\cdots\right].$$

$$= \frac{-1}{\cancel{z}}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right]+$$

$$\frac{2}{z}\left[1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z}+\cdots\right].$$

$$= \frac{-1}{z}+\frac{2}{z^2}-\frac{4}{z^3}+\frac{8}{z^4}+\cdots+\frac{2}{z}-\frac{6}{z^2}+$$

$$\frac{18}{z^3}-\frac{54}{z^2}+\cdots$$

<u>Condition 2</u>:-

(b) $2<|z|<3$ $\quad |z|>2 \;\&\; |z|<3$

$$\frac{|z|}{2}>1 \;\&\; \frac{|z|}{3}<1$$

$$\frac{2}{|z|}<1 \;\&\; \frac{|z|}{3}<1$$

$$\cancel{= \frac{-1}{z}\left(1+\frac{2}{z}\right)^{-1}+\frac{2}{3\left(\frac{z}{3}+1\right)}}$$

$$= \frac{-1}{z\left(1+\frac{2}{z}\right)} + \frac{2}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{-1}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{2}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{-1}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]$$

$$+ \frac{2}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

$$= \frac{-1}{z}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right]$$

$$+ \frac{2}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right].$$

Q- Expand $\frac{z-1}{(z-2)(z-3)^2}$ as Laurent's Series.

(a) $|z| > 3$.

(b) $2 < |z| < 3$.

Soln. $\frac{\cancel{A}-1}{(z-2)(z-3)^2}$

$$= \frac{A}{z-2} + \frac{B}{(z-3)} + \frac{C}{(z-3)^2}$$

$$\Rightarrow (z-2)(z-3)^2 = A(z-3)^2 + B(z-2)(z-3) + C(z-2).$$

$$\Rightarrow (z-2)(z^2+9-6z) = A(z^2+9-6z) + B(z^2-3z-2z+6) + C(z-2$$

$$\Rightarrow z^3 - 2z^2 + 9z - 18 - 6z^2 + 12z = Az^2 + 9A - 6Az + Bz^2 - 5Bz + 6B + Cz - 2C.$$

$$z^2 : -2z^2 - 6z^2 = Az^2 + Bz^2$$

$$\Rightarrow -8z^2 = Az^2 + Bz^2$$

$$A + B = 8$$

$$z: \quad 9z + 12z = -6Az - 5Bz + Cz$$

$$\cancel{-6} - 5 +$$

$$21 = -6A - 5B + C$$

$$9A - 2C = -18$$

$$(z-1) = A(z-3)^2 + B(z-2)(z-3) + C(z-2).$$

Put $z = 2 = 0 \Rightarrow z = 2$

$$\boxed{A = 1}$$

$$z - 3 = 0 \Rightarrow \boxed{C = 2}$$

$$z = 3$$

$$z^2 \to 0 = A + B \Rightarrow \boxed{B = -1.}$$

$$= \frac{1}{z-2} - \frac{1}{z-3} + \frac{2}{(z-3)^2}.$$

$|z| > 3$

$|z| > 2 \quad |z| > 3$

$$\frac{2}{|z|} < 1 \quad \frac{3}{|z|} < 1.$$

$$= \frac{1}{z\left(1 - \frac{2}{z}\right)} + \frac{-1}{z\left(1 - \frac{3}{z}\right)} + \frac{2}{z^2\left(1 - \frac{3}{z}\right)^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{(-1)}{z}\left(1-\frac{3}{z}\right)^{-1} + \frac{2}{z^2}\left(1-\frac{3}{z}\right)^{-2}$$

$$= \frac{1}{z}\left[1 \cancel{+} -\frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right]$$

$$-\frac{1}{z}\left[1 \cancel{+} -\frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right]$$

$$+ \frac{2}{z^2}\left[1 + \frac{6}{z} + \frac{27}{z^2} + \cdots\right]$$

(b) $2 < |z| < 3$

$|z| > 2$ $\qquad$ $|z| < 3$

$\frac{|z|}{2} > 1$ $\qquad$ $\frac{|z|}{3} < 1$.

$\frac{2}{|z|} < 1$

$$= \frac{1}{z-2} + \frac{(-1)}{z-3} + \frac{2}{(z-3)^2}$$

$$= \frac{1}{z\left(1-\frac{2}{z}\right)} + \frac{(-1)}{3\left(\frac{z}{3}-1\right)} + \frac{2}{3^2\left(\frac{z}{3}-1\right)^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{(-1)}{3}\left(\frac{z}{3}-1\right)^{-1} + \frac{2}{9}\left(\frac{z}{3}-1\right)^{-2}$$

$$= \frac{1}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]$$

$$-\frac{1}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

Q. ~~In Laurent~~ Expand $\frac{z}{(z-1)(2-z)}$ in Laurent series in $|z-1|<1$.

Soln:- Put $\cancel{|z-1|<}$ $z-1=u$
$|u|<1$

$$\frac{u+1}{u(1-u)} = \frac{u+1}{u-u^2}$$

$$= \frac{A}{u} + \frac{B}{1-u}$$

Q. Expand $\frac{1}{(z+1)(z+3)}$ in Laurent series for the region $0<|z+1|<2$

Soln. Put $z+1=u \Rightarrow 0<|u|<2$

$$= \frac{u-z}{u(u+2)}$$

$z+1+2$

Zeros & Singularities :-

$f(x) = x-2$ $f(z)=0$

$f(z)=0$.

$z_1, z_2 \ldots$ are the ~~func of zeros~~ zeros of the fun^n

$\sin z = 0 \Rightarrow z = n\pi$ be the zeroes of the given fun^n (i.e, $\sin z$).

$\cos z = 0 \Rightarrow z = (2n+1)\frac{\pi}{2}$.

Singular Point:- In which point the fun^n is not analytic.

$f(z) = \frac{1}{z-2}$ Point $z=a$ is called singular pt. of $f(z)$ if $f(z)$ is not analytic at $z=a$.

Here, $z-2=0$
$\Rightarrow z=2$ → not analytic.

Clearly $z=2$ is the singular point

Q. $f(z) = \frac{1}{z}$ At $z=0$ it is not analytic

$z=0$ is the singular point.

Q.:- $\frac{1}{z(z+2)(z-1)}$.

Sing. pt- $z=0, -2, 1$.

Isolated Sing

$\frac{1}{z(z+2)(z-1)}$

A Point $z=a$ is singularity of not analytic point in its except 'a'.

i) Removal
ii) Pole.
iii) Essential

Date:- 17|02|

$f(z) = \sum_{n=0}^{\infty} a_n$
Anal

1. Removal Si

If a

expanded in Series, ther contains singularity singularity.

Isolated Singularity :-

$$\frac{1}{z(z+2)(z-1)}$$

A Point $z=a$ is said to be isolated singularity of $f(z)$ if $f(z)$ is not analytic but analytic every point in its neighbourhood points except 'a'.

i) Removal singularity.
ii) Pole. singularity.
iii) Essential singularity.

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$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{\text{Principal part}}$$

(i) Removal singularity :-

If a func $f(z)$ can be expanded in terms of a Laurent series, then the principal part contains no terms, then the singularity $z=a$ is called removal singularity.

$$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!}}{z}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^5}{5!}$$

$z=0$ is a ~~so~~ removal singularity of the given funⁿ.

iii) <u>Essential Singularity</u>:-

If a funⁿ $f(z)$ can be expanded in terms of Laurent's series, then the principal part contain infinite no. of series, the singularity $z=a$ is called essential singularity.

$$f(z) = \frac{e^{1/z}}{z}$$

$$= \frac{1 + \left(\frac{1}{z}\right) + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!}}{z}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{2!\,z^3} + \frac{1}{3!\,z^4} + \cdots$$

(iii) **Pole singularity:-** If a funⁿ $f(z)$ can be expanded in terms of Laurent's series, then the principal part contains finite (no.) of series, then the singularity $z=a$ is called pole singularity.

$$f(z) = \frac{e^z}{z} = \frac{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots}{z}$$

$$= \frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots$$

Qs-

$\frac{1-\sin z}{z}$, $\frac{1-\cos z}{z}$, $\frac{\log(1+z)}{2}$, $\frac{\cos z}{2}$

Decide the type of singularity.

★ Pole:-

$$f(z) = \frac{g(z)}{(z-a)^m (z-b)^n}$$

Clearly, $g(z)$ is analytic where $g(a) \neq 0$ & $g(b) \neq 0$, then $z=a$ & $z=b$ be the poles of order m & n respectively. In other words we can say the poles of a given funⁿ can be calculated by putting denominator $= 0$.

$$f(z) = \frac{z^2}{(z-1)(z-2)}$$

Pole : $(z-1)(z-2) = 0$

$z = 1,\ z = 2$ be the simple poles of order 1. (SP)

$$f(z) = \frac{z^3+2}{(z-2)^3(z-1)}$$

Pole : $(z-2)^3 = 0$ & $(z-1) = 0$

So, $z = 2, 2, 2$ & $z = 1$.

Clearly, $z = 2$ is a multiple pole of order 3. (MP)

& clearly, $z = 1$ is a SP.

Residue :- In a Laurent series expansion the coefficient of $\frac{1}{z^n}$ is b_n in the fun $f(z)$ is called residue to $f(z)$ at $z = a$

Residue at Simple Poles :- Suppose $z = z_0$ be the simple pole, then residue of $f(z)$ at $z = z_0$ is given by

$$\boxed{\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \left[(z-z_0) f(z)\right]}$$

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